Subordination chains and Loewner differential equations in several complex variables

Abstract. Let $B$ be the unit ball of $\mathbb{C}^n$ with respect to the Euclidean norm and $f(z, t)$ be a Loewner chain. In this paper we study certain properties of $f(z, t)$ and we obtain a sufficient condition for the transition mapping associated to $f(z, t)$ to satisfy the Loewner differential equation.

1. Introduction and preliminaries. Let $\mathbb{C}^n$ be the space of $n$ complex variables $z = (z_1, \ldots, z_n)'$ with the usual inner product $\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w_j}$ and the Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$, $z \in \mathbb{C}^n$. The symbol $'$ means transpose of vectors and matrices. Let $B_r = \{z \in \mathbb{C}^n : \|z\| < r\}$ and let $B = B_1$ be the unit ball in $\mathbb{C}^n$. The closed ball $\{z \in \mathbb{C}^n : \|z\| \leq r\}$ is denoted by $\overline{B_r}$. In the case of one variable $B_r$ is denoted by $U_r$ and the unit disc $U_1$ by $U$. If $G$ is an open set in $\mathbb{C}^n$, let $H(G)$ be the set of holomorphic maps from $G$ into $\mathbb{C}^n$.

By $L(\mathbb{C}^n, \mathbb{C}^m)$ we denote the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm. Let $I$ be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$.

2000 Mathematics Subject Classification. 32H, 30C45.

Key words and phrases. Subordination chain, transition mapping, Loewner differential equation.
We say that \( f \in H(B) \) is locally biholomorphic on \( B \) if \( f \) has a local holomorphic inverse at each point in \( B \).

If \( f, g \in H(B) \), we say that \( f \) is subordinate to \( g \) if there is a Schwarz mapping \( v \) (i.e. \( v \in H(B), \ v(0) = 0 \) and \( \|v(z)\| < 1, \ z \in B \)) such that \( f(z) = g(v(z)) \), \( z \in B \). We shall write \( f \prec g \) to mean that \( f \) is subordinate to \( g \).

**Definition 1.1.** The mapping \( f : B \times [0, \infty) \to \mathbb{C}^n \) is called a Loewner chain (or a subordination chain) if

(i) \( f(\cdot, t) \) is holomorphic and univalent on \( B, \ t \geq 0 \);
(ii) \( f(0, t) = 0, \ Df(0, t) = e^tI, \ t \geq 0 \);
(iii) \( f(\cdot, s) \prec f(\cdot, t) \) for \( 0 \leq s \leq t < \infty \);
(iv) \( f(z, t) \) is a locally absolutely continuous function of \( t \in [0, \infty) \) locally uniformly with respect to \( z \in B \).

Note that in the case of one variable, the assumption (iv) is always satisfied, as a consequence of the distortion result for the class of normalized univalent functions on \( U \).

The subordination condition (iii) is equivalent to the condition

\[
(1.1) \quad f(z, s) = f(v(z, s, t), t), \quad z \in B, \ 0 \leq s \leq t < \infty,
\]

where \( v = v(z, s, t) \) is a univalent Schwarz mapping, normalized by \( v(0, s, t) = 0 \) and \( Dv(0, s, t) = e^{s-t}I \).

A key role in our discussion is played by the \( n \)-dimensional version of the Carathéodory set:

\[
\mathcal{M} = \{ h \in H(B) : h(0) = 0, \ Dh(0) = I, \ \text{Re} \langle h(z), z \rangle \geq 0, \ z \in B \}.
\]

Recently the authors proved in [2] the following result.

**Lemma 1.2.** Let \( p \in \mathcal{M} \). Then for each \( r \in (0, 1) \) there is a constant

\[
M = M(r) \leq \frac{4}{(1-r)^2},
\]

which is independent of \( p \), such that \( \|p(z)\| \leq M(r) \) for \( \|z\| \leq r \).

Also in [1] the present authors have recently proved that the transition mapping \( v = v(z, s, t) \) associated to a Loewner chain is locally Lipschitz continuous in \( t \in [s, \infty) \) locally uniformly with respect to \( z \in B \).

**Lemma 1.3.** Let \( f(z, t) \) be a Loewner chain and \( v = v(z, s, t) \) be the transition mapping associated to \( f(z, t) \). Then for all \( r \in (0, 1) \) and \( 0 \leq s \leq t_1 < t_2 < \infty \),

\[
(1.2) \quad \|v(z, s, t_1) - v(z, s, t_2)\| \leq \frac{4}{(1-r)^2}(t_2 - t_1), \quad \|z\| \leq r.
\]
Also for all \( r \in (0, 1) \) and \( 0 \leq s_1 < s_2 \leq t < \infty \),

\[
\|v(z, s_1, t) - v(z, s_2, t)\| \leq \frac{4(1 + r)}{(1 - r)^3}(s_2 - s_1), \quad \|z\| \leq r.
\]

2. Main results. We begin this section with the following result.

**Lemma 2.1.** Let \( f(z, t) = e^t z + \ldots \) be a Loewner chain. Then for each \( r \in (0, 1) \) and \( t_0 > 0 \) there is \( K = K(r, t_0) > 0 \), such that

\[
\|f(z, t) - f(z, s)\| \leq K(r, t_0)(t - s), \quad \|z\| \leq r, \quad 0 \leq s \leq t \leq t_0.
\]

Thus \( f(z, t) \) is locally Lipschitz in \( t \), locally uniformly with respect to \( z \in B \).

**Proof.** It is easy to see that \( f \) is a continuous mapping on \( B \times [0, \infty) \), since \( f(z, \cdot) \) is locally absolutely continuous function of \( t \in [0, \infty) \) locally uniformly with respect to \( z \in B \). Hence, for each \( r \in (0, 1) \) and \( t_0 > 0 \) there exists \( M = M(r, t_0) > 0 \) such that

\[
\|f(z, t)\| \leq M(r, t_0), \quad \|z\| \leq r, \quad t \in [0, t_0].
\]

On the other hand, using the Cauchy integral formula, it is not difficult to prove that there exists \( L = L(r, t_0) > 0 \) such that

\[
\|Df(z, t)(u)\| \leq L(r, t_0), \quad \|z\| \leq r, \quad t \in [0, t_0].
\]

Indeed,

\[
Df(z, t)(u) = \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f(z + \zeta u, t)}{\zeta^2} d\zeta,
\]

for all \( u \in \mathbb{C}^n \), \( \|u\| = 1 \) and \( \rho \in (0, 1) \), such that \( z + \zeta u \in B \), \( |\zeta| = \rho \). For example, if \( \rho = \frac{1 - r}{2} \), then \( \|z + \zeta u\| \leq \frac{1 + r}{2} < 1 \) and the above equality yields that

\[
\|Df(z, t)(u)\| \leq M\left(\frac{1 + r}{2}, t_0\right) \frac{2}{1 - r} = L(r, t_0),
\]

for \( \|z\| \leq r < 1 \), \( \|u\| = 1 \) and \( t \in [0, t_0] \). Thus (2.3) follows. Moreover, since

\[
\|f(z, t) - f(w, t)\| \leq \|z - w\| \int_0^1 \|Df((1 - \tau)z + \tau w, t)\| d\tau,
\]
for \( \|z\| \leq r, \|w\| \leq r \), and \( t \in [0, t_0] \), we obtain in view of (2.3) that
\[
\|f(z, t) - f(w, t)\| \leq L(r, t_0)\|z - w\|, \quad \|z\| \leq r, \|w\| \leq r,
\]
for all \( t \in [0, t_0] \). Taking into account (1.1), (1.2) and the above relation, we deduce that
\[
\|f(z, s) - f(z, t)\| = \|f(v(z, s, t), t) - f(z, t)\|
\leq L(r, t_0)\|z - v(z, s, t)\| \leq \frac{4}{(1 - r)^2} L(r, t_0)(t - s)
\]
\[
= \frac{8}{(1 - r)^3} M \left( \frac{1 + r}{2}, t_0 \right) (t - s),
\]
for \( \|z\| \leq r, 0 \leq s \leq t \leq t_0 \). Here we have used the fact that \( \|v(z, s, t)\| \leq \|z\| \leq r \), since \( v(z, s, t) \) is a Schwarz map. This completes the proof. □

Next, we show the following result.

**Lemma 2.2.** Let \( f(z, t) = e^t z + \ldots \) be a Loewner chain and let \( v = v(z, s, t) \) be the transition mapping associated to \( f(z, t) \). Then the following conditions hold:

(i) There exists a subset \( E \) of \((0, \infty)\) of measure zero such that for each \( t \in (0, \infty) \setminus E \), the limit
\[
(2.4) \quad \frac{\partial f}{\partial t}(z, t) = \lim_{h \to 0} \frac{f(z, t + h) - f(z, t)}{h}
\]
does not exist uniformly on compact sets in \( B \). Moreover, the mapping \( \frac{\partial f}{\partial t}(\cdot, t) \) given by (2.4) is holomorphic on \( B \) for each \( t \in (0, \infty) \setminus E \).

(ii) For each \( s \geq 0 \), there exists a subset \( E' \) of \([s, \infty)\) of measure zero such that for each \( t \in [s, \infty) \setminus E' \), the limit
\[
(2.5) \quad \frac{\partial v}{\partial t}(z, s, t) = \lim_{h \to 0} \frac{v(z, s, t + h) - v(z, s, t)}{h}
\]
does not exist uniformly on compact sets in \( B \). Also, the mapping \( \frac{\partial v}{\partial t}(\cdot, s, t) \) given by (2.5) is holomorphic on \( B \) for each \( t \in [s, \infty) \setminus E' \).

(iii) For each \( t > 0 \), there exists a subset \( E'' \) of \((0, t]\) of measure zero such that for each \( s \in (0, t] \setminus E'' \), the limit
\[
\frac{\partial v}{\partial s}(z, s, t) = \lim_{h \to 0} \frac{v(z, s + h, t) - v(z, s, t)}{h}
\]
exists uniformly on compact sets in $B$. The above mapping $\frac{\partial v}{\partial s}(\cdot, s, t)$ is holomorphic on $B$ for each $s \in (0, t] \setminus E''$.

**Proof.** It suffices to prove the first condition, for the latter and third conditions it would be possible to use similar arguments as to (i).

Since $f(z, t)$ is subordination chain, $f(z, t)$ is a locally absolutely continuous function of $t$ locally uniformly with respect to $z \in B$. Hence the limit
\[
\frac{\partial f}{\partial t}(z, t) = \lim_{h \to 0} \frac{f(z, t + h) - f(z, t)}{h}
\]
exists for a.e. $t \geq 0$. The exceptional null set depends on $z$, but we can choose a set $E \subset (0, \infty)$ of measure zero such that $\frac{\partial f}{\partial t}(z, t)$ exists for all $t \in (0, \infty) \setminus E$ and $z \in Q$, where $Q$ is a countable set of uniqueness for the holomorphic functions on $B$ (for example, any countable dense subset of $B$ may be chosen as $Q$). Next, fix $t \in (0, \infty) \setminus E$. Since $f(z, t)$ is locally Lipschitz in $t$, the set
\[
\left\{ \frac{f(z, t + h) - f(z, t)}{h} \right\}_{0 < |h| < \frac{1}{2}}
\]
is locally uniformly bounded on $B$. In view of Vitali’s theorem for holomorphic functions in higher dimensions [3], we conclude that the limit
\[
\lim_{m \to \infty} \frac{f(z, t + h_m) - f(z, t)}{h_m}
\]
exists uniformly on compact sets for any sequence $\{h_m\}_{m \geq 0}$ such that $\lim_{m \to \infty} h_m = 0$. Since $Q$ is a set of uniqueness and all two such limits coincide on $Q$, (2.4) follows. Finally, since the limit
\[
\frac{\partial f}{\partial t}(z, t) = \lim_{h \to 0} \frac{f(z, t + h) - f(z, t)}{h}
\]
exists uniformly on compact sets in $B$, we deduce that $\frac{\partial f}{\partial t}(\cdot, t)$ is holomorphic on $B$ for $t \in (0, \infty) \setminus E$. $\square$

We are now able to prove that Loewner chains satisfy the generalized Loewner differential equation. A part of this result was also obtained in [2, Theorem 1.10], but here we give another proof.
Theorem 2.3. Let \( f : B \times [0, \infty) \to \mathbb{C}^n \) be a Loewner chain. Then there exists a set \( E \subset (0, \infty) \) of measure zero such that for each \( t \in (0, \infty) \setminus E \), there exists a mapping \( h = h(z, t) \) such that

(i) \( h(\cdot, t) \in \mathcal{M}, \; t \in (0, \infty) \setminus E \),

(ii) \( h(z, \cdot) \) is measurable on \([0, \infty)\) for each \( z \in B \), and

\[
\tag{2.6}
\frac{\partial f}{\partial t}(z, t) = Df(z, t)h(z, t), \quad t \in (0, \infty) \setminus E, \; \forall \; z \in B.
\]

Proof. Let \( v = v(z, s, t) \) be the transition mapping associated to \( f(z, t) \). Then

\[
f(z, s) = f(v(z, s, t), t), \quad z \in B, \; 0 \leq s \leq t < \infty,
\]

and hence

\[
f(z, s) - f(z, t) = f(v(z, s, t), t) - f(z, t) = Df(z, t)(v(z, s, t) - z) + o(v(z, s, t), z),
\]

where \( \frac{o(w, z)}{\|w - z\|} \to 0 \) as \( \|w - z\| \to 0 \).

In view of Lemma 2.2, there is a null set \( E \subset (0, \infty) \) such that \( \frac{\partial f}{\partial t}(\cdot, t) \) exists and is holomorphic on \( B \) for each \( t \in (0, \infty) \setminus E \). For such \( t \in (0, \infty) \setminus E \), we have

\[
\tag{2.7}
[Df(z, t)]^{-1} \frac{\partial f}{\partial t}(z, t) = [Df(z, t)]^{-1} \lim_{s \uparrow t} \frac{f(z, s) - f(z, t)}{s - t} = \lim_{s \uparrow t} \left[ \frac{v(z, s, t) - z}{s - t} - [Df(z, t)]^{-1} o(v(z, s, t), z) \right].
\]

First, we show that for \( t \in (0, \infty) \setminus E \) and \( z \in B \),

\[
\tag{2.8}
\lim_{s \uparrow t} \frac{o(v(z, s, t), z)}{t - s} = 0.
\]

Indeed,

\[
\lim_{s \uparrow t} \frac{o(v(z, s, t), z)}{t - s} = \lim_{s \uparrow t} \frac{o(v(z, s, t), z)}{\|v(z, s, t) - z\|} \cdot \frac{\|v(z, s, t) - z\|}{t - s} = 0,
\]

since \( \frac{o(v(z, s, t), z)}{\|v(z, s, t) - z\|} \to 0 \) and \( \frac{\|v(z, s, t) - z\|}{t - s} \) is bounded in view of (1.2).

Hence, from (2.7) and (2.8) we deduce that for each \( t \in (0, \infty) \setminus E \), the limit

\[
\lim_{s \uparrow t} \frac{z - v(z, s, t)}{t - s}
\]
exists for each \( z \in B \). Further, if

\[
h(z, t) = \lim_{s \to t} \frac{z - v(z, s, t)}{t - s},
\]

for all \( z \in B \) and \( t \in (0, \infty) \setminus E \), then \( h(\cdot, t) \in H(B) \) and in view of the fact that \( v = v(\cdot, s, t) \) is a Schwarz mapping, we obtain

\[
\Re \langle h(z, t), z \rangle = \lim_{s \to t} \frac{1}{t - s} \left[ \| z \|^2 - \Re \langle v(z, s, t), z \rangle \right] \geq 0.
\]

Also, since \( v(0, s, t) = 0 \) and \( Dv(0, s, t) = e^{s-t}I \), it is obvious that \( h(0, t) = 0 \) and \( Dh(0, t) = I \) for \( t \in (0, \infty) \setminus E \). Therefore, \( h(\cdot, t) \in \mathcal{M} \).

Moreover, using (2.7), the equality (2.6) now follows.

Finally, we show that \( h(z, t) \) is a measurable function of \( t \in [0, \infty) \). Indeed, since

\[
h(z, t) = \lim_{m \to \infty} m \left( z - v \left( z, t - \frac{1}{m}, t \right) \right),
\]

it suffices to prove that for each \( m \in \mathbb{N} \), \( v \left( z, t - \frac{1}{m}, t \right) \) is a continuous function of \( t \). For this purpose, we observe that

\[
\| v \left( z, t - \frac{1}{m}, t \right) - v \left( z, \tau - \frac{1}{m}, \tau \right) \| \leq \| v \left( z, t - \frac{1}{m}, t \right) - v \left( z, t - \frac{1}{m}, \tau \right) \| + \| v \left( z, \tau - \frac{1}{m}, \tau \right) - v \left( z, \tau - \frac{1}{m}, \tau \right) \| \leq \frac{8}{(1 - \| z \|)^3} |t - \tau|,
\]

where for the last inequality we have used the relations (1.2) and (1.3). This completes the proof. \( \square \)

In order to prove that the transition mappings associated to Loewner chains satisfy the Loewner differential equation (compare with [5, Theorem 6.3]), we use the lemma below. For the proof, it suffices to use similar kind of arguments as in [6, p.192-193].

**Lemma 2.4.** Let \( f : [a, b] \to \mathbb{C}^n \) given by \( f(t) = g(h(t), t) \), where \( g : B_r \times [a, b] \to \mathbb{C}^n \) and \( h : [a, b] \to B_r \). Assume for each \( t \in [a, b] \), \( g(\cdot, t) \in H(B_r) \) and there exist some constants \( M, K > 0 \) such that

\[
\| g(z, t) - g(w, t) \| \leq M \| z - w \|
\]
and
\[ \|g(z, s) - g(z, t)\| \leq K|s - t| \]
for all \( s, t \in [a, b] \) and \( z, w \in B_r \).

If \( h \) is absolutely continuous, then \( f \) is also absolutely continuous and
\[ \frac{df}{dt}(t) = Dg(h(t), t) \frac{dh}{dt}(t) + \frac{\partial g}{\partial t}(h(t), t), \text{ a.e. } t \in [a, b]. \]

**Theorem 2.5.** Let \( f : B \times [0, \infty) \to \mathbb{C}^n \) be a Loewner chain and let \( v = v(z, s, t) \) be the transition mapping associated to \( f(z, t) \). Also let \( h = h(z, t) \) be given by Theorem 2.3. Then for each \( s \geq 0 \) and for almost all \( t \geq s \),
\[ \frac{\partial v}{\partial t}(z, s, t) = -h(v(z, s, t), t), \forall z \in B. \]

**Proof.** Fix \( s \geq 0 \). In view of Lemmas 2.1 and 2.2 we deduce that for almost all \( t \geq s \) there exist the mappings \( \frac{\partial f}{\partial t}(\cdot, t) \) and \( \frac{\partial v}{\partial t}(\cdot, s, t) \) which are holomorphic on \( B \). Also for any \( \tau > s \) and for almost all \( t \in [s, \tau] \), there exist the mappings \( \frac{\partial v}{\partial t}(\cdot, t, \tau) \) which are holomorphic on \( B \). Moreover,
\[ v(z, s, \tau) = v(v(z, s, t), t, \tau), \quad z \in B, \; 0 \leq s \leq t \leq \tau < \infty, \]
(see for example [1]) and
\[ f(w, t) = f(v(w, s, t), t), \quad w \in B, \; 0 \leq s \leq t < \infty. \]

Differentiating the first equality with respect to \( t \) and the second equality with respect to \( t \) and \( w \), we deduce that
\[ 0 = \frac{\partial v}{\partial t}(z, s, \tau) = Dv(v(z, s, t), t, \tau) \frac{\partial v}{\partial t}(z, s, \tau) + \frac{\partial v}{\partial t}(v(z, s, t), t, \tau), \]
for all \( z \in B \) and a.e. \( t \in [s, \tau] \), and
\[ \frac{\partial f}{\partial t}(w, t) = Df(v(w, t, \tau), t) \frac{\partial v}{\partial t}(w, t, \tau) \]
\[ = Df(w, t)[Dv(w, t, \tau)]^{-1} \frac{\partial v}{\partial t}(w, t, \tau), \]
for all \( w \in B \) and a.e. \( t \in [s, \tau] \). Next, combining the relations (2.6) and (2.11), we obtain
\[ \frac{\partial v}{\partial t}(w, t, \tau) = Dv(w, t, \tau)h(w, t), \quad w \in B, \text{ a.e. } t \in [s, \tau]. \]
Letting \( w = v(z, s, t) \) in the above relation and taking into account (2.10), we conclude that

\[
0 = Dv(v(z, s, t), t, \tau) \left[ \frac{\partial v}{\partial t}(z, s, t) + h(v(z, s, t), t) \right],
\]

for all \( z \in B \) and a.e. \( t \in [s, \tau] \). Since \( Dv(v(z, s, t), t, \tau) \) is nonsingular and \( \tau \) was arbitrarily chosen, (2.9) follows, as desired. This completes the proof. \( \square \)

References