Canonical vector valued 1-forms
on higher order adapted frame bundles
over category of fibered squares

Abstract. Let $Y$ be a fibered square of dimension $(m_1, m_2, n_1, n_2)$. Let $V$ be a finite dimensional vector space over $\mathbb{R}$. We describe all $F^2 M_{m_1, m_2, n_1, n_2}$- canonical $V$-valued 1-form $\Theta: TP_r^2(Y) \to V$ on the $r$-th order adapted frame bundle $P_r^2(Y)$.

A fibered square (or fibered-fibered manifold) is any commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\pi} & X \\
\downarrow q & & \downarrow p \\
N & \xrightarrow{\pi_0} & M
\end{array}
$$

where maps $\pi, \pi_0, q, p$ are surjective submersions and induced map $Y \to X \times_M N$, $y \mapsto (\pi(y), q(y))$ is a surjective submersion. We will denote a fibered square (1) by $Y$ in short, [3], [5].

A fibered square (1) has dimension $(m_1, m_2, n_1, n_2)$, if $\dim Y = m_1 + m_2 + n_1 + n_2$, $\dim X = m_1 + m_2$, $\dim N = m_1 + n_1$, $\dim M = m_1$. For two fibered squares $Y_1, Y_2$ of the same dimension $(m_1, m_2, n_1, n_2)$, a fibered squares morphism $f: Y_1 \to Y_2$ is quadruple of local diffeomorphisms $f: Y_1 \to Y_2$.

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All fibered squares of given dimension \((m_1, m_2, n_1, n_2)\) and their morphisms form a category which we denote by \(\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}\).

Every object from the category \(\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}\) is locally isomorphic to the standard fibered square

\[
\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \longrightarrow \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}
\]

which we denote by \(\mathbb{R}^{m_1, m_2, n_1, n_2}\), where arrows are obvious projections.

Let \(Y\) be a fibered square \((1)\) of dimension \((m_1, m_2, n_1, n_2)\). We define the \(r\)-th order adapted frame bundle

\[
P_r^A(Y) = \{ j^r_{(0,0,0,0)} \varphi \mid \varphi: \mathbb{R}^{m_1, m_2, n_1, n_2} \to Y \}
\]

over \(Y\) with the projection \(\beta: P_r^A(Y) \to Y\), \(\beta(j^r_{(0,0,0,0)} \varphi) = \varphi(0, 0, 0, 0)\).

The adapted frame bundle \(P^A_r(Y)\) is a principal bundle with Lie group \(G_{m_1, m_2, n_1, n_2}^r = P_r^A(\mathbb{R}^{m_1, m_2, n_1, n_2} \times (0,0,0,0))\) (with multiplication given by the composition of jets) acting on the right on \(P^A_r(Y)\) by composition of jets. Every \(\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}\)-morphism \(\Phi: Y_1 \to Y_2\) induces a local diffeomorphism \(P^A_r \Phi: P^A_r(Y_1) \to P^A_r(Y_2)\) given by \(P^A_r \Phi(j^r_{(0,0,0,0)} \varphi) = j^r_{(0,0,0,0)}(\Phi \circ \varphi)\), \([1], [4]\).

**Definition 1.** Let \(V\) be a finite dimensional vector space over \(\mathbb{R}\). A \(\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}\)-canonical \(V\)-valued 1-form \(\Theta\) on \(P^r_A\) is any \(\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}\)-invariant family \(\Theta = \{ \Theta_Y \}\) of \(V\)-valued 1-forms \(\Theta_Y: TP^r_A(Y) \to V\) on \(P^r_A(Y)\) for any \(\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}\)-object \(Y\), \([2], [4]\).

The invariance of canonical 1-form \(\Theta\) means that two \(V\)-valued forms \(\Theta_Y\) and \(\Theta_Y'\) are \(P^A_r \Phi\)-related (that is \(P^A_r \Phi^* \Theta_Y' = \Theta_Y\), where \(P^A_r \Phi^* \Theta_Y = \Theta_Y \circ TP^r_A \Phi\)) for any \(\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}\)-morphism \(\Phi: Y_1 \to Y_2\).

**Example 1.** For every \(\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}\)-object \(Y\) we define \(\mathbb{R}^{m_1+m_2+n_1+n_2}\)-valued 1-form \(\theta_Y\) on \(P^1_A(Y)\) as follows. Consider the projection \(\beta: P^1_A(Y) \to Y\) given by \(\beta(j^1_{(0,0,0,0)} \varphi) = \varphi(0, 0, 0, 0)\), an element \(u = j^1_{(0,0,0,0)} \psi \in P^1_A(Y)\) and a tangent vector \(W = j^1_0 c \in T_u P^1_A(Y)\). We define the form \(\theta_Y\) by

\[
\theta_Y(W) = u^{-1} \circ T \beta(W)
\]

\[
= j^1_0(\psi^{-1} \circ \beta \circ c) \in T_{(0,0,0,0)} \mathbb{R}^{m_1+m_2+n_1+n_2} = \mathbb{R}^{m_1+m_2+n_1+n_2}.
\]

Obviously, \(\theta = \{ \theta_Y \}\) is a \(\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}\)-canonical 1-form on \(P^1_A\).

A vector field \(W\) on \(Y\) is projectable-projectable on \(\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}\)-object \((1)\), if there exists vector fields \(W_1\) on \(X\) and \(W_2\) on \(N\) and \(W_0\) on
We can assume that $W,W_1$ are $\pi$-related and $W,W_2$ are $q$-related and $W_1,W_0$ are $\pi_0$-related, [5].

We therefore see that vector field $W$ on $Y$ is projectable-projectable on $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object (1) if and only if the flow $\{\Phi_t\}$ of vector field $W$ is formed by local $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-maps.

The space of all projectable-projectable vector fields on $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object $Y$ will be denoted by $\mathfrak{X}_{\text{proj-proj}}(Y)$. It is Lie subalgebra of Lie algebra $\mathfrak{X}(Y)$ of all vector fields on $Y$.

For projectable-projectable vector field $W \in \mathfrak{X}_{\text{proj-proj}}(Y)$ the flow lifting $P^r_A W$ is vector field on $P^r_A(Y)$ such that if $\{\Phi_t\}$ is the flow of field $W$, then $\{P^r_A(\Phi_t)\}$ is the flow of field $P^r_A W$. (Since $\Phi_t$ are $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-maps, we can apply functor $P^r_A$ to $\Phi_t$).

To present a general example of a $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-canonical $V$-valued 1-form on $P^r_A$ we need the following lemma, which is an obvious modification of the known fact for usual manifolds.

**Lemma 1.** Let $Y$ be a fibered square (1) from the category $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$. Then any vector $w \in T_yP^r_A(Y)$, where $v \in (P^r_A(Y))_y$, $y \in Y$, is of the form $w = P^r_A W_v$ for any projectable-projectable vector field $W \in \mathfrak{X}_{\text{proj-proj}}(Y)$, where $P^r_A W \in \mathfrak{X}(P^r_A(Y))$ is the flow lifting of field $W$ to $P^r_A(Y)$. Moreover $j^r_0 W$ is uniquely determined.

**Proof.** We can assume that $Y = \mathbb{R}^{m_1,m_2,n_1,n_2}$ and $y = (0,0,0,0) \in \mathbb{R}^{m_1+m_2+n_1+n_2}$. Since $P^r_A(\mathbb{R}^{m_1,m_2,n_1,n_2})$ is obviously a principal subbundle of the $r$-th order frame bundle $P^r(\mathbb{R}^{m_1+m_2+n_1+n_2})$, by the well-known manifolds version of Lemma 1, we find $W \in \mathfrak{X}(\mathbb{R}^{m_1+m_2+n_1+n_2})$ such that $w = P^r_A W_v$ and $j^r_{(0,0,0,0)} W$ is uniquely determined, where $P^r A$ is a vector field on $P^r(\mathbb{R}^{m_1+m_2+n_1+n_2})$ being a flow lifting of vector field $W$ and $v \in P^r_A(\mathbb{R}^{m_1,m_2,n_1,n_2})$.

For a projectable-projectable vector field $\widetilde{W} \in \mathfrak{X}_{\text{proj-proj}}(\mathbb{R}^{m_1,m_2,n_1,n_2})$ the vector $P^r A \widetilde{W}_v \in T_yP^r(\mathbb{R}^{m_1+m_2+n_1+n_2})$ is tangent to $P^r_A(\mathbb{R}^{m_1,m_2,n_1,n_2})$ at the point $v$. On the other hand, the dimension of space $P^r_A(\mathbb{R}^{m_1,m_2,n_1,n_2})$ and the dimension of space of $r$-jets $j^r_{(0,0,0,0)} \widetilde{W}$ of projectable-projectable vector fields $\widetilde{W} \in \mathfrak{X}_{\text{proj-proj}}(\mathbb{R}^{m_1,m_2,n_1,n_2})$ are equal. Then Lemma 1 follows from dimension equality, since flow operators are linear.

**Example 2.** Let

\begin{equation}
\lambda : J^{r-1}_{(0,0,0,0)}(T_{\text{proj-proj}}(\mathbb{R}^{m_1,m_2,n_1,n_2})) \to V
\end{equation}

be a $\mathbb{R}$-linear map, where $J^{r-1}_{(0,0,0,0)}(T_{\text{proj-proj}}(\mathbb{R}^{m_1,m_2,n_1,n_2}))$ is the vector space of all $(r-1)$-jets $j^{r-1}_{(0,0,0,0)} W$ at point $(0,0,0,0) \in \mathbb{R}^{m_1+m_2+n_1+n_2}$ of projectable-projectable vector fields $W \in \mathfrak{X}_{\text{proj-proj}}(\mathbb{R}^{m_1,m_2,n_1,n_2})$. Given...
a fibered square $Y$, (1), from the category $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ we define $V$-valued 1-form $\Theta^\lambda_Y: TP^r_A(Y) \to V$ on $P^r_A(Y)$ as follows. Let $w \in T_vP^r_A(Y)$, where $w = j_{(0,0,0,0)}^r \varphi \in (P^r_A(Y))_y$, $y \in Y$. By Lemma 1, we have $w = P^r_AW_v$ for some projectable-projectable vector field $W \in \mathfrak{X}_{\text{proj-proj}}(Y)$ and $j_y^rW$ is uniquely determined. Then is uniquely determined the $(r - 1)$-jet $j_{(0,0,0,0)}^{r-1}((\varphi^{-1})_*W)$, where $(\varphi^{-1})_*W = T\varphi^{-1} \circ W \circ \varphi$. We define

$$\Theta^\lambda_Y(w) := \lambda(j_{(0,0,0,0)}^{r-1}((\varphi^{-1})_*W)).$$

Obviously, $\Theta^\lambda = \{\Theta^\lambda_Y\}$ is $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-canonical $V$-valued 1-form on $P^r_A$.

The main result of this note is the following classification theorem.

**Theorem 1.** Any $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-canonical $V$-valued 1-form on $P^r_A$ is of the form $\Theta^\lambda$ for some uniquely determined $\mathbb{R}$-linear map

$$\lambda: J_{(0,0,0,0)}^{r-1}(T_{\text{proj-proj}}(\mathbb{R}^{m_1,m_2,n_1,n_2})) \to V.$$

In the proof of Theorem 1 we use the following fact.

**Lemma 2.** Let $W_1, W_2 \in \mathfrak{X}_{\text{proj-proj}}(Y)$ be projectable-projectable vector fields on $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object $Y$ and let $y \in Y$ be a point. We suppose that $j_{(0,0,0,0)}^rW_1 = j_{(0,0,0,0)}^rW_2$ and $W_1(y)$ is not vertical vector with respect to composition of projections $\pi: Y \to X$ and $p: X \to M$. Then there exists a (locally defined) $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-map $\Phi: Y \to Y$ such that $j_{(0,0,0,0)}^r(\Phi) = j_{(0,0,0,0)}^r(id_Y)$ and $\Phi_*W_1 = W_2$ near $y$.

**Proof.** It is a direct modification of the proof of Lemma 42.4 in [2].

**Proof of Theorem 1.** Let $\Theta$ be $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-canonical $V$-valued 1-form on $P^r_A$. We must define $\lambda: J_{(0,0,0,0)}^{r-1}(T_{\text{proj-proj}}(\mathbb{R}^{m_1,m_2,n_1,n_2})) \to V$ by

$$\lambda(\xi) := (\Theta_{\mathbb{R}^{m_1,m_2,n_1,n_2}})(P^rA\tilde{W}_{j_{(0,0,0,0)}^r(id_{\mathbb{R}^{m_1,m_2,n_1,n_2}})}),$$

for all $\xi \in J_{(0,0,0,0)}^{r-1}(T_{\text{proj-proj}}(\mathbb{R}^{m_1,m_2,n_1,n_2}))$, where $\tilde{W}$ is a unique (germ at $(0,0,0,0)$) of projectable-projectable vector field on $\mathbb{R}^{m_1,m_2,n_1,n_2}$ such that $j_{(0,0,0,0)}^r\tilde{W} = \xi$ and coefficients of $\tilde{W}$ with respect to the basis of space $\mathfrak{X}_{\text{proj-proj}}(\mathbb{R}^{m_1,m_2,n_1,n_2})$ composed of canonical vector fields are polynomials of degree $\leq r - 1$. We are going to show that $\Theta = \Theta^\lambda$. Because of the $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-invariance of $\Theta$ and $\Theta^\lambda$ it remains to show that

$$\Theta_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(w) = (\Theta^\lambda_{\mathbb{R}^{m_1,m_2,n_1,n_2}})(w)$$

for any $w \in T_{(0,0,0,0)}(id_{\mathbb{R}^{m_1,m_2,n_1,n_2}})P^r_A(\mathbb{R}^{m_1,m_2,n_1,n_2})$.

By the definition of $\lambda$ and $\Theta^\lambda$ we have (8) for any $w$ of the form

$$w = P^r_A\tilde{W}_{j_{(0,0,0,0)}^r(id_{\mathbb{R}^{m_1,m_2,n_1,n_2}})},$$
where \( \tilde{W} \in \mathfrak{X}_{\text{proj-proj}}(\mathbb{R}^{m_1,m_2,n_1,n_2}) \) is a projectable-projectable vector field such that coefficients \( \tilde{W} \) with respect to the above mentioned basis of the space \( \mathfrak{X}_{\text{proj-proj}}(\mathbb{R}^{m_1,m_2,n_1,n_2}) \) are polynomials of degree \( \leq r - 1 \).

Now, let \( w \in T_{j^{r}}(0,0,0,0)( \text{id}_{\mathbb{R}^{m_1,m_2,n_1,n_2}}) \mathcal{P}^{r}_{A}(\mathbb{R}^{m_1,m_2,n_1,n_2}) \). Then by Lemma 1, \( w \) is of the form \( w = \mathcal{P}^{r}_{A}W_{j^{r}}(0,0,0,0)( \text{id}_{\mathbb{R}^{m_1,m_2,n_1,n_2}}) \) for some projectable-projectable vector field \( W \in \mathfrak{X}_{\text{proj-proj}}(\mathbb{R}^{m_1,m_2,n_1,n_2}) \) and \( j^{r}_{(0,0,0,0)}W \) is uniquely determined. We can additionally assume that \( W(0,0,0,0) \) is not vertical vector with respect to projection \( \mathbb{R}^{m_1+m_2+n_1+n_2} \to \mathbb{R}^{m_1} \). Let \( \tilde{W} \in \mathfrak{X}_{\text{proj-proj}}(\mathbb{R}^{m_1,m_2,n_1,n_2}) \) be projectable-projectable vector field such that \( j^{r-1}_{(0,0,0,0)} \tilde{W} = j^{r-1}_{(0,0,0,0)}W \) and coefficients of field \( \tilde{W} \) with respect to the basis of constant vector fields on \( \mathbb{R}^{m_1,m_2,n_1,n_2} \) are polynomials of degree \( \leq r - 1 \). Let \( \tilde{w} = \mathcal{P}^{r}_{A}W_{j^{r}}(0,0,0,0)( \text{id}_{\mathbb{R}^{m_1,m_2,n_1,n_2}}) \). Then (see above) it holds

\[
(\Theta_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(w))(\tilde{w}) = (\Theta_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(w))(\tilde{w}) = (\Theta_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(w))(\tilde{w})
= (\Theta_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(w))(w).
\]

On the other hand by Lemma 2 there exists a \( \mathcal{F}^2 \mathcal{M}_{m_1,m_2,n_1,n_2} \)-map \( \Phi: \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2} \) such that \( j^{r}_{(0,0,0,0)} \Phi = j^{r}_{(0,0,0,0)}( \text{id}_{\mathbb{R}^{m_1,m_2,n_1,n_2}}) \) and \( \Phi_{*}\tilde{W} = W \) near \( (0,0,0,0) \in \mathbb{R}^{m_1+m_2+n_1+n_2} \). Since \( j^{r}_{(0,0,0,0)} \Phi = \text{id} \), then \( \Phi \) preserves \( j^{r}_{(0,0,0,0)}( \text{id}_{\mathbb{R}^{m_1,m_2,n_1,n_2}}) \). Then since \( \Phi_{*}\tilde{W} = W \), so \( \Phi \) sends \( \tilde{w} \) into \( w \). Then because of invariance of \( \Theta \) and \( \Theta^{\lambda} \) with respect to \( \Phi \), we obtain

\[
(\Theta_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(w))(\tilde{w}) = (\Theta_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(w))(\tilde{w}) = (\Theta_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(w))(w).
\]

For \( r = 1 \) we have \( j^{0}_{(0,0,0,0)}(T_{\text{proj-proj}}\mathbb{R}^{m_1+m_2+n_1+n_2}) \cong \mathbb{R}^{m_1+m_2+n_1+n_2} \). Then by Theorem 1, the vector space of \( \mathcal{F}^2 \mathcal{M}_{m_1,m_2,n_1,n_2} \)-canonical \( V \)-valued 1-forms is of dimension \( (m_1 + m_2 + n_1 + n_2) \dim V \). Then we have:

**Corollary 1.** Any \( \mathcal{F}^2 \mathcal{M}_{m_1,m_2,n_1,n_2} \)-canonical 1-form \( \Theta \) = \{ \Theta_{Y} \} \) on \( P^{1}_{A} \) is of the form

\[
(9) \quad \Theta_{Y} = \lambda \circ \theta_{Y}: TP^{1}_{A}(Y) \to V
\]

for some unique linear map \( \lambda: \mathbb{R}^{m_1+m_2+n_1+n_2} \to V \), where \( \theta = \{ \theta_{Y} \} \) is a canonical \( \mathbb{R}^{m_1+m_2+n_1+n_2} \)-valued 1-form on \( P^{1}_{A} \) from Example 1.

**Example 3.** Notice that it holds

\[
(10) \quad j^{r-1}_{(0,0,0,0)}(T_{\text{proj-proj}}\mathbb{R}^{m_1+m_2+n_1+n_2}) \cong \mathbb{R}^{m_1+m_2+n_1+n_2} \oplus g_{m_1,m_2,n_1,n_2}^{r-1},
\]

where \( g_{m_1,m_2,n_1,n_2}^{r-1} = \text{Lie}(G_{m_1,m_2,n_1,n_2}) \).
In this way for \( \lambda = id_{\mathbb{R}^{m_1+m_2+n_1+n_2} \oplus g_{m_1,m_2,n_1,n_2}^{-1}} \) we have \( \mathcal{F}^2 M_{m_1,m_2,n_1,n_2} \) canonical 1-form

\[
\theta_Y := \Theta id_{\mathbb{R}^{m_1+m_2+n_1+n_2} \oplus g_{m_1,m_2,n_1,n_2}^{-1}} \colon T P^r_A(Y) \to \mathbb{R}^{m_1+m_2+n_1+n_2} \oplus g_{m_1,m_2,n_1,n_2}^{-1}
\]
on \( P_A \) (see Example 2). For \( r = 1 \), we have \( \theta^1 = \theta \) as in Example 1.

Analogously as in Corollary 1 we have

**Corollary 2.** Any \( \mathcal{F}^2 M_{m_1,m_2,n_1,n_2} \) canonical \( V \)-valued 1-form \( \Theta = \{ \Theta_Y \} \) on \( P_A \) is of the form:

\[
\Theta_Y = \lambda \circ \theta^r_Y : TP^r_A(Y) \to V
\]
for some uniquely determined linear map \( \lambda : \mathbb{R}^{m_1+m_2+n_1+n_2} \oplus g_{m_1,m_2,n_1,n_2}^{-1} \to V \), where \( \theta^r \) is from Example 3.

**Remark 1.** A notion of fibered square is a generalization of a fibered manifold. The theory of projectable natural bundles over fibered manifolds is essentially related with the idea of fibered square, [2], [3], [5].

**REFERENCES**


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