Abstract. New class $\mathcal{MA}_{n}^{k_{1}, \ldots, k_{n}}, n \geq 1$, of $n$-dimensional Markov-like algorithms is introduced. The equivalence of this class of algorithms and the class $\mathcal{MNA}$ of Markov normal algorithms is discussed.

1. Introduction. The intensive studies on the formalization of the notion of algorithm were conducted from 1930 on [2,7,10,12]. The majority of classical algorithms, such as partial recursive functions, Turing machines, Herbrand-Gödel computability, Markov normal algorithms are collected in Mendelson’s monograph [9]. The equivalence of particular classes of algorithms were shown earlier by several authors [1,4,7] but all results are collected in Mendelson’s monograph [9]. The next class of algorithms, for example the unlimited register machines ($\mathcal{URM}$), was also introduced[3]. The equivalence of the class $\mathcal{URM}$ and the class $\mathcal{PRF}$ of partial recursive functions was shown in [3].

A few classes of Markov-like algorithms were introduced by the authors in [5] where also the equivalence of these algorithms to the class $\mathcal{MNA}$ of Markov normal algorithms were shown. Only a few papers related to
algorithms of higher dimension were published [6,10]. The equivalence of
the class of two-dimensional Markov-like algorithms and the class $MNA$
was shown in [6].

This paper deals with the class $MA^{k_1,\ldots,k_n}$ of $n$-dimensional Markov-like
algorithms with respect to the order $x_{k_1},\ldots,x_{k_n}$ of axes. Every algorithm
of $MA^{k_1,\ldots,k_n}$ is defined by means of a set $\{P_1,\ldots,P_m\}$ of $n$-dimensional
equally shaped productions which are labelled by elements of any set $L$
(for simplicity we assume that $L = \{1,\ldots,m\}$). The succession of the
use of $n$-dimensional productions to the transformed words is almost the
same as for classical Markov normal algorithms, but the manner of use of
the productions depends on the choice of the subwords in the transformed
words. We choose a production $P_i : x_i \rightarrow (\cdot)y_i$, with the least label $i \leq m$,
such that its left-hand side word $x_i$ occurs in a transformed $n$-dimensional
word $t_1$. If such a production exists then we replace the first occurrence of
$x_i$ with respect to the order $x_{k_1},\ldots,x_{k_n}$ of axes by $y_i$ of $P_i$. If a production
$P_i$ is final then the algorithm stops, otherwise we should proceed with the
newly obtained word $t_2$ analogously as with $t_1$.

Notice that every labelled set of $n$-dimensional productions determines
$n!$ different algorithms of the classes $MA^{k_1,\ldots,k_n}$ with respect to the choice
of the orders $x_{k_1},\ldots,x_{k_n}$ of axes.

In this paper only the concept of the proof of a theorem relating to the
equivalence of the above class of $n$-dimensional Markov-like algorithms to
the class of $MNA$ of Markov normal algorithms is given. The complete
proof is very long and troublesome. Therefore we omit the proof.

This paper is the first step in the description of $n$-dimensional Markov-
like algorithms.

The following reasons motivate the introduction of this class of algo-
rithms:

(1) This paper is the first step of developments on $n$-dimensional formal
algorithms, which can be used to study the complexity problems of
$n$-dimensional structures.

(2) One is able to define $n$-dimensional partial recursive functions by
analogy with word or graph recursive functions;

(3) The formalism used here allows to introduce other classes of $n$-di-
mensional Markov-like algorithms, for example parallel algorithms;

(4) One can define $n$-dimensional (not necessarily Markov-like) algo-
rithms by a slight modification of the transformation and control
functions. These algorithms may be useful for the description of
real processes (biological, chemical, physical, medical and some oth-
ers).
2. \textit{n-dimensional words and productions.} Let $\Sigma$ be a nonempty (finite) alphabet. By a \textit{n-dimensional word} in an alphabet $\Sigma$ we mean a partial function $\psi : \mathbb{N}^n \longrightarrow \Sigma$ ($\mathbb{N}$ is the set of all nonnegative integers) satisfying the following conditions:

1. $0 < \text{Dom}(\psi) < \infty$, where $\text{Dom}(\psi)$ denotes the domain of $\psi$;
2. $(0, i_2, \ldots, i_n) \in \text{Dom}(\psi)$, $(j_1, 0, j_3 \ldots j_n) \in \text{Dom}(\psi)$, $(k_1, \ldots, k_{n-1}, 0) \in \text{Dom}(\psi)$;
3. for arbitrary $(i_1, \ldots, i_n) \in \text{Dom}(\psi)$ and $(k_1, \ldots, k_n) \in \text{Dom}(\psi)$ there exists a sequence $(j_1^1, \ldots, j_n^1) \in \text{Dom}(\psi)$, $1 \leq s \leq m$ where $(j_1^1, \ldots, j_n^1) = (i_1, \ldots, i_n)$, $(j_1^m, \ldots, j_n^m) = (k_1, \ldots, k_n)$, and for every $s \in \{1, \ldots, m - 1\}$ and for all $t \in \{1, \ldots, n\}$ we have: $(j_t^s = j_t^{s+1} \text{ or } j_t^s = j_t^{s+1} \pm 1)$.

Statements given in (1) and (2) mean that we consider only finite n-dimensional words having at least one coordinate on particular axis, condition (3) means that every point of a word is connected with another arbitrary one.

Let $\Sigma^*_n$ denote a class of all n-dimensional words in an alphabet $\Sigma$ including the empty n-dimensional word $\lambda_n$ (the function $\psi$ describing $\lambda_n$ has the empty domain).

In the majority of cases the n-dimensional words of $\Sigma^*_n$ will be denoted by lower case Latin letters $t, u, v, w, x, y, z$ (possibly with subscripts). If a word $t$ is described by a function $\psi$ then we will write $t(i_1, \ldots, i_n)$ or $t_{i_1,\ldots,i_n}$ instead of $\psi(i_1, \ldots, i_n)$ and $\text{Dom}(t)$ instead of $\text{Dom}(\psi)$.

A n-dimensional word $t$ will be called over an alphabet $\Sigma$ iff $t$ is in an alphabet $\Sigma'$, where $\Sigma$ is a subset of $\Sigma'$.

Let us define a shape of a n-dimensional word $t \in \Sigma^*_n$. The $n$-tuple $(m_1, \ldots, m_n)$ is said to be a \textit{shape} of a n-dimensional word $t$ ($\text{sh}(t)$) iff

$$m_j = \sup\{i_j \in \mathbb{N} : \exists(i_1 \ldots i_j \ldots i_n) \in \text{Dom}(t)\} + 1$$

for every $1 \leq j \leq n$

We assume the $\text{sh}(\lambda_n) = (0, \ldots, 0)$.

Let us consider two arbitrary n-dimensional words $u$ and $v$ of $\Sigma^*_n$ and let

$$P = \{(i_1, \ldots, i_n) : \exists(i'_1, \ldots, i'_n) \in \text{Dom}(v) \exists(k_1, \ldots, k_n) \in \mathbb{N}^n \forall(j \leq n)[i_j = i'_j + k_j]\}.$$

Then a restricted function $v|_P$ is said to be \textit{an occurrence} of $u$ in $v$. 
A restricted function \( v\mid_P \) is said to be the first occurrence of \( u \) in \( v \) with respect to the order of axes \( x_{k_1}, \ldots, x_{k_n} \) iff the following conditions are satisfied\(^1\):

1. \((k_1, \ldots, k_n)\) is any permutation of \((1, \ldots, n)\);
2. \( v\mid_P \) is an occurrence of \( u \) in \( v \);
3. For every \( P' \subset \text{Dom}(v) \) such that \( v\mid_{P'} \) is an occurrence of \( u \) in \( v \) there exists \((i_1, \ldots, i_n) \in P\) such that for every \((i'_1, \ldots, i'_n) \in P\) we have:
   \[ i_{k_1} < i'_{k_1} \]
   or if there exists \( m \leq n \) such that
   \[ i_{k_j} = i'_{k_j}, \text{ for all } 1 \leq j < m, \text{ then } i_{k_m} < i'_{k_m}. \]

**Example 2.1.** Let us consider the 2-dimensional word \( v \) in the following form:

\[
\begin{array}{ccc}
  a & b & a \\
  a & b & a \\
  a & b & a \\
\end{array}
\]

Then the word \( u \) such that \( u(0, 0) = a, u(1, 0) = b, u(1, 1) = a \) has three occurrences in the word \( v \). Namely, the restricted sequences \( v\mid_P \), \( v\mid_{P'} \), and \( v\mid_{P''} \) are the occurrences of \( u \) in \( v \), where \( P = \{(1, 1), (2, 1), (2, 2)\}, P' = \{(2, 2), (3, 2), (3, 3)\} \) and \( P'' = \{(3, 0), (4, 0), (4, 1)\} \).

\( v\mid_P \) is the first occurrence of \( u \) in \( v \) with respect to the order of axes \((x_1, x_2)\) but \( v\mid_{P'} \) is the first occurrence of \( u \) in \( v \) with respect to the order of axes \((x_2, x_1)\).

Now let us define the concatenation of \( n \)-dimensional words \( u \) and \( v \) of shapes \((p_1, \ldots, p_n)\) and \((q_1, \ldots, q_n)\), respectively.

By a concatenation \( u \circ_j v \) of the words \( u \) and \( v \) in the direction of \( j \)-th axis we mean a \( n \)-dimensional word \( w \) which is defined as follows:

4. \( sh(w) = (\max(p_1, q_1), \ldots, p_j + q_j, \ldots, \max(p_n, q_n)) \);
5. For every \((i_1, \ldots, i_j, \ldots, i_n) \in \text{Dom}(u)\) we have \((i_1, \ldots, i_j, \ldots, i_n) \in \text{Dom}(w)\) and \( u_{i_1, \ldots, i_j, \ldots, i_n} = w_{i_1, \ldots, i_j, \ldots, i_n} \);
6. For every \((s_1, \ldots, s_j, \ldots, s_n) \in \text{Dom}(v)\) we have \((s_1, \ldots, s_j + p_j, \ldots, s_n) \in \text{Dom}(w)\) and \( v_{s_1, \ldots, s_j, \ldots, s_n} = w_{s_1, \ldots, s_j + p_j, \ldots, s_n} \).

\(^1\)The axes of the \( n \)-dimensional Cartesian space \( N^n \) will be denoted by \( x_1, \ldots, x_n \)

\(^2\)We assume the convention that all \( n \)-dimensional words and productions will be written without axes.
Example 2.2. Let us consider the 2-dimensional word \( v \) of Example 2.1. Then the concatenations \( u_1 = v \circ_1 v \) and \( u_2 = v \circ_2 v \) have the forms:

\[
\begin{array}{cccccc}
  a & a \\
  b & a & b & a & b & a \\
  a & b & a & b & a & b & a & b & a \\
  b & a & b & a & b & a & b & a & b
\end{array}
\]

\[
\begin{array}{cccc}
  a & a \\
  b & a & b & a & b & a & b & a \\
  a & b & a & b & a & b & a & b & a & b & a & b & a & b & a & b
\end{array}
\]

Now let us introduce a notion of \( n \)-dimensional production in an alphabet \( \Sigma \).

By an \( n \)-dimensional production in an alphabet \( \Sigma \) we mean a pair \((x, y)\) of \( n \)-dimensional words \( x, y \) in an alphabet \( \Sigma \), \((m_1, \ldots, m_n)\) is a shape of \( x \) and \((p_1, \ldots, p_n)\) is a shape of \( y \), with the properties:

\[
\text{Dom}(x) = \{(i_1, \ldots, i_n) : 0 \leq i_k \leq m_k, 1 \leq k \leq n\}
\]

\[
\text{Dom}(y) = \{(j_1, \ldots, j_n) : 0 \leq j_k \leq p_k, 1 \leq k \leq n\}
\]

and \( m_k \leq p_k \) for all \( 1 \leq k \leq n \) or \( m_k \geq p_k \) for all \( 1 \leq k \leq n \).

The above conditions mean that we consider only such productions, whose left-hand side word \( x \) and right-hand side word \( y \) are "full" \( n \)-dimensional cubes and domain of one word contains the domain of the second one.

The set of all \( n \)-dimensional productions in an alphabet \( \Sigma \) will be denoted by \( \mathcal{P}_n^\Sigma \). Let us distinguish a nonempty subset \( \mathcal{P}_n^{\Sigma \cup \{\delta\}} \) whose elements are called final whereas of \( \mathcal{P}_n^{\Sigma \cup \{\delta\}} \)-nonfinal ones.

As in Markov’s monograph [8] the elements of \( \mathcal{P}_n^{\Sigma \cup \{\delta\}} \) will be denoted by \( x \rightarrow y \) (possibly with subscripts) while of \( \mathcal{P}_n^{\Sigma \cup \{\delta\}} \)-nonfinal ones.

Now let us define an extending function \( \text{ex}_\delta : \Sigma_n^* \rightarrow \{\Sigma \cup \delta\}_n^*, \delta \notin \Sigma \) as follows:

for arbitrary words \( u \in \Sigma_n^* \), \( v \in \{\Sigma \cup \delta\}_n^* \) we have:

\[
\text{ex}_\delta(u) = v
\]

iff the following conditions hold:

(7) if \((i_1, \ldots, i_n) \in \text{Dom}(u)\) then \( v_{i_1, \ldots, i_n} = u_{i_1, \ldots, i_n} \);

(8) if \((i_1, \ldots, i_n) \notin \text{Dom}(u)\) then \( v_{i_1, \ldots, i_n} = \delta \) for all \((i_1, \ldots, i_n) : 0 \leq i_k \leq m_k, 1 \leq k \leq n, (m_1, \ldots, m_n) \) is a shape of \( u \).
Now let us define a cutting function $ct_\delta : \{\Sigma \cup \delta\}^*_n \rightarrow \Sigma^*_n$ as follows: for arbitrary words $v \in \Sigma^*_n, u \in \{\Sigma \cup \delta\}^*_n$ we have:

$$ct_\delta(u) = v$$

iff the following conditions hold:

1. if $(i_1, \ldots, i_n) \in \text{Dom}(u)$ and $u(i_1, \ldots, i_n) \neq \delta$ then $u(i_1, \ldots, i_n) = v(i_1, \ldots, i_n)$;
2. if $(i_1, \ldots, i_n) \in \text{Dom}(u)$ and $u(i_1, \ldots, i_n) = \delta$ then $(i_1, \ldots, i_n) \notin \text{Dom}(v)$, for all $(i_1, \ldots, i_n) \in \text{Dom}(u)$.

Example 2.3. Let us consider the 2-dimensional word

$$t = \begin{array}{cccc}
  b & a & b & a \\
  a & b & a & b \\
  b & a & a & b
\end{array}$$

Then $ex_\delta(t) = \begin{array}{cccc}
  \delta & a & b & a \\
  \delta & \delta & b & a \\
  b & a & a & b
\end{array}$ and $ct_\delta(ex_\delta(t)) = t$.

Now let us define a resulting function $\text{Res}^{k_1 \cdots k_n}_n : \mathcal{P}_\Sigma^n \times \Sigma^*_n \rightarrow \Sigma^*_n$ as follows:

For arbitrary $n$-dimensional production $x \rightarrow (\cdot)y \in \mathcal{P}_\Sigma^n$ and a $n$-dimensional word $t \in \Sigma^*_n$ we have:

$$\text{Res}^{k_1 \cdots k_n}_n(x \rightarrow (\cdot)y, t) = \begin{cases} 
  u & \text{if } x \text{ occurs in } t \\
  t & \text{otherwise,}
\end{cases}$$

where $u$ is a $n$-dimensional word of $\Sigma^*_n$ which is obtained from $t$ in such a way that

$$u = ct_\delta(t_{k_1}^p \circ_{k_1} t_{k_2}^p \circ_{k_2} \ldots t_{k_n}^p \circ_{k_n} y \circ_{k_n} t_{k_n}^s \circ_{k_n} \ldots \circ_{k_1} t_{k_1}^s)$$

where

$$t' = ex_\delta(t)$$

$$t' = t_{k_1}^p \circ_{k_1} t_{k_2}^p \circ_{k_2} \ldots t_{k_n}^p \circ_{k_n} t_{k_n}^s \circ_{k_n} \ldots \circ_{k_1} t_{k_1}^s,$$

$t_{k_i}^p, t_{k_i}^s$ for all $1 \leq i \leq n$ are maximal "prefix" and "suffix" subwords and $t_{k_i}^P$ is the first occurrence of $x$ in $t$ with respect to the order of the axes $(x_{k_1}, \ldots, x_{k_n})$, $\delta \notin \Sigma$.

We can distinguish in the above construction of $u$ a few steps:

1. extending the word $t$ to the "full" $n$-dimensional cube $t'$;
(2) distinguishing all \( t_{k_1}^{p}, t_{k_1}^{s} \), which are "prefix" and "suffix" subwords of \( t' \) in all \( n \) dimensions surrounding the occurrence of \( x \) in \( t' \);
(3) replacing the occurrence \( x \) by \( y \);
(4) creating \( u \) by cutting all additional symbols in the word which is result of step (3).

In the first case (if \( x \) occurs in \( t \)) the production is said to be \textit{effectively used}, whereas in the second one \textit{noneffectively used} to a word \( t \).

**Example 2.4.** Let us present here an example of action of the function \( \text{Res}_{k_1 \ldots k_n}^n : P^n_\Sigma \times \Sigma^n \rightarrow \Sigma^n \). Let us consider the 2-dimensional word \( t \) from Example 2.3 and the production:

\[
P : \begin{array}{ccc}
a & b & \rightarrow & c & c & c \\
a & b & \rightarrow & c & c & c \end{array}
\]

Then we have

\[
t_{1}^{p} = \delta \ a \ b \\
t_{1}^{s} = \delta \ b \\
t_{2}^{p} = a \\
t_{2}^{s} = b \ a \ b \ a \ a \ a \\
t_{2}^{p} = b \ a \ a \ b \ \delta \ \delta
\]

Hence

\[
\text{Res}_{2}^{2,1}(P,t) = ct_{\delta}(t_{2}^{p} \circ_{2} t_{1}^{p} \circ_{1} y \circ_{1} t_{1}^{s} \circ_{2} t_{2}^{s}) = \begin{array}{cccccc}
b & a & b & a & a & a \\
c & c & c \end{array}
\]

3. **The \( n \)-dimensional Markov-like algorithms.** New class \( MA_{k_1 \ldots k_n}^{n}, n \geq 1 \) of \( n \)-dimensional Markov-like with respect to the order \((x_{k_1}, \ldots, x_{k_n})\) of the axes will be introduced.

By a \textit{n-dimensional Markov-like algorithm} of the class \( MA_{k_1 \ldots k_n}^{n} \) (the order of the axes \( x_{k_1}, \ldots, x_{k_n} \) is fixed) in an alphabet \( \Sigma \) we mean a sixtuple

\[
A = (P^n_\Sigma, L, L_i, L_f, \text{Contr}^{k_1 \ldots k_n}, \text{Tr}^{k_1 \ldots k_n}),
\]

where

\( P^n_\Sigma \) is a finite (nonempty) subset of \( P_\Sigma \) of \( n \)-dimensional productions in an alphabet \( \Sigma \).
L is a set of labels of $P^0_n$ (we assume $L = \{1, \ldots, |P_\Sigma|\}$; 
$L_i = \{1\}$ and $L_f$ are the subsets of $L$ whose elements are called \textit{initial} and \textit{final} labels, respectively\(^3\).

A partial function $\text{Contr}^{k_1, \ldots, k_n} : \Sigma^*_n \times L \rightarrow L$, called a \textit{control} of $A$, and a total function $\text{Tr}^{k_1, \ldots, k_n} : \Sigma^*_n \times L \rightarrow \Sigma^*_n$, called a \textit{transformation} of $A$, are defined as follows:

For arbitrary $n$-dimensional words $t, u \in \Sigma^*_n$ and $i \in L$ we have\(^4\): 

\[
\text{Contr}^{k_1, \ldots, k_n}(t, i) = \begin{cases} 
1 & \text{if } x_i \text{ occurs in } t \text{ and } i \notin L_f \\
i + 1 & \text{if } x_i \text{ doesn't occur in } t \text{ and } i \leq |L| \\
\text{undefined} & \text{if } x_i \text{ occurs in } t \text{ and } i \in L_f \\
& \text{or } x_i \text{ doesn't occur in } t \text{ and } i = |L|, 
\end{cases}
\]

\[
\text{Tr}^{k_1, \ldots, k_n}(t, i) = \begin{cases} 
\text{Res}^{k_1, \ldots, k_n}(x_i \rightarrow (\cdot)y_i,t) & \text{if } x_i \text{ occurs in } t \\
t & \text{otherwise.}
\end{cases}
\]

We denote some production from $P^0_n$ by $P_i$ iff $\phi(i) = P_i$, where $\phi$ is one-to-one mapping of $P^0_n$ onto $L$.

Thus if a production $P_i$ has been effectively used to a word $t$ and it is nonfinal then $\text{Contr}^{k_1, \ldots, k_n}(t, i) = 1$ or if $x_i$ doesn’t occur in $t$ and $i < |L|$ then $\text{Contr}^{k_1, \ldots, k_n}(t, i) = i + 1$. $\text{Contr}^{k_1, \ldots, k_n}$ is undefined if a production $P_i$ has been effectively used to a word $t$ and it is final ($i \in L_f$) or if $x_i$ doesn’t occur in $t$ and $i = |L|$.

If a production $P_i : x_i \rightarrow (\cdot)y_i$ has been effectively used to a word $t$ then $\text{Tr}^{k_1, \ldots, k_n}$ transforms a word $t$ in a such way that the first occurrence with respect to the order $x_{k_1}, \ldots, x_{k_n}$ of the axes of $x_i$ of a production $P_i$ in $t$ is replaced by $y_i$. If a production $P_i$ has been noneffectively used to a word $t$ then $\text{Tr}^{k_1, \ldots, k_n}(t, i) = t$ and we continue a computation in every case (if $P_i$ is final or nonfinal) with such only a restriction that $i < |L|$. Let us observe that the process stops iff the effectively used lately production is final or if the used lately production $P_j$ has been noneffectively used and $j = |L|$.

In some examples there is a need to introduce some additional letters (separators, parenthesis). This leads to the following definition.

A $n$-dimensional Markov-like algorithm is said to be over an alphabet $\Sigma$ iff it is an algorithm in some alphabet $\Sigma'$ such that $\Sigma \subset \Sigma'$.

Now let us introduce a notion of a computation of a $n$-dimensional Markov-like algorithm.

A sequence $T = t_1, t_2, \ldots \in (\Sigma^*_n)^\omega$ (finite or infinite) is said to be a \textit{computation} of a $n$-dimensional algorithm $A = (P^0_\Sigma, L, L_i, L_f, \text{Contr}^{k_1, \ldots, k_n},$
$T_{r_{n}^{k_{1}, \ldots, k_{n}}}$ of the class $\mathcal{MA}_{n}^{k_{1}, \ldots, k_{n}}$ iff there exists a sequence $I = i_{1}, i_{2}, \ldots \in L_{\infty}$, called a trace of $T$, such that the following conditions hold:

1. Both sequences $T$ and $I$ are infinite, $i_{1} \in L_{i}$, and for every $j \geq 1$ we have: $t_{j+1} = T_{r_{n}^{k_{1}, \ldots, k_{n}}}(t_{j}, i_{j})$ and $i_{j+1} = C_{n}^{k_{1}, \ldots, k_{n}}(t_{j}, i_{j})$.

2. Both sequences $T$ and $I$ are finite of lengths equal to $m$ for some $m > 1$, $i_{1} \in L_{i}$ and for every $1 \leq j < m$ we have: $t_{j+1} = T_{r_{n}^{k_{1}, \ldots, k_{n}}}(t_{j}, i_{j})$ and $i_{j+1} = C_{n}^{k_{1}, \ldots, k_{n}}(t_{j}, i_{j})$. We additionally assume that $i_{m} = |L| + 1$ and this label indicates the fact that a computation $T$ stops.

Two cases imply that $T$ is finite. The first one is when a production $P_{m-1}^{i_{m-1}}$ with the label $i_{m-1}$ has been effectively used to a word $t_{m-1}$ and it is final or if $P_{m-1}^{i_{m-1}}$ has been noneffectively used to a word $t_{m-1}$ and $i_{m-1} = |L|$.

The set of all computations of a $n$-dimensional Markov-like algorithm $A$ is said to be its computation set and denoted by $C(A)$.

As for all algorithms of $\mathcal{MA}_{n}^{k_{1}, \ldots, k_{n}}$ the control $C_{n}^{k_{1}, \ldots, k_{n}}$ and transformation $T_{r_{n}^{k_{1}, \ldots, k_{n}}}$ are defined in the same manner therefore to define an algorithm $A \in \mathcal{MA}_{n}^{k_{1}, \ldots, k_{n}}$ it is sufficient to define a sequence of productions in (or over) an alphabet $\Sigma$ by omitting their labels (such a sequence will be called a schema of productions).

Example 3.1. Let us consider the 2-dimensional algorithms $A \in \mathcal{MA}_{1}^{1,2}$ and $A' \in \mathcal{MA}_{2}^{2,1}$ in the alphabet $\Sigma = \{a, b, c, d\}$ with the same schema of productions:

$$
P_{1} : 
\begin{array}{c}
d \ c \\
d \ c \\
\end{array} 
\rightarrow 
\begin{array}{c}
a \ a \\
a \ a \\
\end{array}
$$

$$
P_{2} : 
\begin{array}{c}
c \ c \\
d \ d \\
\end{array} 
\rightarrow 
\begin{array}{c}
a \ a \\
a \ a \\
\end{array}
$$

$$
P_{3} : 
\begin{array}{c}
c \ c \\
a \ b \\
\end{array} 
\rightarrow 
\begin{array}{c}
d \ d \\
d \ d \\
\end{array}
$$

Let us consider the word $v$ of the following form:

$\begin{array}{c}
c \ c \ c \ c \\
a \ b \ c \ c \\
\end{array} 
\begin{array}{c}
d \ d \ c \ c \\
d \ d \ c \ c \\
\end{array} 
\begin{array}{c}
da \ a \ c \\
da \ a \ c \\
\end{array}$

Let us consider the word $v$ of the following form:

$\begin{array}{c}
c \ c \ c \ c \\
a \ b \ c \ c \\
\end{array} 
\begin{array}{c}
d \ d \ c \ c \\
d \ d \ c \ c \\
\end{array} 
\begin{array}{c}
da \ a \ c \\
da \ a \ c \\
\end{array}$

Then the computation of $A$ with the initial word $v$ has the form:

$\begin{array}{c}
c \ c \ c \ c \\
a \ b \ c \ c \\
\end{array} 
\begin{array}{c}
d \ d \ c \ c \\
d \ d \ c \ c \\
\end{array} 
\begin{array}{c}
da \ a \ c \\
da \ a \ c \\
\end{array}$

This is a screenshot of a page from a document, containing text in natural language. The document discusses the concept of traces in the context of Markov-like algorithms, defining conditions under which a computation is finite and providing an example to illustrate these conditions. The example involves two-dimensional algorithms defined over a specific alphabet with a given schema of productions.
The computation of $A'$ with the same initial word $v$ has the form:

\[
\begin{align*}
&c\ c\ c\ c\ c\ c\ a\ a\ d\ d\ a\ a \\
a\ b\ c\ c\ a\ b\ d\ d\ a\ b\ a\ a\ d\ d\ a\ a
\end{align*}
\]

4. The permutation $n$-dimensional words and algorithms. Let us give at the beginning the necessary definitions connected with permutations of axis of $n$-dimensional words and productions.

A word $v'$ of $\Sigma_n^*$ will be called an $(j_1, \ldots, j_n)$-permutation word of the word $v \in \Sigma_n^*$ (and denoted $v^{j_1, \ldots, j_n}$) iff the following conditions are satisfied:

(1) for all $(i_1, \ldots, i_n) \in \text{Dom}(v)$ we have $(i_{j_1}, \ldots, i_{j_n}) \in \text{Dom}(v')$ and $v_{i_1, \ldots, i_n} = v'_{i_{j_1}, \ldots, i_{j_n}}$;

(2) for all $(i_{j_1}, \ldots, i_{j_n}) \in \text{Dom}(v')$ we have $(i_1, \ldots, i_n) \in \text{Dom}(v)$ and $v_{i_1, \ldots, i_n} = v'_{i_{j_1}, \ldots, i_{j_n}}$;

Example 4.1. Let us consider the 2-dimensional word $v$ from Example 3.1. Then the word $v'$ which is $(2,1)$-permutations word of $v$ has the form:

\[
\begin{align*}
v' &= b\ c\ c \\
&\ a\ c\ c \\
&\ b\ c \\
&\ a\ c
\end{align*}
\]

Let $(x, y)$ be an arbitrary $n$-dimensional production and let $(j_1, \ldots, j_n)$ be a permutation of $(1, \ldots, n)$.

Then a production $P' = (x', y')$ is said to be a $n$-dimensional $(j_1, \ldots, j_n)$-permutation production in an alphabet $\Sigma$ of the production $P = (x, y)$ in an alphabet $\Sigma$ ($P'$ is denoted as $P^{j_1, \ldots, j_n}$) iff $x' = x^{j_1, \ldots, j_n}$ and $y' = y^{j_1, \ldots, j_n}$.

A $n$-dimensional algorithm $A' \in \mathcal{MA}_n^{k_1, \ldots, k_n}$ in an alphabet $\Sigma$ is said to be a $n$-dimensional $(j_1, \ldots, j_n)$-permutation algorithm of the algorithm $A \in \mathcal{MA}_n^{k_1, \ldots, k_n}$ in an alphabet $\Sigma$ iff $L = L'$, $L_i = L'_i$, $L_f = L'_f$, $|P^\Sigma_n| = |P'^\Sigma_n|$ and $P'_i = P^{j_1, \ldots, j_n}_i, 1 \leq i \leq |P^\Sigma_n|$ (Contr and Tr of $A'$ are the same as for the whole class of algorithms $\mathcal{MA}_n^{k_1, \ldots, k_n}$).

Example 4.2. Let us consider the algorithm $A \in \mathcal{MA}_n^{1,2}$ from the Example 3.1. Then the schema of productions of 2, 1-permutation algorithm $A' \in \mathcal{MA}_2^{2,1}$ in the alphabet $\Sigma = \{a, b, c, d\}$ has the form:

\[
\begin{align*}
P'_1 : &\ c\ c \quad \rightarrow \quad a\ a \\
&\ d\ d \quad \rightarrow \quad a\ a \\
P'_2 : &\ d\ c \quad \rightarrow \quad a\ a \\
&\ d\ c \quad \rightarrow \quad a\ a
\end{align*}
\]
\[ P_3': \begin{align*} & b \ c \ c \ \rightarrow \ d \ d \ d \\ & a \ c \ c \ \rightarrow \ d \ d \ d \ . \end{align*} \]

Then computation of \( A' \) for the word \( v' \) from Example 4.1 has the form:

\[
\begin{align*}
& b \ c \ c \ b \ c \ c \\
& a \ c \ c \ a \ c \ a \\
& b \ c \ c \ d \ d \\
& a \ c \ c \ d \ d \\
& a \ a \ a \ d \ d \\
& a \ a \ a \ d \ d
\end{align*}
\]

Let us point that the computation of \( A' \) for the initial word \( v' \) is "symmetrical" to the computation of \( A \) for the initial word \( v \).

So, we can give here the following lemma.

**Lemma 4.4.** For every \( A \in \mathcal{M}A_n^{k_1,...,k_n} \) and the word \( v \in \Sigma_\infty^n \) we have

\[ A'(v^{j_1,...,j_n}) = [A(v)]^{j_1,...,j_n} \]

where \( (j_1,...,j_n) \) is a permutation of \( (1,...,n) \) and \( A' \in \mathcal{M}A_n^{k_1,...,k_j} \) is an permutation algorithm of \( A \).

Proof is obvious.

5. The equivalence of the classes \( \mathcal{M}A_n^{k_1,...,k_j} \) and \( M\mathcal{N}A \). We will use in this section the notion of a representation of \( n \)-dimensional word \( t \) by \( n-1 \) dimensional word \( u \). Intuitively, to create a representation of a given word \( t \in \Sigma_n \), we will place \( t \) in a \( n \)-dimensional cube \( v \) whereas in empty places any element outside the alphabet \( \Sigma \) will be located. Then a cube covering a word \( t \) is cut into \((n-1)\)-dimensional layers with respect to direction of the \( x_{kn} \) axis. The representation of a word \( t \) will be equal to the concatenation \( v = \theta_1v_1,\ldots,\theta_nv_n \) (\( v_i \) is the \( i \)-th layer of \( v \), \( \theta_i \) is \((n-1)\)-dimensional separators).

Of course, a notion of representation can be inductively extended to representation of \( n \)-dimensional word by \( 1 \)-dimensional word.

We can also represent \( n-1 \)-dimensional word \( t \) by \( n \)-dimensional word \( u \) by extending word \( t \) to \( u \) in such a way that the \((n-1)\)-dimensional layer of \( u \) with respect to direction of the \( x_{kn} \) axis on the zero coordinate is equal \( t \) and other layers of \( u \) with respect to \( x_{kn} \) are empty.

Now we can say about equivalence of two classes \( A_1, A_2 \) of \( n \)-dimensional algorithms and \( m \)-dimensional algorithms, when:

1) for every algorithm \( A_1 \in A_1 \) there exists \( A_2 \in A_2 \) such that for each \( n \)-dimensional word \( t \) the \( m \)-dimensional representant of \( A_1(t) \) is equal to the value of \( A_2 \) for \( m \)-dimensional representant of \( t \)

and

2) for every algorithm \( A_2 \in A_2 \) there exists \( A_1 \in A_1 \) such that for each \( m \)-dimensional word \( u \) the \( n \)-dimensional representant of \( A_2(u) \) is equal to the value of \( A_1 \) for \( n \)-dimensional representant of \( u \).
Theorem 5.1. For every order \(x_{k_1}, \ldots, x_{k_n}\) of the axes the classes \(\mathcal{MA}_{k_1}^{k_1}, \ldots, \mathcal{MA}_{k_n}^{k_n}\) and \(\mathcal{MNA}\) (where \(\mathcal{MNA}\) denotes the class of Markov normal algorithms) are equivalent.

We shall give only a short outline of the proof of a lemma relating to the equivalence of the classes \(\mathcal{MA}_{k_1}^{k_1}, \ldots, \mathcal{MA}_{k_n}^{k_n}\) and \(\mathcal{MA}_{k_1}^{k_1}, \ldots, \mathcal{MA}_{k_n-1}^{k_n-1}\) of Markov-like algorithms, which is a main part in the inductive proof of Theorem 5.1.

This proof is supported of two lemmas.

Lemma 5.2. For arbitrary algorithm \(A \in \mathcal{MA}_{k_1}^{k_1}, \ldots, \mathcal{MA}_{k_n}^{k_n}\) in an alphabet \(\Sigma\) there exists an algorithm \(B \in \mathcal{MA}_{k_1}^{k_1}, \ldots, \mathcal{MA}_{k_n-1}^{k_n-1}\) over an alphabet \(\Sigma\) such that \(A(v) = B(\overline{v})\), for every \(v \in \Sigma_n^*\) where \(\overline{v}\) is a representation of \(n\)-dimensional word \(v \in \Sigma_n^*\) in \((n-1)\)-dimensional word of \(\Sigma_{n-1}^*\). Analogously \(B(v)\) denotes a representation of \(n\)-dimensional word \(B(v)\) in \((n-1)\)-dimensional word of \(\Sigma_{n-1}^*\).

Lemma 5.3. For arbitrary algorithm \(B \in \mathcal{MA}_{k_1}^{k_1}, \ldots, \mathcal{MA}_{k_n-1}^{k_n-1}\) in an alphabet \(\Sigma\) there exists an algorithm \(A \in \mathcal{MA}_{k_1}^{k_1}, \ldots, \mathcal{MA}_{k_n}^{k_n}\) over an alphabet \(\Sigma\) such that \(B(v') = A(v')\) for every \(v' \in \Sigma_{n-1}^*\).

The Lemma 5.3 can be easily proved by transformation of every word \(v' \in \Sigma_{n-1}^*\) into a word \(w \in \Sigma_n^*\) by adding a new \(x_{k_n}\) coordinate and by extending word \(v\) to \(v'\) by locating \(v\) on the zero coordinate of the new \(x_{k_n}\) axis.

We proceed analogously with \((n-1)\)-dimensional productions.

The proof of Lemma 5.2 is more complicated. Two problems should be solved:

1. Representation of all \(n\)-dimensional words of \(\Sigma_n^*\) by means of \((n-1)\)-dimensional words over \(\Sigma_{n-1}^*\), and analogously we follow \(n\)-dimensional productions;

2. We have to transform a schema of \(n\)-dimensional productions into a schema of \((n-1)\)-dimensional productions.

The problem (1) can be solved in the following way. A given word \(t \in \Sigma_n^*\) is placed in a \(n\)-dimensional cube whereas in free places any element outside the alphabet \(\Sigma\) is located. Then a cube covering a word \(t\) is cut into \((n-1)\)-dimensional layers with respect to direction of the \(x_{k_n}\) axis. Then we assign to a pair of \(n\)-dimensional cubes corresponding to \(n\)-dimensional production \(\mathcal{P}_i : x_i \rightarrow (\cdot)y_i\) the sequence of \((n-1)\)-dimensional cubes corresponding to the successive layers. The transformed \(n\)-dimensional word \(t\) should be replaced by a concatenation \(\overline{v} = \theta_1 v_1, \ldots, \theta_n v_n\) (\(v_i\) is the \(i\)-th layer of \(v\), \(\theta_i\)-\((n-1)\)-dimensional separators).
The problem (2) can be solved by adding to productions of a sequence $\mathcal{P}_i (1 \leq i \leq m)$ new symbols outside $\Sigma$ and some additional productions transforming these additional symbols such as following conditions hold:

1. If a sequence $\mathcal{P}_i$ has been effectively used to a transformed word $t$ then we must return to first element of $\mathcal{P}_1$ if $i \notin L_F$ or an algorithm stops if $i \in L_F$;
2. If a sequence $\mathcal{P}_i$ corresponding to a production $\mathcal{P}_i$ has not been effectively used to a transformed word then we have to go to $\mathcal{P}_{i+1}$ (if $i < m$).

Let us add that the complete proof of theorem relating to the equivalence of 2-dimensional Markov-like algorithms and Markov normal algorithms has been given in [6].

**Open problems.** Let us put forward some open problems relating to $n$-dimensional algorithms:

1. One is able to define classes $\mathcal{M}_{A_{n}^{k_{1}, \ldots, k_{n}}}$ of $n$-dimensional Markov-like $j$-algorithms for which the $j$-th left-hand side occurrence of the left side of the productions in the transformed words is replaced by the right side of the respective productions (taking into account some order of axes);
2. A class of $n$-dimensional weighed Markov-like algorithms can be introduced that to every production $\mathcal{P}_i$ a weight $w_i$ is assigned, indicating that the $w_i$-occurrence of the left-hand side of productions with respect to some order of axes is replaced by the right-hand side of $\mathcal{P}_i$.
3. One is able to introduce other classes of $n$-dimensional algorithms (not necessarily Markov-like) only insignificantly modifying the transformation and control functions.
4. There is a need to study different aspects of complexity of $n$-dimensional algorithms.
5. By analogy with $n$-dimensional algorithms one is able to define $n$-dimensional word recursive functions.
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