On the growth of the derivative of $Q_p$ functions

Abstract. In this paper we investigate some properties of the derivative of functions in the $Q_p$ spaces. We first show that $T(r, f')$, the Nevanlinna characteristic of the derivative of a function $f \in Q_p$, $0 < p < 1$, satisfies

$$\int_0^1 (1 - r)^p \exp(2T(r, f')) \, dr < \infty,$$

and that this estimate is sharp in a very strong sense, extending thus a similar result of Kennedy for functions in the Nevanlinna class.

We also obtain several results concerning the radial growth of the derivative of $Q_p$ functions.

1. Introduction and statements of results. Let $\Delta$ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. The Nevanlinna characteristic of an analytic function $f$

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in $\Delta$ is defined by

$$T(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta, \quad 0 \leq r < 1.$$  

The Nevanlinna class $N$ consists of functions $f$ analytic in $\Delta$ such that

$$\sup_{0 \leq r < 1} T(r, f) < \infty.$$  

It is well known that the condition $f \in N$ does not imply $f' \in N$. This was first proved by O. Frostman [11], who showed the existence of a Blaschke product whose derivative is not of bounded characteristic. Subsequently many other examples have been given. Kennedy [17] obtained the sharp bound on the growth of $T(r, f')$ for $f \in N$. Namely, he proved that if $f \in N$, then

$$\int_0^1 (1 - r) \exp(2T(r, f')) dr < \infty,$$

and showed that this result is sharp in the sense that if $\phi$ is a positive increasing function in $(0, 1)$ which satisfies certain “regularity conditions” and is such that

$$\int_0^1 (1 - r) \exp(2\phi(r)) dr < \infty,$$

then there exists $f \in N$ such that $T(r, f') > \phi(r)$ for all $r$ sufficiently close to 1.

Since $T(r, f')$ is an increasing function of $r$, (1) easily implies for $f \in N$

$$\log \frac{1}{1 - r} - T(r, f') \longrightarrow \infty \text{ as } r \rightarrow 1.$$  

For $0 < p < \infty$ the following spaces are defined:

$$Q_p = \left\{ f \text{ analytic in } \Delta : \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^2 g(z, a)^p dx dy < \infty \right\},$$

$$Q_{p, 0} = \left\{ f \text{ analytic in } \Delta : \lim_{|a| \rightarrow 1} \iint_{\Delta} |f'(z)|^2 g(z, a)^p dx dy = 0 \right\},$$

where $g(z, a)$ is the Green function of $\Delta$, given by

$$g(z, a) = \log \left| \frac{1 - \overline{a}z}{z - a} \right|.$$
These spaces were introduced by R. Aulaskari and P. Lappan in [3] while looking for new characterizations of Bloch functions. They proved that for $p > 1$,

$$Q_p = B, \quad \text{and} \quad Q_{p,0} = B_0.$$  

Recall that the Bloch space $B$ and the little Bloch space $B_0$ consist, respectively, of those functions $f$ analytic in $\Delta$ for which (see [1] for more information on these spaces)

$$\sup_{z \in \Delta} (1 - |z|^2) |f'(z)| < \infty, \quad \text{and} \quad \lim_{|z| \to 1} (1 - |z|^2) |f'(z)| = 0.$$  

In fact, $Q_p$ spaces put under the same frame a number of important spaces of functions analytic in $\Delta$. We have, using one of the many characterizations of the spaces $BMOA$ and $VMOA$ (see, e.g., [6,12]):

$$Q_1 = BMOA, \quad \text{and} \quad Q_{1,0} = VMOA.$$  

We refer to [2,5,4,9] for more properties of $Q_p$ spaces. It is shown in [5], that $Q_p$ spaces increase with increasing $p$,

$$Q_p \subset Q_q \subset BMOA, \quad 0 < p < q < 1,$$

all the inclusions being strict.

The first object of this paper is to study the possibility of extending Kennedy’s results to $Q_p$ spaces. First of all, let us notice that the function $f$ constructed by Kennedy to show the sharpness of (1) was given by a power series with Hadamard gaps, i.e., of the form

$$f(z) = \sum_{k=0}^{\infty} c_k z^{n_k}, \quad \frac{n_{k+1}}{n_k} \geq \lambda > 1,$$

and such that $\sum |c_k|^2 < \infty$. Such a function belongs to $BMOA$ (see [6, p. 25]) and, even more, to $VMOA$. Since $VMOA \subset BMOA \subset H^p \subset N$, $0 < p < \infty$, (we refer to [8] for the theory of $H^p$ spaces,) it follows that (1) is sharp for $VMOA = Q_{1,0}$ and, hence, for $BMOA = Q_1$ and for all $H^p$ spaces with $0 < p < \infty$. On the other hand, we remark that Girela [13] showed that (1) can be improved for the Dirichlet class $D$, consisting of all analytic functions in $\Delta$ with a finite Dirichlet integral, i.e., such that

$$\int_{\Delta} |f'(z)|^2 dx dy < \infty.$$  

It is worth noticing that $D \subset Q_{p,0}$ for all $p > 0$, the inclusion being strict, see [5].
Now we turn to $Q_p$ spaces with $p > 1$. As said before, $Q_p = B$ and $Q_{p,0} = B_0$ for all $p > 1$. We have the following trivial estimate:

$$f \in B \implies T(r, f') \leq \log \frac{1}{1-r} + O(1), \quad \text{as } r \to 1.$$ 

Girela [14] proved that this is sharp in the sense that there exists $f \in B$ such that

$$\log \frac{1}{1-r} - T(r, f') = O(1), \quad \text{as } r \to 1,$$

and, consequently,

$$\int_0^1 (1-r)^p \exp(2T(r, f')) dr = \infty.$$

Hence, neither (1) nor (2) is true for the Bloch space.

On the other hand, if $f \in B_0$ then it trivially satisfies (2). However, Girela [14] proved that there exists $f \in B_0$ which does not satisfy (1).

Hence, it remains to consider $Q_p$ spaces with $0 < p < 1$. We can prove the following results.

**Theorem 1.** If $f \in Q_p$, $0 < p < 1$, then

$$(4) \quad \int_0^1 (1-r)^p \exp(2T(r, f')) dr < \infty.$$

**Corollary.** If $f \in Q_p$, $0 < p < 1$, then

$$(5) \quad \frac{p+1}{2} \log \frac{1}{1-r} - T(r, f') \underset{r \to 1}{\to} \infty. \quad \square$$

The following theorem shows the sharpness of Theorem 1.

**Theorem 2.** Let $0 < p < 1$, and let $\phi$ be a positive increasing function in $(0,1)$ satisfying:

(i) $(1-r)^{\frac{p+1}{2}} \exp \phi(r)$ decreases as $r$ increases in $(0,1)$;

(ii) $\phi(r) - \phi(\rho) \to \infty$, as $\frac{1-r}{1-\rho} \to 0$;

(iii) $\int_0^1 (1-r)^p \exp(2\phi(r)) dr < \infty.$
Then there exists a function $f \in Q_p$ such that, for all $r$ sufficiently close to 1,

\[ T(r, f') > \phi(r). \]

Now we turn our attention to study the radial growth of the derivative of $Q_p$ functions. If $p > 1$ and $f \in Q_p = B$ then, trivially,

\[ |f'(re^{i\theta})| = O\left((1 - r)^{-1}\right), \quad \text{as } r \to 1, \text{ for every } \theta \in \mathbb{R}. \]

This is the best that can be said. Indeed, if $q \in \mathbb{N}$ is sufficiently large, there is $C_q > 0$ such that

\[ f(z) = C_q \sum_{k=0}^{\infty} z^{q^k}, \quad z \in \Delta, \]

satisfies $f \in B$ and

\[ |f'(z)| \geq \frac{1}{1 - |z|^2} \quad \text{if} \quad 1 - \frac{1}{q^k} \leq |z| \leq 1 - \frac{1}{q^{k+1}}, \]

(see [19]) which implies

\[ \limsup_{r \to 1} (1 - r^2)|f'(re^{i\theta})| \geq 1, \quad \text{for every } \theta. \]

If $f \in BMOA$, then it has a finite non-tangential limit $f(e^{i\theta})$ for almost every $\theta \in \mathbb{R}$, so, by a result of Zygmund [22, p. 181], it follows that for almost every $\theta$,

\[ |f'(re^{i\theta})| = o\left((1 - r)^{-1}\right), \quad \text{as } r \to 1. \]

This result is also sharp in the sense that the right hand side of (7) cannot be substituted by $O((1 - r)^{-\alpha})$ for any $\alpha < 1$. Indeed, if

\[ f(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^{2^k}, \quad z \in \Delta, \]

then, since $f$ is given by a power series with Hadamard gaps in $H^2$, we have $f \in BMOA$. Also, by Lemma 1 [22, p. 197], the fact $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$ implies

\[ \int_0^1 |f'(re^{i\theta})| \, dr = \infty, \quad \text{for every } \theta \in \mathbb{R}. \]

Consequently, we have proved the following
Proposition 1. There exists \( f \in BMOA \) such that, for any \( \alpha < 1 \) and any \( \theta \)
\[
|f'(re^{i\theta})| \neq O((1-r)^{-\alpha}), \quad \text{as } r \to 1.
\]

However, an estimate which is much stronger than (7) is true for the Dirichlet space \( D \). Seidel and Walsh [20, Thm. 6] proved that if \( f \in D \) then, for a.e. \( \theta \),
\[
|f'(re^{i\theta})| = o\left((1-r)^{-1/2}\right), \quad \text{as } r \to 1,
\]
and Girela [13] proved that this is sharp in a very strong sense.

Now, we shall consider these questions for \( Q_p \) spaces, \( 0 < p \leq 1 \). We can prove the following results.

Theorem 3. If \( f \in Q_p \), \( 0 < p < 1 \), then for a.e. \( \theta \),
\[
|f'(re^{i\theta})| = o\left((1-r)^{-(p+1)/2}\right), \quad \text{as } r \to 1.
\]

Theorem 4. Let \( 0 < p \leq 1 \), and let \( \phi \) be a positive increasing function in \((0,1)\) such that
\[
\int_0^1 (1-r)^p \phi^2(r)dr < \infty.
\]
Then there exists \( f \in Q_p \) such that, for every \( \theta \),
\[
\limsup_{r \to 1^-} \frac{|f'(re^{i\theta})|}{\phi(r)} = \infty.
\]

We remark that Theorem 4 for \( p = 1 \) represents an improvement of Proposition 1.

Finally, let us mention that the techniques used in this work are related to those used by Kennedy [17] and by Girela [13]. Also, we will adopt the convention that \( C \) will always denote a positive constant, independent of \( r \), which may be different on other occasion.

2. Proofs of Theorems 1 and 2. Let \( f \in Q_p \), with \( 0 < p < 1 \). By Jensen’s inequality, we have
\[
\exp(2T(r,f')) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} 2 \log |f'(re^{i\theta})|d\theta\right) \\
\leq \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left(1 + |f'(re^{i\theta})|^2\right) d\theta\right) \\
\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 + |f'(re^{i\theta})|^2\right) d\theta.
\]
Multiplying by \((1 - r)^p\) and integrating, we obtain
\[
\int_0^1 (1 - r)^p \exp(2T(r, f')) \, dr \leq \frac{1}{2\pi} \int_0^\pi \int_{-\pi}^\pi (1 - r)^p \left( 1 + |f'(re^{i\theta})|^2 \right) \, d\theta \, dr.
\]

We now refer to [4, Thm. 1.1], where it is shown that a function \(f\) is in \(Q_p\), \(0 < p \leq 1\), if and only if \(d\mu(z) = (1 - |z|)^p |f'(z)|^2 \, dx \, dy\) is a \(p\)-Carleson measure. A \(p\)-Carleson measure is a finite Borel measure \(\mu\) in \(\Delta\) for which there exists a constant \(c > 0\) such that for all intervals \(I\) of the form \(I = (\theta_0, \theta_0 + h), \theta_0 \in \mathbb{R}\) and \(0 < h < 1\), we have
\[
\mu(S(I)) \leq ch^p,
\]
where \(S(I)\) is the classical Carleson square,
\[
S(I) = \{re^{i\theta} : \theta_0 < \theta < \theta_0 + h, \ 1 - h < r < 1\}.
\]

All this tells us that the term on the right hand side of the above inequality is finite, and therefore Theorem 1 follows. \(\square\)

To prove Theorem 2, take \(0 < p < 1\), and \(\phi\) as in the statement. Since \(\phi\) is increasing, (iii) implies
\[
\infty > \int_0^1 (1 - r)^p \exp(2\phi(r)) \, dr \geq \sum_{k=1}^{\infty} \int_{1-2^{-k}}^{1-2^{-(k+1)}} (1 - r)^p \exp(2\phi(r)) \, dr
\]
\[
\geq \sum_{k=1}^{\infty} 2^{-(k+1)(p+1)} \exp(2\phi(1 - 2^{-k}))
\]
\[
= 2^{-(p+1)} \sum_{k=1}^{\infty} 2^{-k(p+1)} \exp(2\phi(1 - 2^{-k})).
\]

So (see for instance [18, Dini’s Thm, p. 297] there exists an increasing sequence \(\{\alpha_k\}\) of integers greater than 2, such that
\[
\sum_{k=1}^{\infty} \alpha_k^2 2^{-k(p+1)} \exp(2\phi(1 - 2^{-k})) < \infty,
\]
and
\[
\alpha_k \longrightarrow \infty, \quad \alpha_{k+1}/\alpha_k \longrightarrow 1 \quad \text{as} \quad k \to \infty.
\]

Observe that condition (13) implies
\[
\sum_{k=1}^{\infty} \alpha_k^{p+1} 2^{-k(p+1)} \exp(2\phi(1 - 2^{-k})) < \infty.
\]
Define now

\[ n_1 = 1, \quad n_{k+1} = \alpha_k n_k, \quad k = 1, 2, \ldots. \]

Clearly \( n_{k+1} > 2^k \) for \( k \geq 1 \) and by (i) we obtain

\[ \alpha_k n_{k+1}^{-p+1} \exp(2\phi(1 - n_{k+1}^{-1})) \leq \alpha_k n_k^{-p+1} 2^{-(p+1)} \exp(2\phi(1 - 2^{-k})), \]

which, together with (15) and (16), yields

\[ \sum_{k=1}^{\infty} n_k^{-(p+1)} \exp(2\phi(1 - n_{k+1}^{-1})) < \infty. \]

For each \( k = 1, 2, \ldots, \) set

\[ c_k = 10 n_k^{-1} \exp(\phi(1 - n_{k+1}^{-1})), \]

and define the function

\[ f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}, \quad z \in \Delta. \]

The way in which \( n_k \) and \( c_k \) have been chosen shows that \( f \) is a power series with Hadamard gaps defined in \( \Delta \). So in order to see that \( f \in Q_p \), we will use the following result proved in [5].

**Theorem A.** If \( 0 < p \leq 1 \), and \( f(z) = \sum_{k=0}^{\infty} c_k z^{n_k} \) is a power series with Hadamard gaps, then

\[ f \in Q_p \iff f \in Q_{p,0} \iff \sum_{k=0}^{\infty} 2^{k(1-p)} \sum_{\{j: n_j \in I_k\}} |c_j|^2 < \infty, \]

where \( I_k = \{n \in \mathbb{N} : 2^k \leq n < 2^{k+1}\}, \ k = 0, 1, \ldots. \)

For each \( j \in \mathbb{N} \), let \( k(j) \) be the unique non-negative integer such that \( 2^{k(j)} \leq n_j < 2^{k(j)+1} \). Bearing in mind this and (17), we have

\[ \sum_{k=0}^{\infty} 2^{k(1-p)} \sum_{\{j: n_j \in I_k\}} |c_j|^2 = \sum_{j=1}^{\infty} 2^{k(j)(1-p)} |c_j|^2 \]

\[ = 10^2 \sum_{j=1}^{\infty} 2^{k(j)(1-p)} n_j^{-2} \exp(2\phi(1 - n_{j+1}^{-1})) \]

\[ \leq 10^2 \sum_{j=1}^{\infty} n_j^{-(p+1)} \exp(2\phi(1 - n_{j+1}^{-1})) < \infty. \]
Hence, \( f \in Q_p \).

Next, we show that \( f \) satisfies (6). Observe that for \( k \geq 2 \) and \( |z| = 1 - \frac{1}{n_k} \),

\[
|f'(z)| \geq |zf'(z)| = \left| \sum_{j=1}^{\infty} c_j n_j z^{n_j} \right| \\
\geq c_k n_k |z|^{n_k} - \sum_{j=1}^{k-1} c_j n_j |z|^{n_j} - \sum_{j=k+1}^{\infty} c_j n_j |z|^{n_j} \\
\geq c_k n_k \left(1 - \frac{1}{n_k}\right)^{n_k} - \sum_{j=1}^{k-1} c_j n_j - \sum_{j=k+1}^{\infty} c_j n_j \left(1 - \frac{1}{n_k}\right)^{n_j} \\
= (I) - (II) - (III).
\]

Since the sequence \((1 - \frac{1}{n})^n\) increases with \( n \), and \( n_k \geq 2 \),

\[
(21) \quad (I) \geq \frac{1}{4} c_k n_k.
\]

Now, in order to estimate (II) and (III), we will use the following lemma stated in [17, p. 339].

**Lemma 1.** If \( \{s_k\} \) is a sequence of positive numbers and \( s_k/s_{k+1} \to 0 \) as \( k \to \infty \), then,

\[
\sum_{j=1}^{k-1} s_j = o(s_k), \quad \text{and} \quad \sum_{j=k+1}^{\infty} s_j^{-1} = o(s_k^{-1}) \quad \text{as} \quad k \to \infty.
\]

Notice that by (18), (ii), (16), and (14),

\[
\frac{c_k n_k}{c_{k+1} n_{k+1}} = \exp(\phi(1 - n_{k+1}^{-1}) - \phi(1 - n_{k+2}^{-1})) \to \infty \quad \text{as} \quad k \to \infty,
\]

so by the lemma,

\[
(22) \quad (II) = o(c_k n_k), \quad \text{as} \quad k \to \infty.
\]

Now using the elementary inequality \((1 - x)^n < 2(nx)^{-2}\), valid for \( 0 < x < 1 \) and \( n \geq 1 \), we obtain

\[
(23) \quad (III) \leq 2n_k^2 \sum_{j=k+1}^{\infty} \frac{c_j}{n_j}.
\]
But also, by (18), (16), (i), and (14),
\[
\frac{n_k/c_k}{n_k+1/c_{k+1}} = \frac{1}{\alpha_k^2} \exp \phi(1-n_k^{-1}) \leq \frac{1}{\alpha_k} \left( \frac{n_k+2}{n_k+1} \right)^{n_k+1}
\]
\[
= \frac{1}{\alpha_k^{(3-p)/2}} \left( \frac{\alpha_{k+1}}{\alpha_k} \right)^{n_k+1} \to 0,
\]
so by (23) and the lemma again,
\[
(24) \quad (\text{III}) = o(c_k n_k), \quad \text{as } k \to \infty.
\]
Therefore, by (21), (22), and (24), there exists \( k_0 \) such that for all \( k \geq k_0 \),
\[
|f'(z)| > \frac{1}{8} c_k n_k > \exp \phi(1 - \frac{1}{n_k+1}), \quad |z| = 1 - \frac{1}{n_k}.
\]
Thus, for \( k \geq k_0 \),
\[
T(1 - \frac{1}{n_k}, f') = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f'((1 - \frac{1}{n_k})e^{i\theta})| d\theta > \phi(1 - \frac{1}{n_k+1}).
\]
Now, if \( r \geq 1 - (n_k)^{-1} \), take \( k \geq k_0 \) such that \( 1 - (n_k)^{-1} \leq r < 1 - (n_{k+1})^{-1} \).
Since \( T \) and \( \phi \) are increasing functions of \( r \), we obtain
\[
T(r, f') \geq T(1 - \frac{1}{n_k}, f') > \phi(1 - \frac{1}{n_k+1}) \geq \phi(r).
\]
This completes the proof of Theorem 2. \( \square \)

2. Proofs of Theorems 3 and 4. We start proving Theorem 3. Let \( f \in Q_p \). Set
\[
F_r(\theta) = \max_{0 \leq \rho \leq r} |f'(\rho e^{i\theta})|^2, \quad 0 < r < 1, \quad \theta \in \mathbb{R}.
\]
By the Hardy-Littlewood Maximal Theorem,
\[
\int_{-\pi}^{\pi} F_r(\theta) d\theta \leq C \int_{-\pi}^{\pi} |f'(re^{i\theta})|^2 d\theta, \quad 0 < r < 1.
\]
Since \( g(z,0) = \log \frac{1}{|z|} \) and \( f \in Q_p \), we have
\[
\int_0^1 \int_{-\pi}^{\pi} F_r(\theta) \left( \log \frac{1}{r} \right)^p r d\theta dr \leq C \int_0^1 \int_{-\pi}^{\pi} |f'(re^{i\theta})|^2 g(re^{i\theta},0)^p r d\theta dr < \infty.
\]
Hence we deduce that
\[
\int_0^1 F_r(\theta) \left( \log \frac{1}{r} \right)^p r dr < \infty, \quad \text{a.e. } \theta,
\]
which yields, by means of the equivalence \( \log \frac{1}{r} \sim (1 - r) \) as \( r \to 1 \),
\[
\lim_{r \to 1} \int_r^1 F_s(\theta)(1 - s)^p ds = 0, \quad \text{a.e. } \theta.
\]
Since \( F \) is an increasing function of \( r \), we have for a.e. \( \theta \)
\[
\left| f'(re^{i\theta}) \right| \left( \frac{1 - r}{p + 1} \right) \leq F_r(\theta) \int_r^1 (1 - s)^p ds \leq \int_r^1 F_s(\theta)(1 - s)^p ds \quad \text{as } r \to 1
\]
and (10) follows. \( \square \)

**Proof of Theorem 4.** We may assume without loss of generality that \( \phi(r) \not\to \infty \) as \( r \not\to 1 \). Also, it suffices to prove that there exist \( f \in Q_p \) and \( C > 0 \) such that for every \( \theta \)
\[
\limsup_{r \to 1} \left| \frac{f'(re^{i\theta})}{\phi(r)} \right| \geq C.
\]
(25)

The reason for this is that if \( \phi \) is a positive increasing function in \((0,1)\)
satisfying (11), then it is possible to find \( \phi_1 \), positive and increasing in
\((0,1)\) with \( \lim_{r \to 1} \phi_1(r) = \infty \), and such that
\[
\int_0^1 (1 - r)^p \phi_1^2(r) dr < \infty.
\]
Clearly, if there are \( f \in Q_p \) and \( C > 0 \) satisfying (25) for every \( \theta \), with \( \phi \) replaced by \( \phi \phi_1 \), then the same \( f \) satisfies equation (12) for every \( \theta \).

With these assumptions we may start the proof. Take a sequence \( \{r_k\} \not\to 1 \), with \( r_1 > 1/4 \), which satisfies
\[
r_{k+1} - r_k > \frac{1}{2} (1 - r_k), \quad \text{for all } k,
\]
(26)
\[
\frac{\phi(r_{k+1})}{\phi(r_k)} \to \infty \quad \text{as } k \to \infty,
\]
(27)
\[
\frac{(1 - r_{k+1})^{\frac{2-p}{2}}}{(1 - r_k)^2} = O(1), \quad \text{as } k \to \infty.
\]
(28)

It follows from (26) that for all \( k \)
\[
1 - r_{k+1} < \frac{1}{2} (1 - r_k) < r_{k+1} - r_k.
\]
(29)

Bearing this in mind, observe that for all \( k \in \mathbb{N} \)
\[
\int_{r_k}^{r_{k+1}} (1-r)^p dr = \frac{1}{1+p} \left( (1-r_k)^{1+p} - (1-r_{k+1})^{1+p} \right) \geq \frac{1-2^{-(1+p)}}{1+p} (1-r_k)^{1+p}.
\]

Since \( \phi \) is increasing, (11) implies
\[
\sum_{k=1}^{\infty} (1-r_k)^{1+p} \phi^2(r_k) \leq \frac{1+p}{1-2^{-(1+p)}} \sum_{k=1}^{\infty} \int_{r_k}^{r_{k+1}} (1-r)^p \phi^2(r) dr 
\]
(30)
\[
\leq \frac{1+p}{1-2^{-(1+p)}} \int_{r_k}^{r_{k+1}} (1-r)^p \phi^2(r) dr 
\]
\[
\leq \frac{1+p}{1-2^{-(1+p)}} \int_0^1 (1-r)^p \phi^2(r) dr < \infty.
\]

Now, for each \( k \), let \( n_k \) be the unique non-negative integer such that
\[
n_k \leq \frac{1}{1-r_k} < n_k + 1.
\]

This implies, together with the facts that \( \{r_k\} \) is increasing and \( r_1 \geq 1/4 \),
(31) \[
1 - \frac{1}{n_k} \leq r_k < 1 - \frac{1}{n_k + 1}, \quad \text{and} \quad \frac{1}{4} < n_k (1-r_k) \leq 1.
\]

Define now
\[
f(z) = \sum_{k=1}^{\infty} (1-r_k) \phi(r_k) z^{n_k}.
\]

By (30), \( f \) is analytic in \( \Delta \). Moreover, \( f \) is a power series with Hadamard gaps. Indeed, by the definition of \( n_k \) and by (29),
\[
\frac{n_{k+1}}{n_k} \geq \frac{1}{1-r_{k+1}} - \frac{1}{1-r_k} = \frac{1-r_k}{1-r_{k+1}} - (1-r_k) > 2 - \frac{3}{4} > 1, \quad \text{all } k.
\]

We now check that \( f \) is in \( Q_p \). To this end we use Theorem A. For each \( j \), let \( k(j) \) be the unique non-negative integer such that
\[
2^{k(j)} \leq n_j < 2^{k(j)+1}.
\]

In this situation, we have by (31) and (30),
\[
\sum_{k=0}^{\infty} 2^{k(1-p)} \sum_{2^k \leq n_j < 2^{k+1}} (1-r_j)^2 \phi^2(r_j) = \sum_{j=1}^{\infty} 2^{k(j)(1-p)} (1-r_j)^2 \phi^2(r_j)
\]
\[
\leq \sum_{j=1}^{\infty} n_j^{1-p} (1-r_j)^2 \phi^2(r_j) \leq \sum_{j=1}^{\infty} (1-r_j)^{1+p} \phi^2(r_j) < \infty.
\]
This shows that \( f \in Q_p \).

Next, to show that \( f \) satisfies (25), it suffices to find a constant \( C > 0 \) and \( k_0 \in \mathbb{N} \) such that

\[
\left| \frac{f'(r_k e^{i\theta})}{\phi(r_k)} \right| \geq C \text{ for every } \theta \text{ and all } k \geq k_0.
\]

If \( |z| = r_k \ (k \geq 2) \) then, (31) and \( r_k^{n_j} \leq 1 \) imply

\[
|f'(z)| \geq |zf'(z)| = \left| \sum_{j=1}^{\infty} n_j (1 - r_j) \phi(r_j) z^{n_j} \right|
\geq n_k (1 - r_k) \phi(r_k) r_k^{n_k} - \sum_{j \neq k} n_j (1 - r_j) \phi(r_j) r_k^{n_j}
\geq \frac{1}{4} \phi(r_k) \left( 1 - \frac{1}{n_k} \right)^{n_k} - \sum_{j=1}^{k-1} \phi(r_j) - \sum_{j=k+1}^{\infty} \phi(r_j) \left( 1 - \frac{1}{n_k + 1} \right)^{n_j}
= (I) - (II) - (III).
\]

The procedure now is basically the same as in the proof of Theorem 2. Since the sequence \( (1 - \frac{1}{n})^n \) increases with \( n \) and \( n_k \geq 2 \), we have \( (I) \geq C \phi(r_k) \). Now, by (27) and Lemma 1 we obtain \( (II) = o(\phi(r_k)) \). Finally, as in (23), we deduce

\[
(III) \leq 2(n_k + 1)^2 \sum_{j=k+1}^{\infty} \frac{\phi(r_j)}{n_j^2}.
\]

But by (31), (28) and (30),

\[
\frac{n_j^2 / \phi(r_j)}{n_{j+1}^2 / \phi(r_{j+1})} \leq \frac{16}{\phi(1/4)} \frac{(1 - r_{j+1})^2 \phi(r_{j+1})}{(1 - r_j)^2}
= \frac{16}{\phi(1/4)} \frac{(1 - r_{j+1})^{\frac{3-p}{2}}}{(1 - r_j)^2} (1 - r_{j+1})^{\frac{1+p}{2}} \phi(r_{j+1}) \xrightarrow{j \to \infty} 0
\]

so by Lemma 1,

\[
\sum_{j=k+1}^{\infty} \frac{\phi(r_j)}{n_j^2} = o\left( \frac{\phi(r_k)}{n_k^2} \right),
\]

which implies \( (III) = o(\phi(r_k)) \). This completes the proof of Theorem 4. \( \square \)
4. Remarks.

**Remark 1.** The estimate given in Theorem 3 allows us to say something about the radial variation of functions in the $Q_p$ spaces. We start recalling some definitions. For a function $f$ analytic in the unit disk $\Delta$ and $\theta \in [-\pi, \pi]$, the quantity

$$V(f, \theta) = \int_0^1 |f'(re^{i\theta})|dr,$$

denotes the radial variation of $f$ along the radius $[0, e^{i\theta}]$, i.e., the length of the image of this radius under the mapping $f$. The exceptional set $E(f)$ associated to $f$ is then defined as

$$E(f) = \{ e^{i\theta} \in \partial \Delta : V(f, \theta) = \infty \}.$$

Since $\int_0^1 (1-r)^{-(p+1)/2}dr$ is finite if and only if $p < 1$, then an immediate consequence of Theorem 3 is the following

**Theorem 5.** If $f \in Q_p$, $0 < p < 1$, then the exceptional set $E(f)$ has linear measure 0.

Observe that nothing of this kind can be stated for $Q_p$ with $p \geq 1$. Indeed, as we have noticed above before Proposition 1, if $f(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^{2^k}$, then $f \in BMOA = Q_1$ and $V(f, \theta) = \infty$ for every $\theta$.

On the other hand, for functions in the Dirichlet class $D \equiv Q_0$ there is a more precise result due to Beurling [7].

**Theorem B.** If $f \in D$, then the exceptional set $E(f)$ has a zero logarithmic capacity.

We refer to [10,16,21] for the definition and basic results about capacities and Hausdorff measures. We do not know whether the conclusion of Theorem B is true for $Q_p$, $0 < p < 1$. However, something can be said. For $0 < p < 1$, let $D_p$ be the space of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, analytic in $\Delta$ such that

$$\sum_{n=1}^{\infty} n^{1-p} |a_n|^2 < \infty.$$

Zygmund proved the following result (see [16, Ch. 4]).

**Theorem C.** If $f \in D_p$, $0 < p < 1$, then the exceptional set $E(f)$ has zero $p$-capacity. Conversely, if $E$ is a set of zero $p$-capacity, then there is $f \in D_p$ whose exceptional set contains $E$.

It is not difficult to see that $f \in Q_p$, $0 < p < 1$ implies $f \in D_p$. In fact, if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in Q_p$, $0 < p < 1$, there exists $C > 0$ such that

$$\int_{\Delta} |f'(z)|^2 g^p(z,a)dxdy < C,$$

for all $a \in \Delta$. 

In particular, we have for $a = 0$, using properties of the Beta function and Stirling’s formula for the Gamma function: $\Gamma(t + 1) \sim t^e e^{-t} (2\pi t)^{1/2}$,

\[
\infty > \int_0^1 \int_{-\pi}^{\pi} |f'(re^{i\theta})|^2 \log^p \frac{1}{r} r dr d\theta = \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_0^1 r^{2n-1} \log^p \frac{1}{r} dr
\]

\[
\geq \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_0^1 r^{2n-1} (1 - r)^p dr = \sum_{n=1}^{\infty} n^2 |a_n|^2 B(2n, p + 1)
\]

\[
= \sum_{n=1}^{\infty} n^2 |a_n|^2 \frac{\Gamma(2n)\Gamma(p+1)}{\Gamma(2n+p+1)} \approx \sum_{n=1}^{\infty} n^{1-p} |a_n|^2.
\]

Therefore, an immediate consequence of Zygmund’s result is the following

**Theorem 6.** If $f \in Q_p$, $0 < p < 1$, then the exceptional set $E(f)$ has zero $p$-capacity.

However, we do not know whether for a given set $E$ of null $p$-capacity there is $f \in Q_p$ whose exceptional set contains $E$.

**Remark 2.** From Beurling’s result (Theorem B), it follows that any $f \in D$ has non-tangential limit everywhere except for a set of null logarithmic capacity, and then

\[(32) \quad |f'(re^{i\theta})| = o((1 - r)^{-1}) \text{ as } r \to 1,
\]

whenever $e^{i\theta}$ is a point at which $f$ has a finite non-tangential limit.

This implies that for $f \in D$ the estimate (32) holds for every $\theta \in (-\pi, \pi]$, except for a set of null logarithmic capacity. Girela [15] showed that this estimate is sharp in a very strong sense. In our case, using Theorem 6 and (32), we obtain a similar result for $Q_p$, $0 < p < 1$, although we do not know whether it is sharp in the sense given by Girela.

**Theorem 7.** If $f \in Q_p$, $0 < p < 1$, then

\[
|f'(re^{i\theta})| = o((1 - r)^{-1}) \text{ as } r \to 1,
\]

for every $\theta \in (-\pi, \pi]$, except for a set of null $p$-capacity.

**References**


