ZBIGNIEW J. JAKUBOWSKI and AGNIESZKA WŁODARCZYK

On some classes of functions of Robertson type

Abstract. Let $\Delta$ be the unit disc $|z| < 1$ and let $G(A, B)$, $-1 < A \leq 1$, $-1 < B \leq 1$ be the class of functions of the form $g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$, holomorphic and nonvanishing in $\Delta$ and such that $\Re\left\{\frac{2zg'(z)}{g(z)} + 1 + \frac{A}{1 - Bz}\right\} > 0$ in $\Delta$. It is known that the class $G = G(1, 1)$ was introduced by M. S. Robertson. A. Lyzzaik has proved the Robertson conjecture on geometric properties of functions $g \in G$, $g \not= 1$.

In this paper we will investigate the properties of functions of the class $G(A, B)$. In particular when $A = B = 1$, we will obtain corresponding results of the class $G$.

1. Introduction. Let $\mathbb{C}$ denote the open complex plane, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc. In the sequel we will use the following well-known definitions. Let $S^*(\alpha)$, $0 \leq \alpha < 1$, denote the class of functions $h$ holomorphic in $\Delta$, normalized by $h(0) = h'(0) - 1 = 0$ and such that $\frac{h(z)}{z} \not= 0$ and

$$\Re\left(\frac{zh'(z)}{h(z)}\right) > \alpha, \quad z \in \Delta.$$
Functions belonging to the class $S^*(\alpha)$ are called *starlike functions of order* $\alpha$, while $S^* = S^*(0)$ is called the class of *starlike functions (with respect to the origin)*.

Let $S^*(\alpha)$, $0 \leq \alpha < 1$, denote the class of functions $h$ of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \Delta$$

such that for every $z \in \Delta$ we have $h'(z) \neq 0$ and

$$\text{Re} \left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} > \alpha. \quad (1.2)$$

Functions belonging to the class $S^c(\alpha)$ are called *convex functions of order* $\alpha$.

It is noted that $h \in S^c(\alpha)$ if and only if $zh'(z) \in S^*(\alpha)$ for $0 \leq \alpha < 1$ (see e.g. [3], vol. I, p. 140).

Let $h$ be a holomorphic function in the disc $\Delta$. We will say that $h$ is *close-to-convex* in the unit disc $\Delta$ if and only if there is a function $\Phi \in S^c = S^c(0)$ such that

$$\text{Re} \left\{ \frac{h'(z)}{\Phi'(z)} \right\} > 0, \quad z \in \Delta. \quad (1.3)$$

It is known that the classes $S^*(\alpha)$ and $S^c(\alpha)$ were introduced by M. S. Robertson [10], while the class of normalized close-to-convex functions – by W. Kaplan [6]. We know also close-to-convex functions generally normalized (see e.g. [3], vol. II, p. 2).

Moreover, let $\varphi$ denote the class of functions $p$ holomorphic in $\Delta$, $p(0) = 1$ and such that $\text{Re} p(z) > 0$ for $z \in \Delta$. This class is called the *class of Carathéodory functions with positive real part*.

In 1981 M. S. Robertson [11] introduced the class $G$ of all functions $g$ of the form

$$g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n,$$  

holomorphic and nonvanishing in $\Delta$ and such that

$$\text{Re} \left\{ \frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} \right\} > 0, \quad z \in \Delta. \quad (1.5)$$

Robertson also advanced a hypothesis (see [11]) on geometric interpretation of the functions of the family $G$. He assumed that if the function $g \in G$ and $g \not\equiv 1$ then $g$ is close-to-convex and univalent in $\Delta$, $g(\Delta)$ is starlike with respect to the origin, $\lim_{r \to 1} g(r) = 0$ and for some $\alpha \in \mathbb{R}$ we have $\text{Re} \{e^{i\alpha} g(z)\} > 0$, $z \in \Delta$. The above hypothesis was confirmed by A. Lyzzaik [8] in 1984.
A new analytic characterization of the class $G$ has been presented in paper [7]. It is worth noticing that the analytic condition (1.5) was known to Styer [13].

In paper [4] there was introduced the class $G(M)$, $M > 1$, of functions $g$ of the form (1.4) holomorphic and nonvanishing in $\Delta$ and such that

$$\text{Re} \left\{ \frac{2zg'(z)}{g(z)} + \frac{zP'(z;M)}{P(z;M)} \right\} > 0, \quad z \in \Delta,$$

where $P(\cdot;M)$ denotes the known Pick function. The class

$$(1.7) \quad G(1) = \left\{ g \text{ of the form (1.4)}: g(z) \neq 0 \text{ and } \text{Re}\left\{ \frac{2zg'(z)}{g(z)} + 1 \right\} > 0, z \in \Delta \right\}$$

was also considered.

Moreover, M. S. Obradović and S. Owa [9] investigated the class $G(\alpha)$, $0 \leq \alpha < 1$, of functions $g$ of the form (1.4) holomorphic in the disc $\Delta$, $g(z) \neq 0$ for $z \in \Delta$ and satisfying the condition

$$(1.8) \quad \text{Re} \left\{ \frac{zg'(z)}{g(z)} + (1 - \alpha) \frac{1 + z}{1 - z} \right\} > 0, \quad z \in \Delta.$$ 

The purpose of this paper is to introduce and investigate a new class of the aforesaid type.

2. Definition and some properties of the class $G(A, B)$.

**Definition 2.1.** Let $G(A, B)$, where $-1 < A \leq 1$, $-A < B \leq 1$, denote the class of functions $g$ of the form (1.4) holomorphic and nonvanishing in disc $\Delta$ and such that

$$(2.1) \quad \text{Re} \left\{ \frac{2zg'(z)}{g(z)} + Q(z;A,B) \right\} > 0, \quad z \in \Delta,$$

where

$$(2.2) \quad Q(z;A,B) = \frac{1 + Az}{1 - Bz}, \quad z \in \Delta.$$ 

We note that the class $G(1,1)$ is identical to the known class $G$. Moreover, it is shown that $G(0,0) = G(1)$. If $B = -A$ then the function $Q(z;A,-A) \equiv 1$, so we have the class $G(1)$.

It is worth reminding in this place, that the function $Q$ of the form (2.2), when $B < 1$ maps conformally the disc $\Delta$ onto a disc situated on the right in the half-plane. If however $B = 1$, $Q(\Delta;A,B)$ is the half-plane $\{ w : \text{Re} w > \frac{1-A}{2} \}$, where $0 \leq \frac{1-A}{2} < 1$. 


Let $Q(z) = Q(z; A, B)$ and let $F(z) = F(z; A, B)$ be a function satisfying the equation

$$z F'(z) = Q(z),$$

where $Q$ is of the form (2.2). Then $F \in S^*(A, B), -1 < A \leq 1, -A < B \leq 1$ (see [5]), where

$$S^*(A, B) = \left\{ F : F(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in \Delta \text{ and } z F'(z) \prec Q(z) \right\}.$$

Furthermore, we have

$$F(z) = z \cdot \exp \left( \int_0^z \frac{Q(\zeta) - 1}{\zeta} \, d\zeta \right), \quad z \in \Delta.$$

If in the above-mentioned formula we put the function $Q$ of the form (2.2), we will obtain the function of the form

$$F(z; A, B) = \begin{cases} z(1 - Bz)^{-\frac{A+B}{2}} & z \in \Delta, \text{ for } B \neq 0, \\ \exp(Az) & z \in \Delta, \text{ for } B = 0. \end{cases}$$

From (2.1) and (2.3) we conclude that for some function $g \in G(A, B)$ there exists a starlike function $h$ of the class $S^* = S^*(1, 1)$ such that

$$g^2(z) \cdot F(z) = h(z), \quad z \in \Delta$$

and conversely. We have:

**Property 2.1.** Let $g$ be a holomorphic function in $\Delta$ such that $g(0) = 1$. Then $g \in G(A, B)$ if and only if there exists a function $h \in S^*$ such that

$$g(z) = \sqrt{\frac{h(z)}{z}} (1 - Bz)^{\frac{A+B}{2}}, \quad z \in \Delta, \text{ for } B \neq 0,$$

$$g(z) = \sqrt{\frac{h(z)}{z}} \exp \left( -\frac{A}{2} z \right), \quad z \in \Delta, \text{ for } B = 0.$$

**Examples.** It follows from Property 2.1 that the functions:

$$g_0(z; A, B) = \begin{cases} (1 - Bz)^{\frac{A+B}{2}} & z \in \Delta, \text{ for } B \neq 0, \\ \exp \left( -\frac{A}{2} z \right) & z \in \Delta, \text{ for } B = 0 \end{cases}$$

and

$$g_1(z; A, B) = \begin{cases} (1 - z)^{-1}(1 - Bz)^{\frac{A+B}{2}} & z \in \Delta, \text{ for } B \neq 0 \\ (1 - z)^{-1} \exp \left( -\frac{A}{2} z \right) & z \in \Delta, \text{ for } B = 0 \end{cases}$$
belong to the class $G(A,B)$. Furthermore, for $-1 < A \leq 1$, $-A < B \leq 1$, we have

$$g_1(z; A, B)$$

$$= 1 + \left(1 - \frac{1}{2} (A+B)\right) z + \left(1 - \frac{1}{2} (A + B) + \frac{1}{8} (A^2 - B^2)\right) z^2 + \cdots, z \in \Delta.$$  

The function

$$g_2(z) = \frac{1}{\sqrt{1 - z^2}}, \quad z \in \Delta,$$

satisfies the condition

$$\text{Re} \left(2z g_2'(z) + 1\right) > 0, \quad z \in \Delta,$$

so from (1.7) it follows that $g_2 \in G(1)$. Moreover, the function $g_2$ is not univalent, so $g_2 \notin G$.

**Remark 2.1.** Let us consider the function $g_3$, $g_3(0) = 1$, satisfying the equation

$$2z g_3'(z) + \frac{1 + Az}{1 - Bz} = \frac{1 + z^2}{1 - z^2}, \quad z \in \Delta.$$  

Because of (2.1) and (2.2) it is shown, that $g_3 \in G(A,B)$. We can check that if $B < 1$ then there exists a point $z_0 \in \Delta$ such that $g_3'(z_0) = 0$, i.e. $g_3$ is not a univalent function in $\Delta$. Therefore $g_3 \notin G$.

We know the property (see e.g. [4], p. 56) that

$$f \in S^\ast \left(\frac{1}{2}\right) \iff h = \frac{f^2}{I}, \quad \text{where } I(z) \equiv z.$$  

Hence from (2.6) and (2.7) we obtain:

**Property 2.2.** Let $g$ be a holomorphic function in $\Delta$ such that $g(0) = 1$. Then $g \in G(A,B)$ if and only if there exists a function $f \in S^\ast \left(\frac{1}{2}\right)$ such that

$$g(z) = \frac{f(z)}{z} (1 - B z)^\frac{A+B}{2} z, \quad z \in \Delta, \quad \text{for } B \neq 0,$$

$$g(z) = \frac{f(z)}{z} \exp \left(-\frac{A}{2} z\right), \quad z \in \Delta, \quad \text{for } B = 0.$$  

From Property 2.1 and from the known estimates of the respective functionals in the class $S^\ast$ we have:

**Property 2.3.** If $g \in G(A,B)$, $-1 < A \leq 1$, $-A < B \leq 1$, $B \neq 0$, $0 \neq z = re^{i\varphi}$, $0 < r < 1$, $0 \leq \varphi \leq 2\pi$, then the following sharp estimates

$$\frac{1}{1 + |z|} \left| (1 - B z)^\frac{A+B}{2} z \right| \leq |g(z)| \leq \frac{1}{1 - |z|} \left| (1 - B z)^\frac{A+B}{2} z \right|, \quad |z| = r,$$
hold. The upper estimate is attained for the function \( g_\varepsilon \) of the form
\[
g_\varepsilon(z) = (1 - Bz)^{\frac{A + B}{4\pi}} \cdot \sqrt{\frac{k_\varepsilon(z)}{z}},
\]
where \( k_\varepsilon(z) = \frac{z}{(1-z)^2} \), \( \varepsilon = e^{-i\varphi} \), and the lower estimate for \( g_\varepsilon \) and \( \varepsilon = -e^{-i\varphi} \).

If \( g \in G(A, 0) \), then for \( 0 \neq z = re^{i\varphi} \), \( 0 < r < 1 \), \( 0 \leq \varphi \leq 2\pi \) we have
\[
\frac{1}{1 + |z|} \exp\left(-\frac{A}{2} \text{Re} z\right) \leq |g(z)| \leq \frac{1}{1 - |z|} \exp\left(-\frac{A}{2} \text{Re} z\right).
\]
The extremal function for the upper estimate (2.14) is the function \( g_\varepsilon^* \) of the form
\[
g_\varepsilon^*(z) = \exp\left(-\frac{A}{2} \frac{z}{2}\right) \sqrt{\frac{k_\varepsilon(z)}{z}},
\]
where \( \varepsilon = e^{-i\varphi} \), while for the lower estimate is the function \( g_\varepsilon^* \) for \( \varepsilon = -e^{-i\varphi} \).

Let \( 0 < B < 1 \). Then from (2.13) we have \( |g(z)| \geq \frac{1}{2} (1 - B)^{\frac{A + B}{4\pi}} \) for \( z \in \Delta \). If \( -A < B < 0 \) so \( |g(z)| \geq \frac{1}{2} (1 - B)^{\frac{A + B}{4\pi}} \), but when \( B = 0 \) then from (2.14) \( |g(z)| \geq \frac{1}{2} \exp\left(-\frac{1}{2} \frac{A}{|\Delta|}\right) \) for \( z \in \Delta \). In consequence we obtain:

**Property 2.4.** If \( g \in G(A, B) \), \( g \neq 1 \), \( B < 1 \), then there exists the constant \( \delta > 0 \) such that \( |g(z)| > \delta \) for \( z \in \Delta \).

The point \( w = 0 \) is not the boundary point of the set \( g(\Delta) \) for any function \( g \) from class \( G(A, B) \), \( B < 1 \), and consequently \( g \notin G \).

**Property 2.5.** If \( g \in G(A, B) \), \( -1 < A \leq 1 \), \( -A < B \leq 1 \), is of the form (1.4) then the sharp estimates
\[
|2d_1 + A + B| \leq 2,
\]
\[
|2d_2 + d_1^2 + 2d_1 (A + B) + \frac{1}{2} (A + B) (A + 2B)| \leq 3
\]
hold. We obtain the equality in the above estimates for the function \( g_1 \) of the form (2.9).

Because for each function \( h \in S^* \) the functions
\[
z \to \frac{1}{\rho} h(\rho z), \quad z \to e^{i\varphi} h(e^{-i\varphi} z), \quad 0 < \rho < 1, \quad \varphi \in \mathbb{R}, \quad z \in \Delta,
\]
also belong to \( S^* \), from Property 2.1 and estimation (2.15) we obtain:

**Property 2.6.** The region of values of the coefficient \( d_1 \), i.e. \( \{d_1 : g \in G(A, B), g(z) = 1 + d_1 z + \cdots\} \) has the form
\[
\left\{ w \in \mathbb{C} : \left| w + \frac{A + B}{2}\right| \leq 1 \right\}.
\]
From the global formula (2.4) and Property 2.1 it follows:

**Property 2.7.** If \( g \in G(A, B), -1 < A \leq 1, -A < B \leq 1, \) then for \( B \neq 0 \)

\[
(2.18) \quad g(z) = (1 - Bz)^{A + B \frac{z}{2}} \cdot \exp \left( \frac{1}{2} \int_{0}^{z} \frac{p(\zeta) - 1}{\zeta} \, d\zeta \right), \quad z \in \Delta, \ p \in \wp,
\]

and for \( B = 0 \)

\[
(2.19) \quad g(z) = \exp \left( \frac{1}{2} \left( -Az + \frac{1}{2} \int_{0}^{z} \frac{p(\zeta) - 1}{\zeta} \, d\zeta \right) \right), \quad z \in \Delta, \ p \in \wp,
\]

and conversely, where \( \wp \) denotes the aforesaid class of Carathéodory functions with positive real part.

We know that if \( f \in S^{*} \left( \frac{1}{2} \right) \) then the function \( \Phi \) defined by the formula

\[
(2.20) \quad \Phi(z, \xi) = \frac{\xi - f(z) - f(\xi)}{z - \xi}, \quad z, \xi \in \Delta
\]
satisfies the condition \( \text{Re} \, \Phi(z, \xi) > 1/2 \) (see [12], p. 121). Moreover, if \( g \in G(A, B), B \neq 0 \) then from (2.11) the function

\[
(2.21) \quad f(z) = zg(z) (1 - Bz)^{A + B \frac{z}{2}}, \quad z \in \Delta,
\]

belongs to the class \( S^{*} \left( \frac{1}{2} \right). \) We denote

\[
(2.22) \quad d_{0} = 1 = P_{0}(A, B), \quad (1 - Bz)^{-A + B \frac{z}{2}} = 1 + \sum_{k=1}^{\infty} P_{k}(A, B) z^{k}, \quad z \in \Delta,
\]

where

\[
P_{k}(A, B) = \frac{(A + B)(A + 3B) \cdots (A + (2k - 1)B)}{k! 2^{k}}, \quad k = 1, 2, \ldots.
\]

We prove:

**Theorem 2.1.** Let \( g \in G(A, B), -1 < A \leq 1, -A < B \leq 1, B \neq 0, \)

\( g(z) = 1 + \sum_{n=1}^{\infty} d_{n} z^{n}, \ z \in \Delta \) and let \( R_{n}(z; A, B) \) denote the \( n \)-th partial sum of the power series expansion with the centre at the origin of the function \( z \to g(z)(1 - Bz)^{-A + B \frac{z}{2}}, R_{0}(z; A, B) \equiv 1. \) Then the functions

\[
(2.23) \quad \Phi_{n}(z; A, B) = \frac{g(z) - (1 - Bz)^{A + B \frac{z}{2}} \cdot R_{n-1}(z; A, B)}{z^{n} \cdot g(z)},
\]

\( z \in \Delta, \ n = 1, 2, \ldots, \) are holomorphic in \( \Delta \) and

\[
(2.24) \quad |\Phi_{n}(z; A, B)| \leq 1.
\]

In particular

\[
(2.25) \quad |\Phi_{n}(0; A, B)| = \left| \sum_{k=0}^{n} d_{k} P_{n-k}(A, B) \right| \leq 1,
\]
\begin{align}
|\Phi_n'(0; A, B)| \\
&= \left| \sum_{k=0}^{n+1} d_k P_{n+1-k}(A, B) - (P_1(A, B) + d_1) \sum_{k=0}^{n} d_k P_{n-k}(A, B) \right| \\
&\leq 1 - \left| \sum_{k=0}^{n} d_k P_{n-k}(A, B) \right|^2,
\end{align}

(2.26)

\begin{align}
|d_n - g_n(A, B)|^2 + \sum_{k=1}^{p} |d_{n+k} - g_{n+k}(A, B)|^2 &\leq 1 + \sum_{k=1}^{p} |d_k|^2, \quad p \geq 1,
\end{align}

(2.27)

where

\[g_k(A, B) = d_k, \quad k = 0, 1, \ldots, n - 1,
\]

\[d_0 P_{n+k}(A, B) + \cdots + d_{n-1} P_{k+1}(A, B) + g_n(A, B) P_k(A, B)
\]

\[\cdots + g_{n+k}(A, B) P_0(A, B) = 0, \quad k = 0, 1, \ldots.
\]

\textbf{Proof.} By the assumption \(g \in G(A, B), B \neq 0\), so the function \(f\) of the form (2.21) belongs to the class \(S^*\left(\frac{1}{2}\right)\). Let \(z, \xi \in \Delta\). We consider the function \(\Phi\) of the form (2.20). Hence we obtain

\[\Phi(z, \xi) = \frac{1}{1 - \frac{z}{\xi}} - \frac{1}{1 - \frac{z}{\xi}} \cdot \frac{z}{\xi} \cdot \frac{g(z)(1 - Bz)^{-\frac{A+B}{2B}}}{g(\xi)(1 - B\xi)^{-\frac{A+B}{2B}}}, \quad z, \xi \in \Delta.
\]

The expansion of the function \(\Phi\) in powers of \(z\) yields

\[\Phi(z, \xi) = 1 + \sum_{n=1}^{\infty} \Phi_n(\xi; A, B) z^n, \quad z \in \Delta,
\]

where the functions \(\Phi_n(\xi; A, B)\) are defined by formulas (2.23).

We notice that for all \(n = 1, 2, \ldots\) the functions \(\Phi_n\) are holomorphic in \(\Delta\). Moreover, because \(\Re \Phi(z, \xi) > \frac{1}{2}\) then from the known estimate of the coefficients in the class \(\wp\) we obtain the estimates (2.24).

On the other hand, because of (2.22) and the definition the function \(R_{n-1}(\xi; A, B)\), from (2.23) we have

\[\Phi_n(\xi; A, B) = \frac{S_n(A, B) + S_{n+1}(A, B)\xi + \cdots + S_{n+k}(A, B)\xi^k + \cdots}{1 + S_1(A, B)\xi + \cdots},
\]

where

\[S_n(A, B) = d_0 P_n(A, B) + \cdots + d_n P_0(A, B), \quad n = 1, 2, \ldots.
\]

Hence and from inequality (2.24) for \(z = 0\) we obtain (2.25).

The inequality (2.26) is a consequence of the fact that if

\[\Phi_n(\xi; A, B) = a_0 + a_1\xi + a_2\xi^2 + \cdots
\]

and

\[|\Phi_n(\xi; A, B)| < 1 \text{ for } \xi \in \Delta,
\]
then $|a_1| \leq 1 - |a_0|^2$.

From (2.23) we have

$$\Phi_n(z; A, B) \cdot g(z) = \frac{g(z) - (1 - Bz)^{\frac{A + B}{2A}} \cdot R_{n-1}(z; A, B)}{z^n},$$

$z \in \Delta, n = 1, 2, \ldots$.

Put

$$G(z; A, B) = (1 - Bz)^{\frac{A + B}{2A}} \cdot R_{n-1}(z; A, B), \quad z \in \Delta$$

and let

$$G(z; A, B) = \sum_{n=0}^{\infty} g_n(A, B) z^n, \quad z \in \Delta.$$

Equating coefficients at the respective powers of $z$ of identity

$$G(z; A, B) \cdot (1 - Bz)^{-\frac{A + B}{2A}} = R_{n-1}(z; A, B), \quad z \in \Delta,$$

we have (2.28). Then

$$G(z; A, B) = \sum_{k=0}^{n-1} d_k z^k + g_n(A, B) z^n + g_{n+1}(A, B) z^{n+1} + \cdots, \quad z \in \Delta.$$

From this and from (2.23) we have

$$\sum_{k=0}^{p} (d_{n+k} - g_{n+k}(A, B)) z^k + \sum_{k=p+1}^{\infty} a_k(A, B) z^k = \left( \sum_{k=0}^{p} d_k z^k \right) \cdot \Phi_n(z; A, B),$$

where $a_k(A, B)$ are the appropriate coefficients. From the inequality (2.24) we obtain

$$\sum_{k=0}^{p} (d_{n+k} - g_{n+k}(A, B)) z^k + \sum_{k=p+1}^{\infty} a_k(A, B) z^k \leq \sum_{k=0}^{\infty} d_k z^k.$$

Let $z = re^{it}, 0 < r < 1, 0 \leq t \leq 2\pi$. Integrating the above inequality side-wise in the interval $[0, 2\pi]$ and making use of the equality $z \bar{z} = |z|^2$, $z \in \mathbb{C}$, we obtain

$$\sum_{k=0}^{p} |d_{n+k} - g_{n+k}(A, B)|^2 r^{2k} + \sum_{k=p+1}^{\infty} |a_k(A, B)|^2 r^{2k} \leq \sum_{k=0}^{p} |d_k|^2 r^{2k}.$$

Passing to the limit as $r \to 1^-$ and from the fact that $|a_k(A, B)|^2 \geq 0, k = p + 1, \ldots, p \geq 1$ we have (2.27).\)

We know that Theorem 2.1 has its equivalents in the classes $G$ (see [1]) and $G(M), M > 1$ (see [4]).

Similarly we do in case $B = 0$.\)


Let \( g \in G(A, 0) \). Then from (2.12) the function \( f(z) = z \cdot g(z) \exp(\frac{A}{2}z) \), \( z \in \Delta \) belongs to the class \( S^*\left(\frac{1}{2}\right) \). Denote
\[
d_0 = 1 = P_0(A), \quad \exp\left(\frac{A}{2}z\right) = 1 + \sum_{n=1}^{\infty} P_n(A)z^n, \quad z \in \Delta,
\]
where \( P_n(A) = \frac{A^n}{2^n n!} \). It is clear that \( P_0(A) = P_0(A, 0), P_n(A) = P_n(A, 0) \).

We obtain:

**Theorem 2.2.** Let \( g \in G(A, 0), 0 < A \leq 1, g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \), \( z \in \Delta \) and let \( R_n(z; A) \) denote the \( n \)-th partial sum of the power series expansion with the centre at the origin of the function \( z \to g(z) \exp(\frac{A}{2}z) \), \( R_0(z; A) \equiv 1 \). Then the functions
\[
(2.30) \quad \Phi_n(z; A) = g(z) - \exp\left(-\frac{A}{2}z\right) \cdot R_{n-1}(z; A), \quad z \in \Delta, \quad n = 1, 2, \ldots,
\]
are holomorphic in \( \Delta \) and
\[
(2.31) \quad |\Phi_n(z; A)| \leq 1.
\]
In particular
\[
(2.32) \quad |\Phi_n(0; A)| = \left| \sum_{k=0}^{n} d_k P_{n-k}(A) \right| \leq 1,
\]
\[
(2.33) \quad |\Phi'_n(0; A)| = \left| \sum_{k=0}^{n+1} d_k P_{n+1-k}(A) - (P_1(A) + d_1) \sum_{k=0}^{n} d_k P_{n-k}(A) \right|
\leq 1 - \sum_{k=0}^{n} d_k P_{n-k}(A)^2,
\]
\[
(2.34) \quad |d_n - g_n(A)|^2 + \sum_{k=1}^{p} |d_{n+k} - g_{n+k}(A)|^2 \leq 1 + \sum_{k=1}^{p} |d_k|^2, \quad p \geq 1,
\]
where
\[
g_k(A) = d_k, \quad k = 0, 1, \ldots, n - 1,
\]
\[
d_0 P_{n+k}(A) + \cdots + d_{n-1} P_{k+1}(A) + g_n(A) P_k(A) + \cdots + g_{n+k}(A) P_0(A) = 0, \quad k = 0, 1, \ldots.
\]

From (2.23), (2.24) and (2.30), (2.31) for \( n = 1 \) we have:

**Corollary 2.1.** If \( g \in G(A, B), -1 < A \leq 1, -A < B \leq 1, B \neq 0 \), then
\[
(2.36) \quad \left| g(z) - \frac{(1 - Bz)^{\frac{A + B}{2}z}}{1 - |z|^2} \right| \leq \frac{\left| (1 - Bz)^{\frac{A + B}{2}z} \right| |z|}{1 - |z|^2}, \quad z \in \Delta.
\]
If \( g \in G(A,0) \) then

\[
(2.37) \quad \left| g(z) - \frac{\exp \left( -\frac{A}{2} \text{Re} z \right)}{1 - |z|^2} \right| \leq \frac{\exp \left( -\frac{A}{2} \text{Re} z \right) |z|}{1 - |z|^2}, \quad z \in \Delta.
\]

**Remark 2.2.** If in Theorem 2.1 we put \( A = B = 1 \) then we obtain the known theorem for the class \( G \) (see [1] p. 11). Furthermore, from (2.36) for \( g \in G \) we have

\[
\left| g(z) - \frac{1 - z}{1 - |z|^2} \right| \leq \frac{|1 - z||z|}{1 - |z|^2}, \quad z \in \Delta.
\]

**Remark 2.3.** If \( g \in G(A,B) \), \( B \neq 0 \), \( z \in \Delta \) any fixed then the values of the functional \( H(g) = g(z), \ g \in G(A,B) \) belong to \( \overline{K(w_0, |zw_0|)} \) where \( w_0 = \frac{(1-Bz)^{\frac{A+B}{2B}}}{1-|z|^2} \). Since \( w_0 \neq 0 \) and \( |zw_0| < |w_0| \), we have \( 0 \not\in \overline{K(w_0, |zw_0|)} \).

From (2.27), (2.28), (2.22) and (2.34), (2.35), (2.29) we have:

**Corollary 2.2.** If \( g \in G(A,B) \), \( -1 < A \leq 1, \ -A < B \leq 1 \) then

\[
(2.38) \quad \left| d_1 + \frac{A + B}{2} \right|^2 + \left| d_2 - \frac{1}{2} \left( A^2 - B^2 \right) \right|^2 \leq 1 + |d_1|^2.
\]

The extremal function is the function \( g_1 \) of the form (2.9).

3. Application of classical Cluni method. In the following considerations we are using the so-called Cluni method (see [2]), i.e. without using the function (2.20).

Let the function \( g \) of the form (1.4) belong to the class \( G(A,B) \). Thus the conditions (2.1), (2.2) are satisfied. It follows that there exists a function \( p \in \wp \) such that

\[
(3.1) \quad p(z) = \frac{2zg'(z)}{g(z)} + \frac{1 + Az}{1 - Bz}, \quad z \in \Delta.
\]

It is known that if \( p \in \wp \) then the function \( \omega \) of the form

\[
\omega(z) = \frac{p(z) - 1}{p(z) + 1}, \quad z \in \Delta,
\]

belongs to the known class \( \Omega \) (\( \omega \) holomorphic in \( \Delta \), \( \omega(0) = 0, |\omega(z)| < 1 \) for \( z \in \Delta \)). From this fact and from (3.1) we have

\[
\left( 2zg'(z)(1 - Bz) + 2g(z) + (A - B)zg(z) \right) \omega(z) = 2zg'(z)(1 - Bz) + (A + B)zg(z), \quad z \in \Delta.
\]
Let $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n$. Considering the expansion of the function $g$ in power series we get

\[
\left(2 + 2\sum_{n=1}^{\infty} (n+1)d_n z^n + (A-B)\sum_{n=1}^{\infty} d_{n-1} z^n - 2B\sum_{n=1}^{\infty} (n-1)d_{n-1} z^n\right)\left(\sum_{n=1}^{\infty} \omega_n z^n\right)
= 2\sum_{n=1}^{\infty} nd_n z^n - 2B\sum_{n=1}^{\infty} (n-1)d_{n-1} z^n + (A+B)\sum_{n=1}^{\infty} d_{n-1} z^n, \quad z \in \Delta.
\]

From this

\[
\left(2 + \sum_{n=1}^{\infty} (2(n+1)d_n + (A + B - 2Bn)d_{n-1}) z^n\right)\left(\sum_{n=1}^{\infty} \omega_n z^n\right)
= \sum_{n=1}^{\infty} (2nd_n + (A + 3B - 2Bn)d_{n-1}) z^n, \quad z \in \Delta.
\]

Let

\[
p_n(A, B) = 2(n+1)d_n + (A + B - 2Bn)d_{n-1}, \quad n = 1, 2, \ldots,
\]
and

\[
s_n(A, B) = 2nd_n + (A + 3B - 2Bn)d_{n-1}, \quad n = 1, 2, \ldots.
\]

Then we obtain

\[
2\sum_{n=1}^{\infty} \omega_n z^n + \sum_{n=2}^{\infty} \left(p_1(A, B)\omega_{n-1} + \cdots + p_{n-1}(A, B)\omega_1\right) z^n
= \sum_{n=1}^{\infty} s_n(A, B) z^n, \quad z \in \Delta.
\]

Equating coefficients on both sides of the above identity we have

\[
2\omega_1 = 2d_1 + A + B,
\]

\[
2\omega_n + p_1(A, B)\omega_{n-1} + \cdots + p_{n-1}(A, B)\omega_1 = s_n(A, B) \text{ for } n = 2, 3, \ldots.
\]

Since $|\omega_1| \leq 1$, from (3.5) we obtain

\[
|2d_1 + A + B| \leq 2,
\]
which is identical to the estimate (2.15).

Next from (3.2)–(3.4) we have

\[
\left(2 + \sum_{k=1}^{n-1} p_k(A, B) z^k\right)\left(\sum_{k=1}^{\infty} \omega_k z^k\right)
= \sum_{k=1}^{n} s_k(A, B) z^k + \sum_{k=n+1}^{\infty} a_k z^k,
\]
where \(a_k\) are the appropriate coefficients. Since \(|\omega(z)| < 1\) for \(z \in \Delta\),
\[
\left| \sum_{k=1}^{n} s_k(A, B)z^k + \sum_{k=n+1}^{\infty} a_kz^k \right|^2 < \left| 2 + \sum_{k=1}^{n-1} p_k(A, B)z^k \right|^2, \quad z \in \Delta.
\]
Similarly to the proof of the inequality (2.27) we get
\[
(3.7) \quad \sum_{k=1}^{n} |s_k(A, B)|^2 \leq 4 + \sum_{k=1}^{n-1} |p_k(A, B)|^2, \quad n = 2, 3, \ldots.
\]
Since \(|s_k(A, B)|^2 \geq 0\) for \(k = 1, \ldots, n - 1\), then
\[
|s_n(A, B)|^2 \leq 4 + \sum_{k=1}^{n-1} |p_k(A, B)|^2, \quad n = 2, 3, \ldots.
\]
If we adopt the notation (3.3), (3.4), we get:
\[
\text{Theorem 3.1. If the function } g \text{ of the form (1.4) belongs to the class } G(A, B), \text{ then the estimates}
\]
\[
(3.8) \quad \left| 2nd_n + (A + 3B - 2Bn)d_{n-1} \right|^2 \leq 4 + \sum_{k=1}^{n-1} \left| 2(k+1)d_k + (A + B - 2Bk)d_{k-1} \right|^2, \quad n = 2, 3, \ldots.
\]
\[
\text{hold.}
\]
\[
\text{Remark 3.1. If we put } n = 2 \text{ in (3.7), then we have}
\]
\[
(3.9) \quad |2d_1 + A + B|^2 + |4d_2 + (A - B)d_1|^2 \leq 4 + |4d_1 + (A - B)|^2.
\]
This estimate is different from (2.38).

4. The class \(G[\alpha] = G(1 - 2\alpha, 1)\). Relations between classes \(G[0]\), \(G[\alpha]\) and \(G[1]\). We have recalled different applications of the function (2.22) in geometric theory of functions. In particular we often use it when \(B = 1\) and \(A = 1 - 2\alpha, 0 \leq \alpha < 1\). Hence we consider the class
\[
G[\alpha] := G(1 - 2\alpha, 1), \quad 0 \leq \alpha < 1.
\]
Obviously, \(G[0] = G(1, 1) = G\) and \(G[1] = G(0, 0) = G(1)\). Furthermore, from the obtained properties of the class \(G(A, B)\) we get the corresponding properties of the class \(G[\alpha]\). In particular we have:
\[
\text{Property 4.1.}
\]
\[
(4.1) \quad g \in G[\alpha] \iff g(z) = \sqrt{h(z)} \left( 1 - z \right)^{1-\alpha}, \quad h \in S^*;
\]
\[
(4.2) \quad g \in G[\alpha] \implies |d_1 + 1 - \alpha| \leq 1;
\]
\[ g \in G[\alpha] \Rightarrow |g(z) - \frac{(1 - z)^{1-\alpha}}{1 - |z|^2}| \leq \frac{|(1 - z)^{1-\alpha}| |z|}{1 - |z|^2}. \]

Mutual relations between classes \( G[0], G[\alpha], G[1], 0 < \alpha < 1 \) are also worth considering.

Let
\[ g_0(z) = 1, \quad I(z) = z, \quad z \in \Delta. \]

Obviously
\[ g_0 \in G[0] \cap G[\alpha] \cap G[1], \quad 0 < \alpha < 1. \]

Since \( I \in S^* \), from (4.1) the function \( g_1 \) of the form
\[ g_1(z) = (1 - z)^{1-\alpha}, \quad z \in \Delta, \quad 0 \leq \alpha < 1, \]
satisfies the conditions
\[ g_1(\cdot; \alpha) \in G[\alpha], \quad 0 \leq \alpha < 1 \quad \text{and} \quad g_1(\cdot; \alpha) \notin G[1]. \]

On the other hand for the function
\[ g_2(z) = 1 - z, \quad z \in \Delta, \]
we have
\[ g_2 \notin G[0] \quad \text{and} \quad g_2 \notin G[\alpha] \quad \text{for} \quad 0 < \alpha \leq 1. \]

The function \( h(z) = \frac{z}{(1-z)^2} \in S^* \), so from (4.1) for the function \( g_3 \) of the form
\[ g_3(z; \alpha) = \frac{1}{(1-z)^\alpha}, \quad z \in \Delta, \]
the following conditions hold
\[ g_3(\cdot; \alpha) \notin G[0] \quad \text{and} \quad g_3(\cdot; \alpha) \in G[\alpha], \quad 0 < \alpha \leq 1. \]

We see that the point \( z_0 = 0 \) is not the boundary point of the set \( g_3(\Delta) \) (it is an exterior point).

However for the function
\[ g_4(z) = \frac{1}{1-z}, \quad z \in \Delta, \]
we have
\[ g_4 \notin G[\alpha], \quad 0 \leq \alpha < 1 \quad \text{and} \quad g_4 \in G[1]. \]

If we consider in the property (4.1) the function \( h(z) = \frac{z}{(1+z)^2} \in S^* \), then we have the mapping
\[ g_5(z; \alpha) = \frac{(1 - z)^{1-\alpha}}{1 + z}, \quad z \in \Delta, \]
satisfying the conditions
\[ g_5(\cdot; \alpha) \notin G[0], \quad g_5(\cdot; \alpha) \in G[\alpha], \quad 0 < \alpha < 1 \quad \text{and} \quad g_5(\cdot; \alpha) \notin G[1]. \]
From the above-mentioned examples (4.4), (4.6), (4.8), (4.10), (4.12), (4.14) and the obtained conditions (4.5), (4.7), (4.9), (4.11), (4.13), (4.15) we have

\[ G[0] \cap G[\alpha] \cap G[1] \neq \emptyset, \]
\[ G[0] \setminus (G[\alpha] \cup G[1]) \neq \emptyset, \]
\[ G[1] \setminus (G[\alpha] \cup G[0]) \neq \emptyset, \]
\[ G[\alpha] \setminus (G[0] \cup G[1]) \neq \emptyset, \]
\[ (G[0] \cap G[\alpha]) \setminus G[1] \neq \emptyset, \]
\[ (G[1] \cap G[\alpha]) \setminus G[0] \neq \emptyset. \]

From the above relationships we get a question, whether a function \( g \in G[0] \cap G[1], g \neq 1 \) exists. The answer is negative. We have:

**Corollary 4.1.** The function \( g \in G[0] \cap G[1], g \neq 1 \) does not exist. The intersection of classes \( G[0], G[\alpha] \) for \( 0 < \alpha < 1 \), and \( G[1] \) is a singleton, i.e. \( G[0] \cap G[\alpha] \cap G[1] = \{ g_0 \} \).

Indeed, suppose on the contrary that there exists a function \( g \in G[0] \cap G[1], g \neq 1 \). Then from (4.1) we have

\[ g \in G[0] \Leftrightarrow g^2(z) = (1 - z)^2 \frac{h_1(z)}{z}, \quad z \in \Delta, h_1 \in S^*, \]

and

\[ g \in G[1] \Leftrightarrow g^2(z) = \frac{h_2(z)}{z}, \quad z \in \Delta, h_2 \in S^*. \]

From this

\[ h_2(z) = (1 - z)^2 h_1(z), \quad z \in \Delta, h_1, h_2 \in S^*. \]

We see that

\[ \Re \left\{ \frac{zh_2'(z)}{h_2(z)} \right\} = \Re \left\{ \frac{-2z}{1-z} + \frac{zh_1'(z)}{h_1(z)} \right\}, \quad z \in \Delta, h_1, h_2 \in S^*. \]

But for \( z \to 1^- \) we have \( \Re \left\{ \frac{zh_2'(z)}{h_2(z)} \right\} \to -\infty \), which contradicts the definition of the function \( h_2 \in S^* \).

Unfortunately, we do not know so far any mutual relations between \( G[\alpha_1] \) and \( G[\alpha_2] \), where \( \alpha_1 \neq \alpha_2 \).

**References**


Zbigniew J. Jakubowski Agnieszka Włodarczyk
Chair of Special Functions Chair of Special Functions
Faculty of Mathematics Faculty of Mathematics
University of Łódź University of Łódź
Banacha 22, 90-238 Łódź, Poland Banacha 22, 90-238 Łódź, Poland
e-mail: zjakub@math.uni.lodz.pl e-mail: agnieszka@math.uni.lodz.pl

Received September 1, 2004