C. S. BAGEWADI, D. G. PRAKASHA and VENKATESHA

On pseudo projectively flat LP-Sasakian manifold with a coefficient $\alpha$

Abstract. Recently, the notion of Lorentzian almost paracontact manifolds with a coefficient $\alpha$ has been introduced and studied by De et al. [1]. In the present paper we investigate pseudo projectively flat LP-Sasakian manifold with a coefficient $\alpha$.

1. Introduction. In 1989, Matsumoto [2] introduced the notion of LP-Sasakian manifolds. Then Mihai and Rosca [3] introduced the same notion independently and they obtained several results in this manifold. In a recent paper, De, Shaikh and Sengupta [1] introduced the notion of LP-Sasakian manifolds with a coefficient $\alpha$, which generalizes the notion of LP-Sasakian manifolds.

In the present paper we study pseudo projectively flat LP-Sasakian manifold with a coefficient $\alpha$. Here we prove that in a pseudo projectively flat LP-Sasakian manifolds with a coefficient $\alpha$ the characteristic vector field is a concircular vector field if and only if the manifold is $\eta$-Einstein and pseudo projectively flat LP-Sasakian manifold with a coefficient $\alpha$ is a manifold of constant curvature if the scalar curvature $r$ is a constant.

2. Preliminaries. Let $M$ be the $n$-dimensional differential manifold endowed with a $(1,1)$ tensor field $\phi$, a contravariant vector field $\xi$, a covariant
vector field $\eta$ and a Lorentzian metric $g$ of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-, +, +, \ldots, +)$, where $T_pM$ denotes the tangent vector space of $M$ at $p$ and $R$ is the real number space, which satisfies

\begin{equation}
\eta(\xi) = -1, \quad \phi^2 X = X + \eta(X)\xi, \quad (2.1)
\end{equation}

\begin{equation}
g(X, \xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \quad (2.2)
\end{equation}

for all vector fields $X$ and $Y$. Then such a structure $(\phi, \xi, \eta, g)$ is termed as Lorentzian almost paracontact structure and the manifold $M$ with the structure $(\phi, \xi, \eta, g)$ is called Lorentzian almost paracontact manifold $M$ [2].

In the Lorentzian almost paracontact manifold $M$, the following relations hold [2]:

\begin{equation}
\phi\xi = 0, \quad \eta(\phi X) = 0, \quad (2.3)
\end{equation}

\begin{equation}
\omega(X, Y) = \omega(Y, X) \quad (2.4)
\end{equation}

where $\omega(X, Y) = g(X, \phi Y)$. In the Lorentzian almost paracontact manifold $M$, if the relations

\begin{equation}
(\nabla_Z \omega)(X, Y) = \alpha [g(X, Z) + \eta(X)\eta(Z)]\eta(Y) + (g(Y, Z) + \eta(Y)\eta(Z))\eta(X) \quad (2.5)
\end{equation}

and

\begin{equation}
\omega(X, Y) = \frac{1}{\alpha}(\nabla_X \eta)(Y) \quad (2.6)
\end{equation}

hold, where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$, then $M$ is called an LP-Sasakian manifold with a coefficient $\alpha$ [1]. An LP-Sasakian manifold with coefficient 1 is an LP-Sasakian manifold [2].

If a vector field $V$ satisfies the equation of the following form:

\[ \nabla_X V = \beta X + T(X)V, \]

where $\beta$ is a non-zero scalar function and $T$ is a covariant vector field, then $V$ is called a torse-forming vector field [5].

In a Lorentzian manifold $M$, if we assume that $\xi$ is a unit torse-forming vector field, then

\begin{equation}
(\nabla_X \eta)(Y) = \alpha [g(X, Y) + \eta(X)\eta(Y)], \quad (2.7)
\end{equation}

where $\alpha$ is a non-zero scalar function. Hence the manifold admitting a unit torse-forming vector field satisfying (2.7) is an LP-Sasakian manifold with a coefficient $\alpha$. And, if $\eta$ satisfies

\begin{equation}
(\nabla_X \eta)(Y) = \varepsilon [g(X, Y) + \eta(X)\eta(Y)], \quad \varepsilon^2 = 1, \quad (2.8)
\end{equation}

\[ \nabla_X V = \beta X + T(X)V, \]

where $\beta$ is a non-zero scalar function and $T$ is a covariant vector field, then $V$ is called a torse-forming vector field [5].
then $M$ is called an LSP-Sasakian manifold [2]. In particular, if $\alpha$ satisfies (2.7) and the equation of the following form:

$$\alpha(X) = P\eta(X), \quad \alpha(X) = \nabla_X \alpha,$$

where $P$ is a scalar function, then $\xi$ is called a concircular vector field.

Let us consider an LP-Sasakian manifold $M$ with the structure $(\phi, \xi, \eta, g)$ and with a coefficient $\alpha$. Then we have the following relations [1]:

$$\eta(R(X, Y)Z) = -\alpha(X)\omega(Y, Z) + \alpha(Y)\omega(X, Z) + \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)],$$

and

$$S(X, \xi) = -\psi\alpha(X) + (n - 1)\alpha^2\eta(X) + \alpha(\phi X),$$

where $R, S$ denote respectively the curvature tensor and the Ricci tensor of the manifold and $\psi = \text{Trace}(\phi)$.

We now state the following results, which are used in the later section.

**Lemma 2.1** ([1]). In an LP-Sasakian manifold $M$ with a non-constant coefficient $\alpha$, one of the following cases occurs:

i) $\psi^2 = (n - 1)^2$

ii) $\alpha(Y) = -P\eta(Y),$

where $P = \alpha(\xi)$.

**Lemma 2.2** ([1]). In a Lorentzian almost paracontact manifold $M(\phi, \xi, \eta, g)$ with its structure $(\phi, \xi, \eta, g)$ satisfying $\omega(X, Y) = \frac{1}{\alpha}(\nabla_X \eta)(Y)$, where $\alpha$ is a non-zero scalar function, the vector field $\xi$ is torse-forming if and only if the relation $\psi^2 = (n - 1)^2$ holds.

3. Pseudo projectively flat LP-Sasakian manifold with a coefficient $\alpha$. Let us consider a pseudo projectively flat LP-Sasakian manifold $M$ $(n > 3)$ with a coefficient $\alpha$. First suppose that $\alpha$ is not constant. Then since the pseudo projective curvature tensor vanishes, the curvature tensor $'^R$ satisfies [4]

$$'^R(X, Y, Z, W) = -\frac{b}{a}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)]$$

$$+ \frac{r}{n} \left[\frac{1}{n - 1} + \frac{b}{a}\right][g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]$$

and

$$'^R(X, Y, Z, W) = g(R(X, Y)Z, W),$$

where $a, b$ are constants such that $a, b \neq 0$ and $a + b(n - 1) \neq 0$, $r$ is the scalar curvature of the manifold. Putting $W = \xi$ in (3.1) and then using
(2.10) and (2.11), we get

\[
- \alpha(X)\omega(Y,Z) + \alpha(Y)\omega(X,Z) + \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]
\]

\[
(3.2)
\]

\[
= - \frac{b}{a} [S(Y,Z)\eta(X) - S(X,Z)\eta(Y)]
\]

\[
+ \frac{r}{n} \left( \frac{1}{n-1} + \frac{b}{a} \right) [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].
\]

Again if we put \(X = \xi\) in (3.2) and using (2.3) and (2.11), we obtain

\[
S(Y,Z) = \left[ - \frac{a}{b} \alpha^2 + \frac{ar}{bn(n-1)} + \frac{r}{n} \right] g(Y,Z)
\]

\[
+ \left[ - \frac{a}{b} \alpha^2 - (n-1)\alpha^2 + \frac{ar}{bn(n-1)} + \frac{r}{n} \right] \eta(Y)\eta(Z)
\]

\[
+ \psi\alpha(Z) - \alpha(\phi Z)\eta(Y) - \frac{a}{b} P\omega(Y,Z)
\]

where \(P = \alpha(\xi)\).

If an LP-Sasakian manifold \(M\) with the coefficient \(\alpha\) satisfies the relation

\[
S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),
\]

where \(a, b\) are the associated functions on the manifold, then the manifold \(M\) is called an \(\eta\)-Einstein manifold. Then we have [1]

\[
S(X,Y) = \left[ \frac{r}{n-1} - \alpha^2 - \frac{P\psi}{n-1} \right] g(X,Y)
\]

\[
(3.4)
\]

\[
+ \left[ \frac{r}{n-1} - n\alpha^2 - \frac{nP\psi}{n-1} \right] \eta(X)\eta(Y).
\]

Putting \(X = Y = e_i\), in (3.4), where \(\{e_i\}\) is an orthonormal basis of the tangent space at a point of the manifold and taking summation over \(1 \leq i \leq n\), we get

\[
(3.5)
\]

\[
r = n(n-1)\alpha^2 + n\psi P.
\]

By virtue of (3.3) and (3.4) we get

\[
\left[ \frac{\alpha^2}{b}(a-b) + \frac{r(b-a)}{n(n-1)b} - \frac{P\psi}{(n-1)} \right] g(Y,Z) - \psi\alpha(Z) - \alpha(\phi Z)\eta(Y)
\]

\[
+ \left[ \frac{\alpha^2}{b}(a-b) + \frac{r(b-a)}{n(n-1)b} - \frac{nP\psi}{(n-1)} \right] \eta(Y)\eta(Z)
\]

\[
+ \frac{a}{b} P\omega(Y,Z) = 0.
\]

Putting \(Y = \xi\) in (3.6), we obtain

\[
\psi\alpha(Z) - \alpha(\phi Z) = -\psi P\eta(Z),
\]
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for all $Z$. Replace $Z$ by $Y$ in the above equation, we get

$$\psi \alpha(Y) - \alpha(\phi Y) = -\psi P\eta(Y),$$

for all $Y$. Using (3.7) in (3.6) and then by virtue of (3.5) we get

$$Pa^b \left[ \frac{\psi}{n-1} [g(Y, Z) + \eta(Y)\eta(Z)] + \omega(Y, Z) \right] = 0.$$

If $P = 0$, then from (3.7) we have $\alpha(\phi Y) = \psi \alpha(Y)$. Thus $\psi$ is equal to $\pm 1$ as $\psi$ is an eigenvalue of the matrix $\phi$. Hence, by virtue of Lemma 2.1, we get $\alpha(Y) = 0$ for all $Y$ and so $\alpha$ is constant, which contradicts our assumption.

Consequently, we have $P \neq 0$ and hence from (3.8) we get

$$a^b \left[ \frac{\psi}{n-1} [g(Y, Z) + \eta(Y)\eta(Z)] + \omega(Y, Z) \right] = 0.$$

Putting $Y = \phi Y$ in (3.9) and then using (2.3), we obtain

$$a^b \left[ \frac{\psi}{n-1} \omega(Y, Z) + [g(Y, Z) + \eta(Y)\eta(Z)] \right] = 0.$$

Combining (3.9) and (3.10), we get

$$\{ \psi^2 - (n - 1)^2 \} [g(Y, Z) + \eta(Y)\eta(Z)] = 0,$$

which gives by virtue of $n > 1$

$$\psi^2 = (n - 1)^2.$$

Hence Lemma 2.2 proves that $\xi$ is torse-forming.

We have

$$\nabla_X \eta(Y) = \beta \{ g(X, Y) + \eta(X)\eta(Y) \}.$$

Then from (2.6) we get

$$\omega(X, Y) = \frac{\beta}{\alpha} \{ g(X, Y) + \eta(X)\eta(Y) \} = g \left( \frac{\beta}{\alpha} (X + \eta(X)\xi), Y \right)$$

and $\omega(X, Y) = g(\phi X, Y)$.

Since $g$ is non-singular, we have

$$\phi(X) = \frac{\beta}{\alpha} (X + \eta(X)\xi)$$

and

$$\phi^2(X) = \left( \frac{\beta}{\alpha} \right)^2 (X + \eta(X)\xi).$$

It follows from (2.1) that $\left( \frac{\beta}{\alpha} \right)^2 = 1$ and hence, $\alpha = \pm \beta$. Thus we have

$$\phi(X) = \pm (X + \eta(X)\xi).$$

By virtue of (3.7) we see that $\alpha(Y) = -P\eta(Y)$, where $P = \alpha(\xi)$. Thus, we conclude that $\xi$ is a concircular vector field. Conversely, we suppose that
\( \xi \) is a concircular vector field. Then we have the equation of the following form:

\[
(\nabla_X \eta)(Y) = \beta \{g(X, Y) + \eta(X)\eta(Y)\},
\]

where \( \beta \) is a certain function and \( \nabla_X \beta = q\eta(X) \) for a certain scalar function \( q \). Hence by virtue of (2.6) we have \( \alpha = \pm \beta \). Thus

\[
\Omega(X, Y) = \varepsilon \{g(X, Y) + \eta(X)\eta(Y)\}, \quad \varepsilon^2 = 1,
\]

\[
\psi = \varepsilon(n - 1), \quad \nabla X \alpha = \alpha(X) = p\eta(X), \quad p = \varepsilon q.
\]

Using these relations in (3.3) and (3.7), it can be easily seen that \( M \) is \( \eta \)-Einstein. Thus we can state the following:

**Theorem 3.1.** In a pseudo projectively flat LP-Sasakian manifold \( M \) \((n > 1)\) with a non-constant coefficient \( \alpha \), the characteristic vector field \( \xi \) is a concircular vector field if and only if \( M \) is \( \eta \)-Einstein.

Next we consider the case where the coefficient \( \alpha \) is constant. In this case the following relations hold:

(3.12) \[
\eta(R(X, Y)Z) = \alpha^2 \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}
\]

(3.13) \[
S(X, \xi) = (n - 1)\alpha^2\eta(X).
\]

Putting \( W = \xi \) in (3.1) and then using (3.12) and (3.13), we get

(3.14) \[
a \cdot \alpha^2 \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} + b[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)]
\]

Again putting \( X = \xi \) in (3.14) we get by virtue of (3.13) that

(3.15) \[
S(Y, Z) = \left[\frac{r}{n} \left(1 + \frac{a}{b(n - 1)}\right) - \frac{a}{b}\alpha^2\right] g(Y, Z)
\]

\[+ \frac{(a + b(n - 1))}{b} \left[\frac{r}{n(n - 1)} - \alpha^2\right] \eta(Y)\eta(Z).
\]

Hence we can state the following:

**Theorem 3.2.** A pseudo projectively flat LP-Sasakian manifold \( M \) \((n > 1)\) with a constant coefficient \( \alpha \) is an \( \eta \)-Einstein manifold.

Differentiating (3.15) covariantly along \( X \) and making use of (2.6) we get

(\[\nabla_X S\])(Y, Z) = \frac{dr(X)}{n - 1} \left(1 + \frac{a}{b(n - 1)}\right) [g(Y, Z) + \eta(Y)\eta(Z)]
\]

\[+ \alpha(a + b(n - 1)) \left[\frac{r}{n(n - 1)} - \alpha^2\right] \times [\omega(X, Y)\eta(Z) + \omega(X, Z)\eta(Y)].\]
where $dr(X) = \nabla_X r$. This implies that
\[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)\]
\[
= \frac{dr(X)}{n-1} \left(1 + \frac{a}{b(n-1)}\right) [g(Y, Z) + \eta(Y)\eta(Z)]
\]
\[
- \frac{dr(Y)}{n-1} \left(1 + \frac{a}{b(n-1)}\right) [g(X, Z) + \eta(X)\eta(Z)]
\]
\[
+ \frac{\alpha(a + b(n-1))}{b} \left[ \frac{r}{n(n-1)} - \alpha^2 \right]
\]
\[
\times [\omega(X, Z)\eta(Y) - \omega(Y, Z)\eta(X)].
\]

On the other hand, in our case, since we have $(\nabla_X P)(X, Y)Z = 0$, we get $\text{div} P = 0$, where “\text{div}” denotes the divergence. So for $n > 1$, $\text{div} P = 0$ gives
\[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)\]
\[
= \frac{1}{n(a+b)} \left[ \frac{a + (n-1)b}{n-1} \right] [g(Y, Z)dr(X) - g(X, Z)dr(Y)].
\]

It follows from (3.16) and (3.17) that
\[
\frac{1}{n(a+b)} \left[ \frac{a + (n-1)b}{n-1} \right] [g(Y, Z)dr(X) - g(X, Z)dr(Y)]
\]
\[
= \frac{dr(X)}{n-1} \left(1 + \frac{a}{b(n-1)}\right) [g(Y, Z) + \eta(Y)\eta(Z)]
\]
\[
+ \frac{dr(Y)}{n-1} \left(1 + \frac{a}{b(n-1)}\right) [g(X, Z) + \eta(X)\eta(Z)]
\]
\[
+ \frac{\alpha(a + b(n-1))}{b} \left[ \frac{r}{n(n-1)} - \alpha^2 \right]
\]
\[
\times [\omega(X, Z)\eta(Y) + \omega(Y, Z)\eta(X)].
\]

If $r$ is constant, then from (3.18) we obtain
\[
\frac{\alpha(a + b(n-1))}{b} \left[ \frac{r}{n(n-1)} - \alpha^2 \right] = 0.
\]

Since $a + b(n-1) \neq 0$, the above equation gives
\[(3.19) \quad r = n(n-1)\alpha^2.
\]

Now substituting (3.15) in (3.1) we get
\[
'\mathcal{R}(X, Y, Z, W) = \alpha^2 [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]
\]
\[
+ \frac{(a + b(n-1))}{a} \left( \frac{r}{n(n-1)} - \alpha^2 \right)
\]
\[
\times [g(Y, W)\eta(X)\eta(Z) - g(X, W)\eta(Y)\eta(Z)].
\]
Hence by using (3.19) in (3.20) it follows that,
\[ ('R(X,Y,Z,W) = \alpha^2 [g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \]
This shows that the manifold is of constant curvature. Thus we can state the following:

**Theorem 3.3.** In a pseudo projectively flat LP-Sasakian manifold \( M (n > 1) \) with a constant coefficient \( \alpha \), if the scalar curvature \( r \) is constant, then \( M \) is of constant curvature.

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**References**


C. S. Bagewadi, D. G. Prakasha
Department of Mathematics and Computer Science
Kuvempu University
Jnana Sahyadri-577 451, Shimoga
Karnataka, India
e-mail: prof_bagewadi@yahoo.co.in

Venkatesha
Department of Mathematics and Computer Science
Kuvempu University
Jnana Sahyadri-577 451, Shimoga
Karnataka, India
e-mail: vens_2003@rediffmail.com

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