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Estimates of $L_p$ norms for sums of positive functions

Abstract. We present new inequalities of $L_p$ norms for sums of positive functions. These inequalities are useful for investigation of convergence of simple partial fractions in $L_p(\mathbb{R})$.

Let $p_n$ be a polynomial of degree $n$ with zeros $z_1, z_2, \ldots, z_n$. The logarithmic derivative of $p_n$

$$g_n(t) = \frac{p'_n(t)}{p_n(t)} = \sum_{k=1}^{n} \frac{1}{t - z_k}$$

is called a simple partial fraction.

Let $z_k = x_k + iy_k$. V. Yu. Protasov [1] showed that if

$$\sum_{k=1}^{\infty} \frac{1}{|y_k|^{1/q}} < +\infty, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

then the series

$$g_{\infty}(t) = \sum_{k=1}^{\infty} \frac{1}{t - z_k}$$

converges in $L_p(\mathbb{R})$.

In [1] the problem to find necessary and sufficient conditions for convergence of the series $g_{\infty}$ in $L_p(\mathbb{R})$ was posed. Protasov proved that if $g_{\infty}$

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converges in $L_p(\mathbb{R})$ and all $z_k$ lie in the angle $|z| \leq C|y|$ with a fixed $C$, then for all $\varepsilon > 0$ the following condition holds:

$$\sum_{k=1}^{\infty} \frac{1}{|y_k|^{1/q+\varepsilon}} < +\infty$$

Therefore, we see that the sufficient condition (1) is quite close to the necessary condition (2).

In the paper [2] we proved the following theorem.

**Theorem 1.** Let $p > 1$. If

$$\sum_{k=1}^{\infty} \frac{k^{p-1}}{|y_k|^{p-1}} < +\infty,$$

then the series

$$g_\infty(t) = \sum_{k=1}^{\infty} \frac{1}{t - z_k}$$

converges in $L_p(\mathbb{R})$. Conversely, if $g_\infty(t)$ converges in $L_p(\mathbb{R})$, the sequence $|y_n|$ is increasing and $|z_k| \leq C|y_k|$, then the condition (3) holds.

The proof of Theorem 1 is based on the following fact.

For any $p \geq 2$ there exists a positive constant $C_p$ depending only on $p$ such that the following inequality holds

$$\int_{-\infty}^{+\infty} \left( \sum_{k=1}^{n} \frac{y_k}{(t-x_k)^2 + y_k^2} \right)^p \, dt \leq C_p \sum_{k=1}^{n} \frac{k^{p-1}}{|y_k|^{p-1}}.$$

It turns out that there exists a nontrivial generalization of this result for arbitrary positive functions from arbitrary measurable space.

To be precise, let $X$ be a measurable space with positive measure $\mu$. Suppose that $f_k \in L_1(X, \mu) \cap L_\infty(X, \mu)$ and $f_k \geq 0$, $k = 1, 2, \ldots, n$.

We set

$$L = \max_{1 \leq k \leq n} \int_X f_k \, d\mu,$$

$$M_k = \|f_k\|_\infty.$$

The aim of the present paper is the following theorem.

**Theorem 2.** If $p \in (1, 2]$, then there exists $C_p$ such that

$$\int_X \left( \sum_{k=1}^{n} f_k \right)^p \, d\mu \leq C_p L \sum_{j=1}^{n} \left( \sum_{k=j}^{n} M_k \right)^{p-1}.$$
If \( p \in [2, +\infty) \), then there exists \( C_p \) such that

\[
\left( \sum_{k=1}^{n} f_k \right)^p \leq C_p L \sum_{k=1}^{n} (k M_k)^{p-1}.
\]

To prove Theorem 2 we need the following

**Lemma.** For any natural \( p \) the following inequality holds

\[
\left( \sum_{k=1}^{n} f_k \right)^p \leq p! (p - 1)! L \sum_{k=1}^{n} (k M_k)^{p-1}.
\]

**Proof.** We multiply out and then integrate term by term:

\[
\int_X \left( \sum_{k=1}^{n} f_k \right)^p d\mu
\]

\[
= \sum_{k_1, k_2, \ldots, k_p} \int_X f_{k_1} f_{k_2} \cdots f_{k_p} d\mu
\]

\[
\leq p! \sum_{k_1 \geq k_2 \geq \cdots \geq k_p} \int_X f_{k_1} f_{k_2} \cdots f_{k_p} d\mu
\]

\[
\leq p! \sum_{k_1 \geq k_2 \geq \cdots \geq k_p} \int_X M_{k_1} M_{k_2} \cdots M_{k_{p-1}} f_{k_p} d\mu
\]

\[
= p! \sum_{k_1 \geq k_2 \geq \cdots \geq k_{p-1}} M_{k_1} M_{k_2} \cdots M_{k_{p-1}} \sum_{k_p=1}^{k_{p-1}} \int_X f_{k_p} d\mu
\]

\[
\leq p!L \sum_{k_1 \geq k_2 \geq \cdots \geq k_{p-1}} M_{k_1} M_{k_2} \cdots M_{k_{p-2}} M_{k_{p-1}} M_{k_{p-1}}.
\]

In these inequalities the indexes \( k_1, k_2, \ldots, k_p \) are varying from 1 to \( n \). We note that for \( p = 1 \) the last sum is equal to \( n \pi \). For \( p = 2 \) that sum is equal to \( 2 \pi \sum_{k=1}^{n} k M_k \).

It is clear that to prove (6) it is enough to show that

\[
\sum_{k_1=1}^{n} M_{k_1} \sum_{k_2=1}^{n} M_{k_2} \cdots \sum_{k_{p-2}=1}^{n} M_{k_{p-2}} \sum_{k_{p-1}=1}^{n} M_{k_{p-1}} \sum_{k_{p-1}=1}^{n} M_{k_{p-1}} \leq (p - 1)! \sum_{k=1}^{n} (k M_k)^{p-1}.
\]

This inequality was established in the paper \([2]\). Lemma is proved. \( \square \)

**Proof of Theorem 2.** We have

\[
\int_X \left( \sum_{k=1}^{n} f_k \right)^p d\mu = \int_X \left( \sum_{k=1}^{n} f_k \right)^p \left( \sum_{k=1}^{n} f_k \right)^{p-1} d\mu \leq 2^{p-1} (I_1 + I_2),
\]
where
\[ I_1 = \int \sum_{j=1}^{n} f_j \left( \sum_{k=1}^{j} f_k \right)^{p-1} d\mu, \]
\[ I_2 = \int \sum_{j=1}^{n} f_j \left( \sum_{k=j+1}^{n} f_k \right)^{p-1} d\mu. \]

Here we have used the classical inequality \((a + b)^\alpha \leq 2^\alpha (a^\alpha + b^\alpha)\) which holds for all positive \(a, b, \alpha\).

It is easy to see that
\[ I_2 \leq \int \sum_{j=1}^{n} f_j \left( \sum_{k=j+1}^{n} M_k \right)^{p-1} d\mu \leq L \sum_{j=1}^{n} \left( \sum_{k=j+1}^{n} M_k \right)^{p-1}. \]

Further we shall consider the cases \(p \leq 2\) and \(p > 2\) separately.

**Case** \(p \in (1, 2]\).

To get an upper estimate for \(I_1\) we use the Hölder inequality
\[ I_1 \leq \sum_{j=1}^{n} \left( \int f_j^\alpha d\mu \right)^{1/\alpha} \left( \int \left( \sum_{k=1}^{j} f_k \right)^{(p-1)\beta} d\mu \right)^{1/\beta} \]
with parameters \(\alpha = 1/(2 - p), \beta = 1/(p - 1)\). Therefore,
\[ I_1 \leq \sum_{j=1}^{n} \left( \int f_j^\alpha d\mu \right)^{2-p} \left( \int \left( \sum_{k=1}^{j} f_k d\mu \right)^{p-1} \right) \]
\[ = \sum_{j=1}^{n} \left( \int f_j^{\alpha-1} f_j d\mu \right)^{2-p} \left( \int \left( \sum_{k=1}^{j} f_k d\mu \right)^{p-1} \right) \]
\[ \leq \sum_{j=1}^{n} \left( M_j^{\alpha-1} L \right)^{2-p} (jL)^{p-1} = L \sum_{j=1}^{n} (jM_j)^{p-1}. \]

Applying Copson’s inequality ([3], Theorem 344)
\[ \sum_{n=1}^{\infty} (a_n + a_{n+1} + \cdots)^{p-1} > (p - 1)^{p-1} \sum_{n=1}^{\infty} (na_n)^{p-1} \]
we get
\[ I_1 \leq L \sum_{j=1}^{n} \left( \sum_{k=j}^{n} M_k \right)^{p-1}. \]

This inequality together with (7) gives us desired estimate (4) which proves Theorem 2 in case when \(p \leq 2\).
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Case $p \in (2, +\infty)$.

It follows from Copson’s inequality ([3], Theorem 331)

$$
\sum_{n=1}^{\infty} (a_n + a_{n+1} + \cdots)^{p-1} \leq (p-1)^{p-1} \sum_{n=1}^{\infty} (na_n)^{p-1}
$$

that

$$
I_2 \leq L(p-1)^{p-1} \sum_{j=1}^{n} j^{p-1} M_j^{p-1}.
$$

To estimate $I_1$ we again use the Hölder inequality

$$
I_1 \leq \sum_{j=1}^{n} L^{1/\alpha} M_j^{(\alpha-1)/\alpha} \left( \int_X \left( \sum_{k=1}^{j} f_k \right)^{m} \mu \right)^{(p-1)/m}
$$

with parameters $\alpha = m/(m+1-p)$, $\beta = m/(p-1)$ where $m$ is the integer part of $p$. Further, Lemma and Hölder’s inequality yield the following estimates

$$
I_1 \leq L \sum_{j=1}^{n} M_j^{(p-1)/m} \left( \pi m!(m-1)! \sum_{k=1}^{j} (kM_k)^{m-1} \right)^{(p-1)/m}
$$

$$
= LC(m,p) \sum_{j=1}^{n} (jM_j)^{(p-1)/m} \left( \frac{1}{j} \sum_{k=1}^{j} (kM_k)^{m-1} \right)^{(p-1)/m}
$$

$$
\leq LC(m,p) \left( \sum_{j=1}^{n} (jM_j)^{\alpha_1(p-1)/m} \right)^{1/\alpha_1}
$$

$$
\times \left( \sum_{j=1}^{n} \left( \frac{1}{j} \sum_{k=1}^{j} (kM_k)^{m-1} \right)^{\beta_1(p-1)/m} \right)^{1/\beta_1}.
$$

Setting $\alpha_1 = m$, $\beta_1 = m/(m-1)$, we obtain

$$
I_1 \leq LC(m,p) \left( \sum_{j=1}^{n} (jM_j)^{p-1} \right)^{1/m}
$$

$$
\times \left( \sum_{j=1}^{n} \left( \frac{1}{j} \sum_{k=1}^{j} (kM_k)^{m-1} \right)^{(p-1)/(m-1)} \right)^{(m-1)/m}.
$$
Applying Hardy’s inequality ([3], Theorem 326)
\[ \sum_{n=1}^{\infty} \left( \frac{a_1 + a_2 + \cdots + a_k}{k} \right)^s \leq \left( \frac{s}{s-1} \right)^s \sum_{k=1}^{\infty} a_k^s \]
with \( s = (p - 1)/(m - 1) \) and \( a_k = (kM_k)^{m-1} \) we see that
\[ I_1 \leq LC(m, p) \left( \sum_{j=1}^{n} (jM_j)^{p-1} \right)^{1/m} \left( s^s(s-1)^{-s} \sum_{j=1}^{n} (jM_j)^{p-1} \right)^{(m-1)/m} \]
\[ = LC_1(m, p) \sum_{j=1}^{n} (jM_j)^{p-1}. \]
This estimate together with (8) proves (5). Theorem 2 is proved. \( \square \)

Let us remark that the inequalities (4) and (5) are sharp up to some absolute constant depending on \( p \) only. This can be easily seen by setting \( f_j \equiv 1 \) on \( X \). Other examples can be constructed as follows: \( X = \mathbb{R} \) and
\[ f_k(t) = \frac{y_k}{(t-x_k)^2 + y_k^2}. \]
In case when \( |y_k| \) is increasing sequence, it was proved in the paper [2] that the sign \( \leq \) in the inequalities (4) and (5) can replaced by \( \geq \) with some other absolute constant \( c_p > 0 \) depending on \( p \) only.

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References

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