A coefficient product estimate
for bounded univalent functions

Dedicated to Professor Zdzisław Lewandowski
on his 70-th birthday

Abstract. For given integers \( m \) and \( n \), \( 2 < m < n \), we consider the problem \( \max |a_m a_n| \) in the class \( S(M) \) of holomorphic, univalent and bounded functions \( f(z) = z + a_2 z^2 + \ldots \) in the unit disk \( |z| < 1 \). We prove that \( \max |a_m a_n| \) is realized by the Pick function for \( M \) close to 1 iff \((m - 1)\) and \((n - 1)\) are relatively prime.

1. Introduction. In what follows we deal with the class \( S(M) \), \( M > 1 \), of holomorphic functions \( f \) in the unit disk \( D = \{ z : |z| < 1 \} \) which have the form

\[
f(z) = z + a_2 z^2 + \ldots, \quad z \in D,
\]

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and are univalent and bounded by $M$ in $D$, i.e. $|f(z)| < M$, $z \in D$.

Very little is known about the coefficient problem within the class $S(M)$. Among others we suggest [3], [13], [14], [11], [12] as the references.

The important role in this class plays the so-called Pick function $P_M(z)$ given by the formula

$$P_M(z) = 2z \left[ (1-z)^2 + \frac{2}{M} z + (1-z)\sqrt{(1-z)^2 + \frac{4}{M} z} \right]^{-1}, \quad z \in D,$$

which maps the unit disk $D$ onto the disk $D_M = \{w : |w| < M\}$ slit along the segment $[-M, -M(2M - 1 - 2\sqrt{M(M-1)})]$.

If $M \to \infty$, then the class $S(M)$ reduces to the well known class $S$ and the Pick function reduces to the Koebe function $K(z) = z(1-z)^{-2}$ which is known to be extremal for $\max |a_n|$, $f \in S$, $n = 2, 3, \ldots$, by the famous de Branges result [2].

Contrary to that, in the class $S(M)$ the Pick function is not extremal for $\max |a_3|$ if $M \in (1, e)$. However, for the functional $|a_2a_n|$, $n \geq 3$, the Pick function is extremal if $M$ is close to 1, [6]. The last result proved the conjecture [5] characterizing the extremal property of the Pick function for $M$ close to 1.

In this note we extend the result from [6] and we will prove that

$$\max_{f \in S(M)} |a_ma_n|, \quad 2 < m < n,$$

is attained by the Pick function for $M$ close to 1 iff $(m - 1)$ and $(n - 1)$ are relatively prime.

We have

**Theorem 1.** For every given integers $m, n$, $2 < m < n$, such that $(m - 1)$ and $(n - 1)$ are relatively prime there exists $M_{m,n} > 1$ such that for all $M \in (1, M_{m,n})$ the functional $|a_ma_n|$ is maximized in $S(M)$ only by the Pick function $P_M \in S(M)$ or its rotations.

**Theorem 2.** Let integers $m, n$, $2 < m < n$, be such that $(m - 1)$ and $(n - 1)$ are not relatively prime. Then the functional $|a_ma_n|$ is not maximized by rotations of the Pick function, when $M$ is close to 1.

In order to prove the above Theorems, we will apply the Loewner equation for the class $S(M)$ and optimal control theory results adjusted to univalent functions problems as developed in [8]. This method has been successfully applied to other coefficient problems in the class $S(M)$ (e.g. [9], [10]).
2. Auxiliary Theorems and Lemmas.

**Theorem A (Loewner equation)** [4]. Let \( w = w(z, t) \) be the solution of the Loewner equation

\[
\frac{dw}{dt} = -w \frac{e^{iu} + w}{e^{iu} - w}, \quad w|_{t=0} = z, \quad 0 \leq t \leq \log M,
\]

with a piecewise continuous function \( u = u(t) \).

Then

\[
w(z, t) = e^{-t}[z + a_2(t)z^2 + a_3(t)z^3 + \ldots], \quad z \in D, \quad t \geq 0,
\]

is holomorphic and univalent with respect to \( z \in D \) for every \( t \geq 0 \). Moreover, the functions given by the formula

\[
f(z) := Mw(z, \log M) \in S(M),
\]

form a dense subclass of \( S(M) \).

**Remark 1.** In the case \( u(t) = \text{const.} \), the function \( f \) given by (4) is the Pick function \( P_M(z) \) or its rotations and in the case when \( u(t) \) is a smooth function on \( [0, \log M] \), the corresponding \( f \) maps \( D \) onto \( D_M \) minus a smooth slit.

In general, piecewise smooth functions \( u(t) \) correspond to mappings \( f \) of \( D \) onto \( D_M \) with a finite number of smooth slits [8].

The function \( u(t) \) will be called the control function.

**Remark 2.** By the generalized Loewner equation [8] we mean the differential equation of the form

\[
\frac{dw}{dt} = -w \sum_{j=1}^{n} \lambda_j \frac{e^{iu_j} + w}{e^{iu_j} - w}, \quad w|_{t=0} = z, \quad 0 \leq t \leq \log M,
\]

where \( \lambda_j \geq 0, \ j = 1, \ldots, n \), and \( \sum_{j=1}^{n} \lambda_j = 1 \). The functions \( u_j(t), \ j = 1, \ldots, n \), are again piecewise continuous functions [8].

In the case when \( u_j(t), \ j = 1, \ldots, n \), are smooth functions on \( [0, \log M] \) the functions \( f \) of the form (4) obtained from the equation (5) map \( D \) onto \( D_M \) with a finite number of smooth slits.

Moreover, in the case when \( f^* \in S(M) \) is a boundary function of the coefficient region

\[V_n^M = \{ a = (a_2, \ldots, a_n) : f \in S(M) \}\]
there exists the unique system of continuous functions $u_1, \ldots, u_{n-1}$ and non-negative constants $\lambda_1, \ldots, \lambda_{n-1}$, $\sum_{j=1}^{n-1} \lambda_j = 1$, such that $f^*(z) = Mw(z, \log M)$, where $w = w(z, t)$ is the solution of (5).

The parametric representation for the coefficients obtained from the Loewner equation allows us to apply the classical variational methods [1] or the Pontryagin maximum principle [7] to solve extremal problems in the class $S(M)$.

Indeed, let $a_k(t)$ be given by (3), $a_k(t) = x_{2k-1}(t) + ix_{2k}(t)$, $k = 2, \ldots, n$, and $a(t) = (a_1(t), a_2(t), \ldots, a_n(t))^T$, $a_1(t) = 1$, $a_0 = (1, 0, \ldots, 0)^T$, $x(t) = (x_3(t), \ldots, x_{2n}(t))$. Substituting (3) into (2) we obtain the following differential equation for $a(t)$ [8]:

$$\frac{da(t)}{dt} = -2 \sum_{s=1}^{n-1} e^{-s(t+iu)} A^s(t)a(t), \quad a(0) = a^0,$$

where

$$A(t) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_1(t) & 0 & \cdots & 0 & 0 \\ a_2(t) & a_1(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1}(t) & a_{n-2}(t) & \cdots & a_1(t) & 0 \end{pmatrix}.$$  

The $k$-th row in the formula (6) for vector $a(t)$ is a system of two equations for $x_{2k-1}$ and $x_{2k}$. We will write equations for coordinates of $x(t)$ as follows

$$\frac{dx_k}{dt} = g_k(t, x, u), \quad x_k(0) = 0, \quad x_{2k-1}(\log M) + ix_{2k}(\log M) = a_k.$$

Note that

$$g_{2k-1}(0, 0, u) = -2 \cos(k-1)u, \quad g_{2k}(0, 0, u) = 2 \sin(k-1)u, \quad k \geq 2.$$

The coefficient region $V^M_n$ is the reachable set for the system (7), i.e. the set of all possible values of $x(\log M)$ which can be obtained as solutions of (7) with arbitrary piecewise continuous functions $u = u(t)$. To find $V^M_n$ it is sufficient to describe its boundary $\partial V^M_n$. Every boundary point $a \in \partial V^M_n$ is represented by a solution of (7) with $u(t)$ satisfying corresponding variational equations. As proved in [8], in order to describe all the boundary functions $f \in S(M)$ which correspond to the boundary points of the coefficient region $V^M_n$ we have to consider the following Hamilton function

$$H(t, x, \psi, u) = \sum_{k=3}^{2n} g_k(t, x, u)\psi_k,$$
where \( \psi = \psi(t) = (\psi_3(t), \ldots, \psi_{2n}(t)) \) is the nonzero conjugate vector which satisfies the conjugate Hamiltonian system

\[
\frac{d\psi_k}{dt} = -\frac{\partial H}{\partial x_k}, \quad \psi_k(0) = \xi_k, \quad k = 3, \ldots, 2n.
\]

**Theorem B** [8]. Let \( x(t) \) be a solution of the system (7) with a piecewise continuous control function \( u^*(t) \). If \( x = x(\log M) \) is a boundary point of \( V_n^M \), then there exists a solution \( \psi = \psi(t) \) of the system (10) with the same control function \( u^*(t) \) such that

\[
\max_u H(t, x(t), \psi(t), u) = H(t, x(t), \psi(t), u^*(t))
\]

for all \( t \in [0, \log M] \) for which \( u^*(t) \) is continuous.

The condition (11) is called the Pontryagin maximum principle. Evidently \( u^*(t) \) is a root of the equation

\[
H_u(t, x, \psi, u) = 0
\]

at the continuity points of \( u^*(t) \).

Denote \( \xi = (\xi_3, \ldots, \xi_{2n}) \). In particular, at \( t = 0 \) we have

\[
H(0, 0, \xi, u) = -2 \sum_{k=2}^{n} (\xi_{2k-1} \cos(k-1)u - \xi_{2k} \sin(k-1)u).
\]

Varying the initial data \( \xi \) in (10) and solving the systems (7) and (10) with the control functions \( u \) satisfying the Pontryagin maximum principle, we obtain all the boundary points of \( x(\log M) \) of \( V_n^M \).

Since the conjugate system (10) is linear with respect to \( \psi \), the vector \( \psi(t) \) depends linearly on the initial data \( \xi \) in (10). The maximizing property of a control function \( u \) satisfying the Pontryagin maximum principle is also preserved if \( \psi \) is multiplied by a positive number. This allows us to normalize the initial vector \( \xi \) in a suitable form. Such a suitable normalization will be introduced and explained later.

For certain values \( \xi \) the Hamilton function may have several maximum points \( u \in (-\pi, \pi] \). In this case instead of (2) we have to use the generalized Loewner differential equation (5) and the corresponding system of differential equations for \( x_k(t), \psi(t) \) instead of (7) and (10). More precisely, if at \( t = 0 \) the Hamilton function \( H(0, 0, \xi, u) \) attains its maximum at \( m \) different points \( u_1, \ldots, u_m \) in \( (-\pi, \pi] \), then we have to use the generalized Loewner differential equation (5) with index \( m \).
Theorem C [8]. Let \( x(t) \) be a solution of the system (7) with a control function \( u^*(t) \) and \( x = x(\log M) \) be a boundary point of \( V^M_n \). If \( H(0, 0, \xi, u) \) attains its maximal value at exactly one point in \((-\pi, \pi]\) for which \( H_{uu}(0, 0, \xi, u) \neq 0 \), then \( u^*(t) \) is continuous on \([0, \log M]\).

Note that because of possible rotation in \( S(M) \) the extremal problem (1) can be reduced to the following:

\[
\Re a_m a_n \rightarrow \text{max}.
\]

Suppose \( f \in S(M) \) maximizes \( \Re a_m a_n \) in \( S(M) \). Then \( f_\alpha(z) = e^{-i\alpha} f(e^{i\alpha} z) \) with \( \alpha = 2\pi k/(m + n - 2) \), \( k = 0, 1, \ldots, m + n - 3 \), also maximizes \( \Re a_m a_n \) in \( S(M) \). Evidently \( a_m \neq 0 \) for the extremal function. Hence, if \( m - 1 \) and \( n - 1 \) are relatively prime, there exists an extremal function \( f \in S(M) \) for which

\[
-\frac{\pi}{m + n - 2} < \pi - \arg a_m \leq \frac{\pi}{m + n - 2}.
\]

Denote such a function by \( f_0^M \). It is known [3] that \( f_0^M \) maps \( D \) onto \( D_M \) minus finitely many piecewise analytic curves. Hence, it may be represented by the formula (4), where \( w(z, t) \) is the solution of the Loewner differential equation (2).

Lemma 1. Assume that the extremal function \( f_0^M \in S(M) \) is given by the formula (4) where \( w(z, t) \) has the expansion (3) and is a solution of the Loewner differential equation (2). Then

\[
a_m(t) = -2t + o(t), \quad a_n(t) = -2t + o(t), \quad a_m(t)a_n(t) = 4t^2 + o(t^2),
\]

\[
a_m = -2(M - 1) + o(M - 1), \quad a_n = -2(M - 1) + o(M - 1),
\]

\[
a_m a_n = 4(M - 1)^2 + o((M - 1)^2),
\]

as \( t \to 0^+ \) and \( M \to 1^+ \).

Proof. Denote

\[
I(t) = \Re a_m(t) a_n(t) = x_{2m-1}(t)x_{2n-1}(t) - x_{2m}(t)x_{2n}(t).
\]

Using (8) after differentiating \( I(t) \) we have

\[
\frac{dI}{dt} \bigg|_{t=0} = 0,
\]

(17)
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\[
\frac{d^2 I}{dt^2} = 2 \left( \frac{dx_{2m-1}}{dt} \frac{dx_{2n-1}}{dt} \right) t=0 - 2 \left( \frac{dx_{2m}}{dt} \frac{dx_{2n}}{dt} \right) t=0
\]
\[
= 8(\cos(m-1)u(0)\cos(n-1)u(0) - \sin(m-1)u(0)\sin(n-1)u(0))
\]
\[
= 8 \cos(m+n-2)u(0).
\]

The formulas (17) and (18) imply the asymptotic expansion

\[
\Re a_m a_n = I(\log M) = 4 \cos(m+n-2)u(0)(M-1)^2 + o((M-1)^2).
\]

Solving the extremal problem (14) for \( M \) close to 1 we have to maximize the first nonzero term in the asymptotic expansion (19). Hence, we have to put \( \cos(m+n-2)u(0) = 1 \) in (19) or equivalently

\[
u(0) = u_j(0) = \frac{2\pi j}{m+n-2}, \quad j = 0, 1, \ldots, m+n-3.
\]

Therefore, for the extremal function we have

\[
\Re a_m a_n = 4(M-1)^2 + o((M-1)^2).
\]

From (8) we obtain the asymptotic expansion for \( a_m \) and \( a_n \)

\[
a_m = -2 \cos(m-1)u_j(0)(M-1) + o(M-1),
\]
\[
a_n = -2 \cos(n-1)u_j(0)(M-1) + o(M-1).
\]

For the extremal function \( f_0^M \) we can choose \( j = 0 \) or equivalently \( u_j(0) = 0 \), which implies in this case

\[
a_m(t) = -2t + o(t), \quad a_n(t) = -2t + o(t),
\]

and hence

\[
a_m(t)a_n(t) = 4t^2 + o(t^2).
\]

For \( t = \log M \) this gives

\[
a_m = -2(M-1) + o(M-1), \quad a_n = -2(M-1) + o(M-1),
\]

which ends the proof of Lemma 1. □

The coefficient region \( V_n^M \) is the closed set in \( \mathbb{R}^{2n-2} \) whose points have coordinates \( (x_3, \ldots, x_{2n}) \). For \( c \in \mathbb{R} \), let us define the surface \( Q_c \) in \( \mathbb{R}^{2n-2} \) by the equation

\[
x_{2m-1}x_{2n-1} - x_{2m}x_{2n} = c.
\]

In order to solve the extremal problem (14) we have to find the maximal value \( c \) such that the surface \( Q_c \) has a nonempty intersection with \( V_n^M \). Denote \( Q_c = \max Q \) and assume that

\[
x \in Q \cap V_n^M, \text{ where } x = x(\log M) = (x_3(\log M), \ldots, x_{2n}(\log M)).
\]

Without loss of generality we assume that the point \( x \in V_n^M \) which corresponds to the extremal function \( f_0^M \) is such that \( x_{2m-1} < 0 \) and \( x_{2n-1} < 0 \) when \( M \) is close to 1.
Lemma 2. The normal vector $\mathbf{n}$ to the surface $Q$ at the point $x$ has the form

$$
\mathbf{n} = \left[0, \ldots, 0, -\frac{\Re a_n}{\Re a_m}, \frac{\Im a_n}{\Re a_m}, 0, \ldots, 0, -1, \frac{\Im a_m}{\Re a_m}, 0, \ldots, 0, -1, \frac{\Im a_m}{\Re a_m}\right],
$$

where the first two nonzero coordinates correspond to the indices $2m-1$ and $2m$.

Lemma 2 has been proved in [6] for $m = 2$. Its generalization for arbitrary $m > 2$ is evident.

Note that the representation formula (21) for $\mathbf{n}$ follows from the normalization condition $x_{2n-1} = -1$. Since we have $x_{2m-1} < 0$ and $x_{2n-1} < 0$, the chosen direction of $\mathbf{n}$ corresponds to increasing parameter $c$ of the level surfaces $Q_c$.

Remark that functions $g_3, \ldots, g_{2n}$ on the right-hand side of (7) do not depend on $x_{2n-1}, x_{2n}$. Therefore,

$$
\frac{d\psi_{2n-1}}{dt} = \frac{d\psi_{2n}}{dt} = 0
$$

and we may assume that $\psi_{2n-1}(t) = -1$.

When $c$ is the maximal value in (20), the normal vector $\mathbf{n}$ is orthogonal to the tangent or support hyperplane to $V^M_n$ at the point $x$. The boundary point $x$ is delivered by the extremal function $f^M_0 \in S(M)$ represented by an integral of the Loewner equation (2) or the generalized Loewner equation (5). It is known from the calculus of variations [1] or from the optimal control theory [7] that the conjugate vector $\psi(\log M)$ of the system (10) or its analogue which corresponds to (5) is also orthogonal to the tangent or support hyperplane to $V^M_n$ at the point $x$. Therefore, we can normalize $\psi(\log M)$ in such a way that it coincides with $\mathbf{n}$.

The condition

$$
\psi(\log M) = \mathbf{n}
$$

is called the transversality condition at the point $x$. This is the necessary condition for our extremal problem.

Lemma 3. If $M$ is close to 1, then the extremal function $f^M_0$ is given by the formula (4) where $w(z, t)$ is a solution of the Loewner differential equation (2) with a continuous control function $u(t)$.

Proof. Let us observe that since $(m-1)$ and $(n-1)$ are relatively prime, the function

$$
h(u) = 2(\cos(m-1)u + \cos(n-1)u)
$$
attains its maximum on $[-\pi, \pi]$ only at $u = 0$. Indeed, $h(u) \leq 4$ and $h(u) = 4$ only if $\cos(m-1)u = 1$ and $\cos(n-1)u = 1$, which is possible on $[-\pi, \pi]$ only for $u = 0$.

From (16) we see that the nonzero coordinates of the normal vector $\pi$ have the asymptotic expansions

$$-(\Re a_n) = -1 + o(M-1), \quad 3\Re a_n = o(M-1), \quad 3\Re a_m = o(M-1).$$

The functions $-\frac{\partial H}{\partial \xi_k}$ on the right-hand side of (10) are bounded for $0 \leq t \leq \log M$, which means that $\psi_k(t)$ are close to $\xi_k$ if $t$ is close to 0. Therefore, according to the transversality conditions the initial data $\xi$ for the extremal function $f^M_0$ are close to the vector $\pi_0 = (0, \ldots, 0, -1, 0, \ldots, 0, -1, 0)$ where -1 first appears at the $(2m-1)$-th place.

Notice that the function

$$H(0, 0, \xi^0, u) = h(u) = 2(\cos(m-1)u + \cos(n-1)u)$$

has only one absolute maximum in $(-\pi, \pi]$ at $u = 0$ and $H(u, 0, 0, \xi^0, 0) = h'(0) = -2((m-1)^2 + (n-1)^2) < 0$. This property is preserved for $\xi$ close to $\xi^0$, i.e. $H(0, 0, \xi, u)$ attains its maximal value at exactly one point in $(-\pi, \pi]$ for which $H_u(0, 0, \xi, u) < 0$ for all $\xi$ from the neighborhood of $\xi^0$ including the point $\xi$ corresponding to the extremal function $f^M_0$. Applying Theorem C we end the proof of Lemma 3. □

Lemma 3 guarantees that the control function $u$ in the right-hand side of (7) and (10) is the analytic branch of the implicit function $u = u(t, x, \psi)$ determined by the equation (12) with the initial value $u(0, 0, \xi^0) = 0$.

Vectors $x, \psi$ being the solution of the systems (7) and (10) with $u = u(t, x, \psi)$ on their right-hand sides depend only on $t$ and $\xi$, i.e. $x = x(t, \xi), \psi = \psi(t, \xi)$. Put

$$u(t, \xi) = u(t, x(t, \xi), \psi(t, \xi)).$$

**Remark 3.** In particular, the initial value for $u(0, 0, \xi^0)$ means that $u(0, \xi^0) = 0$.

Every control function $u(t)$ corresponding to the extremal function $f$ of the extremal problem (14) has to satisfy two necessary conditions: the Pontryagin maximum principle (11) and the transversality condition (22). Now we are able to show that the control function $u(t) = 0$ which corresponds to the rotation: $-P_M(-z)$ of the Pick function satisfies both of these necessary conditions.
Lemma 4. The control function $u(t) = 0$ satisfies the Pontryagin maximum principle (11) and the transversality condition (22) if $M$ is close to 1.

Proof. The control function $u(t) = 0$ in the Loewner differential equation (2) corresponds to the function $-P_M(-z)$ for all $M > 1$. It is well known that this function has real coefficients. Hence $x_{2k}(t) = 0$, $k = 2, \ldots, n$, in (7) as well as $g_{2k}(t, x, u) = 0$, $k = 2, \ldots, n$.

From the explicit formulas for the right-hand side of the system (10) (see e.g. [8]) it follows that in this case

$$-\frac{\partial H}{\partial x_{2k}} = 0, \quad k = 2, \ldots, n.$$ 

Therefore, the initial equations $\xi_{2k} = 0$, $k = 2, \ldots, n$, are preserved for the whole trajectory $(x(t), \psi(t))$, namely, $\psi_{2k}(t) = 0$, $k = 2, \ldots, n$, $0 \leq t \leq \log M$.

Let $x(t)$ be the solution of the system (7) with $u = 0$ on its right-hand side and $\psi(t)$ be the solution of the Cauchy problem for the system (10) with $x(t)$ given by (7) and $u = 0$ on its right-hand side. Instead of the initial data at $t = 0$ in (10) we consider the initial data

$$\psi(\log M) = \left(0, \ldots, 0, \frac{x_{2n-1}(\log M)}{x_{2n-1}(\log M)}, 0, \ldots, 0, -1, 0\right),$$

which is equivalent to the transversality condition (22). In this case we integrate the system (10) from $\log M$ to 0. Write $\psi(0) = \xi^P = (\xi_1^P, \ldots, \xi_{2n}^P)$. The choice of the initial data $\xi = \xi^P$ at $t = 0$ guarantees that the transversality condition (22) is satisfied. Note that $\psi(\log M)$ and $\psi(0)$ are close to $\xi^0 = (0, \ldots, 0, -1, 0, \ldots, 0, -1, 0)$.

Moreover, $\psi_{2k} = 0$, $\xi_{2k}^P = 0$, $k = 2, \ldots, n$.

The Hamilton function $H(t, x, \psi, u)$ as a function of $u$ is a polynomial of the $(n-1)$-th degree with respect to $\cos u$. It is "close" to $h(u) = 2(\cos(m-1)u + \cos(n-1)u)$ if $M$ is close to 1. As we have pointed out, this is a polynomial with respect to $\cos u$ and polynomials $h_i(u)$ which are close to it have only one absolute maximum at $\cos u = 1$ and $h_i''(0) < 0$. This means that $u(t) = 0$ satisfies the Pontryagin maximum principle (11), which ends the proof of Lemma 4.  □

Recall that the maximum principle generates the function $u(t, \xi)$ by the equation (12). It was shown in [6] that $u(t, \xi)$ has bounded partial derivatives in a neighborhood of the point $(t, \xi) = (0, \xi^0)$. 

3. Proofs of the theorems.

Proof of Theorem 1. We wish to show that there exists the unique point \( \xi^* \) in the neighborhood of \( \xi^0 \) for which the solution of the systems (7) and (10) satisfies the maximum principle (11) and the transversality condition (22). As soon as the point \( \xi^P \) corresponding to the rotation \( -P_M(-z) \) of the Pick function also generates the solution of (7) and (10) satisfying these necessary extremum conditions, we obtain \( \xi^* = \xi^P \).

The problem is difficult due to the fact that the value \( \psi(\log M) \) in the transversality condition (22) depends on unknown coefficients \( a_m, a_n \). To avoid this difficulty we require that according to Lemma 1, \( x_{2m-1}^2 < 0 \) and introduce the vector

\[
(24) \quad \psi^*(t, \xi) = \psi(t, \xi) - m(t, \xi),
\]

where

\[
(25) \quad m(t, \xi) = \left(0, \ldots, 0, -\frac{x_{2n-1}(t, \xi)}{x_{2m-1}(t, \xi)}, \frac{x_{2n}(t, \xi)}{x_{2m-1}(t, \xi)}, 0, \ldots, 0, \frac{x_{2m}(t, \xi)}{x_{2m-1}(t, \xi)}\right).
\]

The transversality condition can now be written in the form

\[
(26) \quad \psi^*(\log M, \xi^*) = (0, \ldots, 0, -1, 0).
\]

From (7) we obtain the differential equations and from (8) the initial data for \( -m(t, \xi) \)

\[
\frac{d}{dt} \frac{x_{2n-1}}{x_{2m-1}} = \frac{g_{2n-1}(t, x, u)x_{2m-1} - g_{2m-1}(t, x, u)x_{2n-1}}{x_{2m-1}^2} = G_{2m-1}(t, x, u),
\]

\[
\left. \frac{x_{2n-1}}{x_{2m-1}} \right|_{t=0} = \frac{\cos(n-1)u(0, \xi)}{\cos(m-1)u(0, \xi)};
\]

\[
- \frac{d}{dt} \frac{x_{2n}}{x_{2m-1}} = -\frac{g_{2n}(t, x, u)x_{2m-1} - g_{2m-1}(t, x, u)x_{2n}}{x_{2m-1}^2} = G_{2m}(t, x, u),
\]

\[
\left. \frac{x_{2n}}{x_{2m-1}} \right|_{t=0} = \frac{\sin(n-1)u(0, \xi)}{\cos(m-1)u(0, \xi)};
\]

\[
- \frac{d}{dt} \frac{x_{2m}}{x_{2m-1}} = -\frac{g_{2m}(t, x, u)x_{2m-1} - g_{2m-1}(t, x, u)x_{2m}}{x_{2m-1}^2} = G_{2n}(t, x, u),
\]

\[
\left. \frac{x_{2m}}{x_{2m-1}} \right|_{t=0} = \frac{\sin(m-1)u(0, \xi)}{\cos(m-1)u(0, \xi)}.
\]
This implies that the function \( \psi^*(t) = (\psi_3^*(t), \ldots, \psi_{2n}^*(t)) \) is the solution of the system of differential equations

\[
\frac{d\psi_{2m-1}^*}{dt} = -\frac{\partial H}{\partial x_{2m-1}} + G_{2m-1}, \quad \psi_{2m-1}^*(0) = \xi_{2m-1} + \frac{\cos(n-1)u(0,\xi)}{\cos(m-1)u(0,\xi)},
\]

\[
\frac{d\psi_{2m}^*}{dt} = -\frac{\partial H}{\partial x_{2m}} + G_{2m}, \quad \psi_{2m}^*(0) = \xi_{2m} + \frac{\sin(n-1)u(0,\xi)}{\cos(m-1)u(0,\xi)},
\]

\[
\frac{d\psi_k^*}{dt} = -\frac{\partial H}{\partial x_k}, \quad \psi_k^*(0) = \xi_k, \quad k = 3, \ldots, 2m - 2, 2m + 1, \ldots, 2n - 2,
\]

\[
\frac{d\psi_{2n}^*}{dt} = G_{2n}, \quad \psi_{2n}^*(0) = \xi_{2n} + \frac{\sin(m-1)u(0,\xi)}{\cos(m-1)u(0,\xi)}.
\]

Let us consider the mapping

\[ F : \xi \to y = \psi^*(\log M, \xi), \]

where \( \xi \) is from a neighborhood of \( \xi^0 \).

The function \( y = F(\xi) \) maps the initial data analytically depending on \( \xi \) onto the solution of the Cauchy problem (27) - (30). Hence, \( F \) is analytic and its derivative \( F_\xi \) is the Jacobi matrix consisting of elements

\[ \psi_{jk}^*(\log M, \xi) = \frac{\partial \psi_{jk}^*(\log M, \xi)}{\partial \xi_k}, \quad j, k = 3, \ldots, 2n - 2, 2n. \]

To determine \( \psi_{jk}^*(t, \xi) \) we differentiate the equations (27)-(30) with respect to \( \xi_k, \quad k = 3, \ldots, 2n - 2, 2n \). The derivatives \( x_\xi \) and \( \psi_\xi \) can be found in the following way. Differentiating the systems (7) and (10) with respect to \( \xi \) gives the system of differential equations

\[
\frac{dx_\xi}{dt} = L(t, x, u, x_\xi, u_\xi), \quad x_\xi(0, \xi) = 0,
\]

\[
\frac{d\psi_\xi}{dt} = N(t, x, \psi, x_\xi, \psi_\xi, u_\xi), \quad \psi_\xi(0, \xi) = 1.
\]

The initial data 0 and 1 in (31) and (32) are the zero matrix and the unit matrix respectively. The right-hand sides \( L \) and \( N \) in (31) and (32)
are linear with respect to $x_\xi$, $\psi_\xi$ and $u_\xi$. To find $u_\xi$ we differentiate the equation (12) with respect to $\xi$, and obtain

$$H_{ux}(t, x, \psi, u)x_\xi + H_{u\psi}(t, x, \psi, u)\psi_\xi + H_{uu}(t, x, \psi, u)u_\xi = 0$$

which implies the formula

$$u_\xi = -\left(H_{ux}(t, x, \psi, u)x_\xi + H_{u\psi}(t, x, \psi, u)\psi_\xi\right) / H_{uu}(t, x, \psi, u).$$  \hspace{1cm} (33)$$

Substituting $u_\xi$ from (33) to (31) and (32) we solve the Cauchy problem for the obtained system of differential equations. The solution $(x_\xi(t, \xi), \psi_\xi(t, \xi))$ of the Cauchy problem is bounded for $\xi$ close to $\xi^0$ and $t$ close to 0. The obtained system of differential equations with respect to $\psi_{jk}$ has a solution as the Cauchy problem smoothly depending on initial data.

There remains to determine the initial data $\eta_{jk} = \psi_{jk}^\ast(0, \xi)$, $j, k = 3, \ldots, 2n - 2, 2n$. Differentiating the initial data in (27)-(30) with respect to $\xi_3, \ldots, \xi_{2n-2}, \xi_{2n}$, we obtain

$$\eta_{(2m-1)k}(\xi) = \delta_{(2m-1)k} - [(n - 1) \sin(n - 1)u(0, \xi) \cos(m - 1)u(0, \xi) -$$

\begin{align*}
(m - 1) \cos(n - 1)u(0, \xi) \sin(m - 1)u(0, \xi)] & \frac{u_{\xi_k}(0, \xi)}{\cos^2(m - 1)u(0, \xi)}, \\
& k = 3, \ldots, 2n - 2, 2n, \hspace{2cm} (34)
\end{align*}

$$\eta_{(2m)k}(\xi) = \delta_{(2m)k} + [(n - 1) \cos(n - 1)u(0, \xi) \cos(m - 1)u(0, \xi) +$$

\begin{align*}
(m - 1) \sin(n - 1)u(0, \xi) \sin(m - 1)u(0, \xi)] & \frac{u_{\xi_k}(0, \xi)}{\cos^2(m - 1)u(0, \xi)}, \\
& k = 3, \ldots, 2n - 2, 2n, \hspace{2cm} (35)
\end{align*}

$$\eta_{jk}(\xi) = \delta_{jk}, \quad j = 3, \ldots, 2m-2, 2m+1, \ldots, 2n-2, 2n, \quad k = 3, \ldots, 2n-2, 2n, \hspace{2cm} (36)$$

$$\eta_{(2n)k}(\xi) = \delta_{(2n)k} + \frac{m - 1}{\cos^2(m - 1)u(0, \xi)}u_{\xi_k}(0, \xi), \quad k = 3, \ldots, 2n - 2, 2n. \hspace{2cm} (37)$$

To determine $u_{\xi_k}(0, \xi)$ we notice that $H_{u\psi_k}(0, 0, \xi, u) = (g_k)_u(0, 0, u)$, $k = 3, \ldots, 2n$, $x_{\xi_k}|_{t=0} = 0$, and obtain from (33) at $\xi = \xi^0$ that

$$u_{\xi_k}(0, \xi^0) = \frac{(g_k)_u(0, 0, u)}{2((m - 1)^2 + (n - 1)^2)}, \quad k = 3, \ldots, 2n - 2, 2n.$$
In particular, taking into account (8) and Remark 3 we have

\[
(38) \quad u_{\xi^{2m-1}}(0, \xi^0) = \frac{(m-1)\sin(m-1)u(0, \xi^0)}{(m-1)^2 + (n-1)^2} = 0, \\
(39) \quad u_{\xi^{2m}}(0, \xi^0) = \frac{(m-1)\cos(m-1)u(0, \xi^0)}{(m-1)^2 + (n-1)^2} = \frac{m-1}{(m-1)^2 + (n-1)^2}, \\
(40) \quad u_{\xi^{2n}}(0, \xi^0) = \frac{(n-1)\cos(n-1)u(0, \xi^0)}{(m-1)^2 + (n-1)^2} = \frac{n-1}{(m-1)^2 + (n-1)^2}.
\]

Substitution of (38) - (40) to (34), (35) and (37) at \( \xi = \xi^0 \) gives

\[
(41) \quad \eta_{(2m-1)k}(\xi^0) = \delta_{(2m-1)k}, \quad k = 3, \ldots, 2n-2, 2n, \\
(42) \quad \eta_{(2m)(2m)}(\xi^0) = 1 + \frac{(m-1)(n-1)}{(m-1)^2 + (n-1)^2}, \\
(43) \quad \eta_{(2n)(2n)}(\xi^0) = 1 + \frac{(m-1)(n-1)}{(m-1)^2 + (n-1)^2}.
\]

Let \( B(t, \xi) \) be the Jacobi matrix consisting of elements \( \psi_{jk}^* (t, \xi) \). According to (36), (41) - (43) we have

\[
det B(0, \xi^0) = 1 + \frac{2(m-1)(n-1)}{(m-1)^2 + (n-1)^2} > 0.
\]

Hence, \( \det B(\log M, \xi^0) > 0 \) if \( M \) is close to 1. This means that \( B(\log M, \xi^0) = F_\xi(\xi^0) \) is invertible and \( F \) maps a neighborhood \( U_\epsilon(\xi^0) = \{ \xi : \| \xi - \xi^0 \| < \epsilon \} \) of \( \xi^0 \) one-to-one onto a neighborhood of \( y^0 = F(\xi^0) \). Therefore, there exists the unique \( \xi \in Q_\epsilon(\xi^0) \) for which the maximum principle (11) and the transversality condition (22) in the form (26) are satisfied. According to Lemma 4 this is \( \xi = \xi^P \) which corresponds to the function \(-P_M(-z)\). This ends the proof of Theorem 1. \( \square \)
Remark 4. We have proved that in the extremal problem (1) or (14) the necessary extremum conditions in the form of the Pontryagin maximum principle (11) - (12) and the transversality conditions (22) or (26) are in fact the sufficient extremum conditions if \( M \) is close to 1.

Proof of Theorem 2. Let \( j \geq 2 \) be the common divisor of \((m-1)\) and \((n-1)\) and the function \( f_1^M(z) \) be given by the formula (4) where \( w(z,t) \) is a solution of the generalized Loewner equation (5) with index 2, \( \lambda_1 = \lambda_2 = 1/2, u_1(t) = 0 \) and \( u_2(t) = 2\pi/j \). The idea of the proof of Theorem 2 is to compare the asymptotic expansions of \( I(t) = \Re a_m(t) a_n(t) \) for two functions: \(-P_M(-z)\) and \( f_1^M(z) \). Denote \( I(t) \) corresponding to \(-P_M(-z)\) and \( f_1^M(z) \) by \( I_P(t) \) and \( I_1(t) \) respectively. Note that \(-P_M(-z)\) can be represented by formula (4) if the solution \( w(z,t) \) of the equation (5) corresponds to \( u_1(t) = u_2(t) = 0 \).

The differential equation for \( a(t) \) generated by the generalized Loewner equation (5) with index 2 has, according to [8], the form

\[
\frac{da(t)}{dt} = -2 \sum_{j=1}^{2} \sum_{s=1}^{n-1} e^{-s(t+iuj)} A^s(t) a(t), \quad a(0) = a^0.
\]

Recall that according to (17) \( I_P'(0) = I_1'(0) = 0 \). For given \( u_1, u_2 \) and \( \lambda_1, \lambda_2 \) we have from (44)

\[
a'_k(0) = -(e^{-i(k-1)uj} + e^{-i(k-1)uj}) = -(1 + e^{-i(k-1)2\pi/j}), \quad k = 2, \ldots, n.
\]

In particular, according to (8) and (45)

\[
a'_m(0) = -2, \quad a'_n(0) = -2,
\]

for both functions \(-P_M(-z)\) and \( f_1^M(z) \). From here we immediately obtain

\[
I''_P(0) = I''_1(0) = 2 \Re a'_m(0)a'_n(0) = 8.
\]

We now establish the equality

\[
I''_1(0) = 3 \Re (a'_m(0)a''_m(0) + a''_m(0)a'_n(0)).
\]

Differentiate (44) with respect to \( t \) at \( t = 0 \) with \( \lambda_1 = \lambda_2 = 1/2 \) and \( u_1 = 0 \) to obtain

\[
\frac{d^2 a(t)}{dt^2} \bigg|_{t=0} = -\sum_{s=1}^{n-1} (1+e^{-isu_2})((-s)A^s(0)a^0 + sA^{s-1}(0)A'(0)a^0 + A^s(0)a'(0)).
\]
The specific triangular form of the matrices $A(0)$ and $A'(0)$ implies

$$A^s(0)a^0 = (0, \ldots, 0, 1, 0, \ldots, 0)^T,$$

where 1 appears at the $(s + 1)$-th place, and

$$A^{s-1}(0)A'(0)a^0 = A^s(0)a'(0) = (0, \ldots, 0, a'_2(0), \ldots, a'_{n-s}(0))^T.$$ 

Therefore, it follows from (48) that

$$a''_m(0) = 2(m - 1) - \sum_{s=1}^{m-2} (1 + e^{-isu^2})(s + 1)a'_{m-s}(0),$$

(49)

$$a''_n(0) = 2(n - 1) - \sum_{s=1}^{n-2} (1 + e^{-isu^2})(s + 1)a'_{n-s}(0).$$

(50)

Substituting (45) into (49) and (50) and taking into account that $e^{-i(m-1)u^2} = e^{-i(n-1)u^2} = 1$ we have

$$a''_m(0) = 2(m - 1) + \sum_{s=1}^{m-2} (s + 1)(2 + e^{-isu^2} + e^{-i(m-s-1)u^2})$$

(51)

$$= 2(m - 1) + \sum_{s=1}^{m-2} (s + 1)(2 + e^{-isu^2} + e^{isu^2}),$$

and

$$a''_n(0) = 2(n - 1) + \sum_{s=1}^{n-2} (s + 1)(2 + e^{-isu^2} + e^{isu^2}).$$

(52)

Finally, (45), (51) and (52) being substituted into (47) give the required formula for $I''_1(0)$

$$I''_1(0) = -6\Re(a''_m(0) + a''_n(0)) = -12(m + n - 2)$$

(53)

$$- 12 \left[ \sum_{s=1}^{m-2} (s + 1)(1 + \cos su^2) + \sum_{s=1}^{n-2} (s + 1)(1 + \cos su^2) \right].$$
To obtain $I''''_P(0)$ it is sufficient to substitute $u_2 = 0$ instead of $u_2$ into (53). Hence we have

\begin{equation}
I''''_P(0) = -12(m + n - 2) - 24 \left[ \sum_{s=1}^{m-2} (s + 1) + \sum_{s=1}^{n-2} (s + 1) \right].
\end{equation}

Evidently

$I'''_1(0) > I'''_P(0)$.

From the asymptotic expansions

\[
I_1(t) = \frac{1}{2!} I_1''(0)t^2 + \frac{1}{3!} I_1'''(0)t^3 + o(t^3),
\]

\[
I_P(t) = \frac{1}{2!} I_P''(0)t^2 + \frac{1}{3!} I_P'''(0)t^3 + o(t^3),
\]

and from (46) we claim that $I_1(t) > I_P(t)$ for $t$ close to 0. Therefore the Pick function does not maximize $\Re a_m a_n$ in the class $S(M)$ for $M$ close to 1 which ends the proof of Theorem 2. □

REFERENCES


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