JANUSZ GODULA and VICTOR STARKOV

Universal linearly invariant families
and Bloch functions in the unit ball

Abstract. In this note we consider universal linearly invariant families of mappings defined in the unit ball. We give a connection of such families with Bloch functions, as well as with Bloch mappings.

1. Preliminaries. Connections between linearly invariant families of functions on the unit disk ([P]) and Bloch functions were studied in several papers (see for example [CCP], [GS1]). In the case of the unit polydisk similar results were obtained in [GS2], [GS3]. In this paper we connect the universal linearly invariant families of locally biholomorphic mappings in the unit ball of \( \mathbb{C}^n \) ([Pf2]) with Bloch functions ([H1], [H2], [T1], [T2]) or Bloch mappings ([L]).

Let \( \mathbb{C}^n \) denote \( n \)-dimensional complex space of all ordered \( n \)-tuples \( z = (z_1, z_2, \ldots, z_n) \) of complex numbers with the inner product \( \langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n \). The unit ball \( \mathbb{B}^n \) of \( \mathbb{C}^n \) is then the set of all \( z \in \mathbb{C}^n \) with \( |z| = \langle (z, z) \rangle^{1/2} < 1 \). For a vector-valued, holomorphic mapping \( f(z) = (f^1(z), \ldots, f^n(z)) \) let \( f_k^j(z) = \frac{\partial f_j(z)}{\partial z_k} \) and \( f_{ik}^j(z) = \frac{\partial^2 f_j(z)}{\partial z_i \partial z_k} \). Then the derivative \( D f(z) \) of \( f \) at \( z \) is represented by a matrix \( (f_k^j(z)) \) and let the

\text{2000 Mathematics Subject Classification.} 32A10, 32A18, 32A99.

\text{Key words and phrases.} Linearly invariant family, universal linearly invariant family, order of a function, order of a mapping, order of a family, Bloch function, Bloch mapping.
second derivative operator be given by the following formula \( D^2 f(z)(w, \cdot) = (\sum_{k=1}^n f'_{ik}(z)w_k) \) and the identity matrix by \( I \). The (complex) Jacobian of \( f \) at \( z \) can be defined by \( J_f(z) = \det D f(z) \). Let

\[
\mathcal{LS}_n = \{ f : f \text{ is holomorphic in } B^n, J_f(z) \neq 0 \text{ for } z \in B^n, f(\mathbb{O}) = \mathbb{O}, D f(\mathbb{O}) = I \}
\]

be the family of normalized, locally biholomorphic mappings of \( B^n \). The operator on \( \mathcal{LS}_n \) that defines the linear invariance is the Koebe transform

\[
\Lambda_{\phi}(f)(z) = (D \phi(\mathbb{O}))^{-1}(D f(\phi(\mathbb{O})))^{-1}\{f(\phi(z)) - f(\phi(\mathbb{O}))\},
\]

where \( \phi \) belongs to the set \( \mathcal{A} \) of biholomorphic automorphisms of \( B^n \) and \( f \in \mathcal{LS}_n \). Up to multiplication by an unitary matrix, the biholomorphic automorphisms of \( B^n \) are

\[
\phi(z) = \phi_a(z) = \frac{a - P_a z - s Q_a z}{1 - \langle z, a \rangle}, \quad a \in B^n,
\]

where \( P_\mathbb{O} = \mathbb{O} \) and \( P_a z = \langle z, a \rangle a \) for \( a \neq \mathbb{O} \), \( Q_a = I - P_a \) and \( s = (1 - \|a\|^2)^{1/2} \). For details see [R]. The following definitions are known ([P\textsc{f}2],[B\textsc{F}G]).

**Definition 1.1.** A family \( \mathcal{F} \) is called linearly invariant if

(i) \( \mathcal{F} \subset \mathcal{LS}_n \),

(ii) \( \Lambda_{\phi}(f) \in \mathcal{F} \) for all \( f \in \mathcal{F} \) and \( \phi \in \mathcal{A} \).

Let the trace of a matrix will be denoted by \( \text{tr} \). The number

\[
\text{ord} \mathcal{F} = \sup_{g \in \mathcal{F}} \sup_{\|w\| = 1} \left| \text{tr} \left\{ \frac{1}{2} D^2 g(\mathbb{O})(w, \cdot) \right\} \right|
\]

\[
= \sup_{g \in \mathcal{F}} \sup_{\|w\| = 1} \left| \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n g_{j,k}(\mathbb{O})w_k \right|
\]

is called ([P\textsc{f}2]) the order of a linearly invariant family \( \mathcal{F} \). Let us introduce the notion of the order of a function.

**Definition 1.2.** For \( f \in \mathcal{LS}_n \) the number

\[
\text{ord} f = \sup_{\phi \in \mathcal{A}} \sup_{\|w\| = 1} \frac{1}{2} \left| \text{tr} \{ D^2 g(\mathbb{O})(w, \cdot) \} \right|
\]

where \( g(z) = \Lambda_{\phi}(f)(z) \), is called the order of \( f \).
**Definition 1.3.** The family

\[ \mathcal{U}_\alpha = \bigcup \{ f \in \mathcal{S}_n : \text{ord } f \leq \alpha \} \]

is called the universal linearly invariant family.

In the paper we will use the following results. If \( \mathcal{F} \subset \mathcal{S}_n \) is a linearly invariant family of order \( \alpha \) and \( f \in \mathcal{F} \) then

\[
(1.2) \quad \frac{(1 - \|z\|)^{\alpha - \frac{n+1}{2}}}{(1 + \|z\|)^{\alpha + \frac{n+1}{2}}} \leq |J_f(z)| \leq \frac{(1 + \|z\|)^{\alpha - \frac{n+1}{2}}}{(1 - \|z\|)^{\alpha + \frac{n+1}{2}}}, \quad z \in \mathbb{B}^n, \quad ([Pf2])
\]

\[
(1.3) \quad |\log((1 - \|z\|^2)^{\frac{n+1}{2}}|J_f(z)|)| \leq \alpha \log \frac{1 + \|z\|}{1 - \|z\|}, \quad z \in \mathbb{B}^n, \quad ([Pf2])
\]

\[
(1.4) \quad \frac{d}{d\rho} \log(J_f(\rho w)) = \text{tr}\{(D f(\rho w))^{-1} D^2 f(\rho w)(w, \cdot)\}, \quad \rho \in [0, 1), \ w \in \overline{\mathbb{B}^n}. \quad ([Pf1])
\]

The above inequalities are rendered by the mappings

\[ K_\alpha(z) = (k_\alpha(z_1), z_2 \sqrt{k'_\alpha(z_1)}, \ldots, z_n \sqrt{k'_\alpha(z_1)}), \quad ([Pf2],[LS2]) \]

where

\[ k_\alpha(z_1) = \frac{n+1}{4\alpha} \left[ \left( \frac{1 + z_1}{1 - z_1} \right)^{\frac{2\alpha}{n+1}} - 1 \right]. \]

In [GLS] it was proved the following theorem.

**Theorem A.** *The family \( \mathcal{U}_\alpha \) coincides with the set of all functions satisfying the conditions of Definition 1.1 and the right hand side inequality in (1.2).*

**2. Bloch functions.** R. Timoney studied ([T1], [T2]) Bloch functions in several complex variables and he gave several equivalent definitions (see also [H1], [H2]). In this paper we will use the following one.

**Definition 2.1.** A holomorphic function \( h : \mathbb{B}^n \to \mathbb{C} \) is called a Bloch function if its norm

\[ \|h\|_B = |h(\emptyset)| + \sup_{\phi \in \mathcal{A}} \|\nabla(h \circ \phi)(\emptyset)\| \]
is finite.

Now let

$$Q_h(z) = \sup_{C^n \ni x \neq O} \frac{|\langle \nabla h(z), \bar{x} \rangle|}{H_z(x, x)^{1/2}},$$

where $H_z(u, v) = \frac{n+1}{2} [(1 - \|z\|^2)(u, v) + (u, z)(z, v)]/((1 - \|z\|^2)^2)$, $u, v \in \mathbb{C}^n$, $z \in \mathbb{B}^n$, is the Bergman metric. Then from Lemma 1 of [H1] it follows that $Q_{h \circ \phi}(z) = Q_h(\phi(z))$ for every automorphism $\phi \in A$. Therefore

$$\sup_{a \in \mathbb{B}^n} Q_h(a) = \frac{2}{n+1} \sup_{\phi \in A, \|x\|=1} |\langle \nabla (h \circ \phi)(O), x \rangle| = \frac{2}{n+1} \sup_{\phi \in A} \|\nabla (h \circ \phi)(O)\|.$$

Thus Definition 2.1 is equivalent to the following definition of Bloch functions given in [H2]: $\sup_{a \in \mathbb{B}^n} Q_h(a) < \infty$. Timoney in [T1] proved that quantities

$$\sup_{a \in \mathbb{B}^n} \sup_{\|w\| \leq 1} \left[ (1 - \|w\|^2) \langle \nabla h(w), \bar{w} \rangle \right]$$

are equivalent. In this way the norms $\|h\|_B$ and

$$(2.1) \quad \|h\|_X = |h(0)| + \sup_{w \in \mathbb{B}^n} (1 - \|w\|^2) |\langle \nabla h(w), \bar{w} \rangle|$$

are equivalent. The family of all Bloch functions will be denoted by $\mathcal{B} = \mathcal{B}(\mathbb{B}^n)$. In the next theorem we give a new condition which is equivalent to the definition of a Bloch function.

**Theorem 2.1.** A holomorphic function $h : \mathbb{B}^n \to \mathbb{C}$ belongs to $\mathcal{B}$ if and only if there exists a mapping $f \in \bigcup_{\alpha < \infty} \mathcal{U}_\alpha$ such that

$$h(z) - h(O) = \log(J_f(z)), \quad z \in \mathbb{B}^n.$$

Moreover, if $h(z) - h(O) = \log(J_f(z)) \in \mathcal{B}$ and ord $f = \alpha$, then

$$2 \left( \alpha - \frac{n+1}{2} \right) \leq \|h - h(O)\|_X \leq 2 \left( \alpha + \frac{n+1}{2} \right)$$

and

$$2 \left( \alpha - \frac{n+1}{2} \right) \leq \|h - h(O)\|_B \leq 2 \left( \alpha + \frac{n+1}{2} \right).$$

The inequalities are sharp.

**Proof.** For $\rho \in [0, 1)$, $w \in \partial \mathbb{B}^n$ define $h(\rho w) = \log(J_f(\rho w))$, where ord $f = \alpha$. Observe that we have

$$\frac{d}{d\rho} h(\rho w) = \langle (\nabla h)(\rho w), \bar{w} \rangle = \frac{d}{d\rho} \log(J_f(\rho w))$$
and

\[(2.2) \quad \langle (\nabla h)(\rho w), \rho w \rangle = \rho \frac{d}{d\rho} \log(J_f(\rho w)).\]

Pfaltzgraff showed ([Pf2]) that for \(g(z) = \Lambda_\phi(f)(z), \phi = \phi_a \) and \(a = \rho w\)

\[(2.3) \quad \rho \frac{d}{d\rho} \log(J_f(\rho w)) = (n + 1) \frac{\|\rho w\|^2}{1 - \|\rho w\|^2} + \text{tr} \left\{ D^2 g(\Omega) \left( \frac{-\rho w}{1 - \|\rho w\|^2}, \cdot \right) \right\} \]

\[= (n + 1) \frac{\rho^2}{1 - \rho^2} + \text{tr} \left\{ D^2 g(\Omega) \left( \frac{-\rho w}{1 - \rho^2}, \cdot \right) \right\}.\]

Therefore by (2.2) we get

\[|\langle (\nabla h)(\rho w), \rho w \rangle| \leq (n + 1) \frac{\rho^2}{1 - \rho^2} + \left| \text{tr} \left\{ D^2 g(\Omega) \left( \frac{-\rho w}{1 - \rho^2}, \cdot \right) \right\} \right|\]

and thus

\[\left(1 - \rho^2 \right) |\langle (\nabla h)(\rho w), \rho w \rangle| \leq (n + 1) \rho^2 + \rho |\text{tr} \{ D^2 g(\Omega)(w, \cdot) \}| \]

\[\leq (n + 1) \rho^2 + 2\rho \alpha \leq ((n + 1) + 2\alpha)\rho.\]

By (2.1) the function \(h\) belongs to the Bloch class \(B\) and \(\|h - h(\Omega)\|_X \leq 2(\alpha + \frac{n+1}{2})\).

Conversely, let \(h \in B\) and let \(f \in \mathcal{LS}_n\), such that \(\log J_f(z) = h(z) - h(\Omega)\). In \(\mathcal{LS}_n\) there is such mapping, for example

\[f(z) = (z_1, \ldots, z_{n-1}, \int_0^{z_n} \exp[h(z_1, \ldots, z_{n-1}, s) - h(\Omega)] ds).\]

Let \(z = w\rho, \) where \(\rho \in [0, 1], \|w\| = 1.\) Let \(\phi \in \mathcal{A}\) be fixed. Then let \(g(z) = \Lambda_\phi(f)(z)\). Now combining (2.2) and (2.3) we get

\[\left(1 - \rho^2 \right) |\langle (\nabla h)(\rho w), \rho \bar{w} \rangle| = (n + 1) \rho^2 - \rho \text{tr} \{ D^2 g(\Omega)(w, \cdot) \}.\]

Thus by (2.1) we obtain

\[\frac{1}{2} \rho |\text{tr} \{ D^2 g(\Omega)(w, \cdot) \}| \leq \frac{n + 1}{2} \rho^2 + \frac{1}{2} (1 - \rho^2) |\langle (\nabla h)(\rho w), \rho \bar{w} \rangle| \leq \frac{n + 1}{2} + \frac{1}{2} \|h - h(\Omega)\|_X.\]
For \( \rho \to 1 \) we get
\[
\frac{1}{2} |\text{tr}\{D^2 g(\mathcal{O})(w, \cdot)\}| \leq \frac{n+1}{2} + \frac{1}{2}\|h - h(\mathcal{O})\|_X.
\]

Therefore \( f \) belongs to a class \( \mathcal{U}_\alpha \). Moreover
\[
\alpha = \text{ord}\, f = \frac{1}{2} \sup_{\|w\| = 1} |\text{tr}\{D^2 g(\mathcal{O})(w, \cdot)\}| \leq \frac{n+1}{2} + \frac{1}{2}\|h - h(\mathcal{O})\|_X.
\]

Thus \( 2\alpha - (n + 1) \leq \|h - h(\mathcal{O})\|_X \). In the above inequality the equality is attained for \( f_0(z) = z \), \( h(z) = \log J_f(z) \equiv 0 \); (ord \( f_0 = \frac{n+1}{2} \)). In the inequality \( \|h - h(\mathcal{O})\|_X \leq 2\alpha + n + 1 \) the equality is attained for \( h = h_\alpha = \log J_{K_\alpha} \), where \( K_\alpha(z) \) was defined before; (ord \( K_\alpha = \alpha \), [Pf2]). Since
\[
J_{K_\alpha}(z) = (k_\alpha(z_1))^{(n+1)/2} = \frac{(1+z_1^{\alpha -(n+1)/2}}{(1-z_1^{\alpha +(n+1)/2}},
\]
we have \( \nabla h_\alpha(z) = \left( \frac{2\alpha + (n+1)z_1}{1-z_1^2}, 0, \ldots, 0 \right) \) and
\[
\|h_\alpha - h_\alpha(\mathcal{O})\|_X = \sup_{|z_1| < 1} \left[ (1 - |z_1|^2)|z_1| \left| \frac{2\alpha + (n+1)z_1}{1-z_1^2} \right| \right] = 2\alpha + n + 1.
\]

Now we will prove suitable inequalities for \( \| \cdot \|_G \). Let ord \( f = \alpha, g = \Lambda_\phi(f), \phi \in \mathcal{A} \) and \( h = \log J_f \). Then \( J_\phi(z) = C J_f(\phi(z)) J_\phi(z) \), where \( C \) is a constant. Therefore
\[
\nabla (\log J_\phi)(\mathcal{O}) = \nabla (h \circ \phi)(\mathcal{O}) + \left\langle \frac{\nabla J_\phi}{J_\phi}(\mathcal{O}) \right\rangle.
\]

For a holomorphic function \( q(z) \) in \( B^n \) we have \( \frac{\partial \text{Re} \, q}{\partial x_k} = \frac{\partial q(z) + \overline{q(z)}}{2} = \frac{1}{2} \frac{\partial q}{\partial x_k} \). Thus \( \nabla \text{Re} \, q = \frac{1}{2} \nabla q \). Moreover \( |J_\phi(z)| = \frac{\left( 1-|z|^2 \right)^{(n+1)/2}}{(1-(z,\partial z)^2)} \), for \( a \in B^n \) (see [R]), and then
\[
(\nabla \log J_\phi)(\mathcal{O}) = 2(\nabla \log |J_\phi|)(\mathcal{O}) = (n + 1)\alpha,
\]
where \( a \) is an arbitrary element in \( B^n \) for arbitrary \( \phi \in \mathcal{A} \).

It is known (see for example [S]) that for a matrix \( (f_{k,j}(z))_{k,j=1}^n \), where \( f_{k,j}(z) \) are analytic functions in a domain,
\[
\frac{d}{dz} \det(f_{k,j})_{k,j=1}^n = \sum_{k=1}^n \det \left( \begin{array}{ccc} f_{11}(z) & \ldots & f_{1n}(z) \\ \vdots & \ddots & \vdots \\ f_{k1}(z) & \ldots & f_{kn}(z) \\ \vdots & \ddots & \vdots \\ f_{n1}(z) & \ldots & f_{nn}(z) \end{array} \right).
\]
From the normalization of \( g(z) = (g^1, \ldots, g^n) \) it follows that

\[
(\nabla J_g)(\mathbb{O}) = \left( \sum_{k=1}^{n} g^k_1(\mathbb{O}), \ldots, \sum_{k=1}^{n} g^n_k(\mathbb{O}) \right)
\]

and \(( (\nabla J_g)(\mathbb{O}), \bar{w} ) = \text{tr}\{D^2 g(0)(w, \cdot)\} \). Therefore

\[
\langle \nabla (\log J_g)(\mathbb{O}), \bar{w} \rangle = \text{tr}\{D^2 g(0)(w, \cdot)\} = \langle \nabla (h \circ \phi)(\mathbb{O}), \bar{w} \rangle + (n+1) \langle a, \bar{w} \rangle,
\]

where \( a \) depends on \( \phi \) and

\[
\sup_{\phi \in A, \|w\|=1} |\langle \nabla (h \circ \phi)(\mathbb{O}), \bar{w} \rangle| - (n+1) \cdot \sup_{a \in B^n, \|w\|=1} |\langle a, \bar{w} \rangle| \leq 2 \alpha = \sup_{\phi \in A, \|w\|=1} |\text{tr}\{D^2 g(0)(w, \cdot)\}| \leq \sup_{\phi \in A} \|\nabla (h \circ \phi)(\mathbb{O})\| + (n+1) \sup_{a \in B^n} \|a\|,
\]

which is equivalent to the following inequalities

\[
2\alpha - n - 1 \leq \|h - h(\mathbb{O})\| \leq 2\alpha + n + 1.
\]

For \( h \equiv 0 \) we have the equality in the left inequality. Similarly as before for \( h = h_\alpha \) we have the equality in the right inequality. It is sufficient to prove that \( \sup_{a \in B^n} \|\nabla (h_\alpha \circ \phi_\alpha)(\mathbb{O})\| = 2\alpha + n + 1 \). Indeed

\[
h_\alpha \circ \phi_\alpha = \left( \alpha - \frac{n+1}{2} \right) \log(1 + \phi^1_\alpha) - \left( \alpha + \frac{n+1}{2} \right) \log(1 - \phi^1_\alpha),
\]

\[
\nabla (h_\alpha \circ \phi_\alpha)(\mathbb{O}) = \frac{2\alpha + a_1(n+1)}{1 - a_1^2} \nabla \phi^1_\alpha(\mathbb{O}), \quad a = (a_1, \ldots, a_n).
\]

Since (see [R])

\[
\phi^1_\alpha(z) = \frac{a_1 - a_1 \langle z, a \rangle}{1 - \langle z, a \rangle} - s(z_1 - a_1 \langle z, a \rangle), \quad s = \sqrt{1 - \|a\|^2},
\]

we get

\[
\nabla \phi^1_\alpha(\mathbb{O}) = \left( \ldots, a_1 \bar{a}_k \frac{s}{s+1} - s\delta^1_k, \ldots \right), \quad 1 \leq k \leq n.
\]
where $\delta_{ik}$ denotes the Kronecker delta. Therefore

$$\|\nabla(h_\alpha \circ \phi_a)(\mathbb{O})\| = \frac{|2\alpha + a_1(n + 1)|}{|1 - a_1^2|} \|\nabla \phi_a(\mathbb{O})\| = \frac{|2\alpha + a_1(n + 1)|}{|1 + a_1^2|} \sqrt{\frac{1 - \|a\|^2}{1 - |a_1|^2}}$$

and

$$\|h_\alpha\|_B \geq \sup_{a \in B} \left[ \frac{|2\alpha + a_1(n + 1)|}{|1 - a_1^2|} \sqrt{(1 - \|a\|^2)(1 - |a_1|^2)} \right] = 2\alpha + n + 1.$$  

This proves the exactness of the inequality $\|h - h(\mathbb{O})\|_B \leq 2\alpha + n + 1$. □

It was proved in [LS1] that for every $f$ from $\mathcal{U}_\alpha$ and every $v \in \mathbb{C}^n$, $\|v\| = 1$, the quantities

$$|J_f(rv)| \left( \frac{(1 - r)^{\alpha + (n + 1)/2}}{(1 + r)^{\alpha - (n + 1)/2}} \right) \quad \text{and} \quad \max_{\|v\|=1} |J_f(rv)| \left( \frac{(1 - r)^{\alpha + (n + 1)/2}}{(1 + r)^{\alpha - (n + 1)/2}} \right)$$

are decreasing with respect to $r \in [0, 1)$ and for $r \to 1^-$ they have limits which belong to the interval $[0, 1]$. From the above and Theorem 2.1 the next result follows.

**Corollary 2.1.** For every function $h \in B$ and every $v \in \mathbb{C}^n$, $\|v\| = 1$ the quantities

$$\text{Re}[h(rv) - h(\mathbb{O})] + \left( \alpha + \frac{n + 1}{2} \right) \log(1 - r) - \left( \alpha - \frac{n + 1}{2} \right) \log(1 + r)$$

and

$$\max_{\|v\|=1} \text{Re}[h(rv) - h(\mathbb{O})] + \left( \alpha + \frac{n + 1}{2} \right) \log(1 - r) - \left( \alpha - \frac{n + 1}{2} \right) \log(1 + r)$$

are decreasing with respect to $r \in [0, 1)$ and for $r \to 1^-$ they have non-positive limits, where $\alpha = \text{ord} f$ for $f \in \cup_{\alpha < \infty} \mathcal{U}_\alpha$ such that $h(z) - h(\mathbb{O}) = \log J_f(z)$.

Since order of $e^{i\lambda} h$ is changing with $\lambda \in \mathbb{R}$ note that it is not possible to replace the real part by the modulus sign in the last corollary.
Theorem 2.2. A holomorphic function $h : B^n \to \mathbb{C}$ belongs to $\mathcal{B}$ if and only if there exists a positive constant $C$ such that for all $z \in B^n$

$$\sup_{\phi \in \mathcal{A}} \left| \text{Re}[h(\phi(z)) - h(\phi(\mathcal{O}))] + \log \left| \frac{J_{\phi}(z)}{J_{\phi}(\mathcal{O})} \right| + \log(1 - \|z\|^2)^{\frac{n+1}{2}} \right| \leq C \log \frac{1 + \|z\|}{1 - \|z\|},$$

where the best value (the smallest) of $C$ is equal to $\text{ord } f$, for a mapping $f$ from $\mathcal{L}S_n$ such that $\log J_f(z) = h(z) - h(\mathcal{O})$.

Proof. Let $h \in \mathcal{B}$. We can assume that $h(\mathcal{O}) = 0$. Then by Theorem 2.1 there exists a mapping $f \in \cup_{\alpha<\infty} U_{\alpha}$ such that $h(z) = \log(J_f(z))$. For $g(z) = \Lambda_{\phi}(f)(z)$ we get

$$Dg(z) = (D \phi(\mathcal{O}))^{-1}(Df(\phi(\mathcal{O})))^{-1}(Df)(\phi(z))D\phi(z).$$

Moreover, it is clear that

$$\log|J_g(z)| = \text{Re}[h(\phi(z)) - h(\phi(\mathcal{O}))] - \log |J_{\phi}(\mathcal{O})| + \log |J_{\phi}(z)|.$$

By (1.3) we have

$$|\log((1 - \|z\|^2)^{\frac{n+1}{2}}|J_g(z)|)| \leq \alpha \log \frac{1 + \|z\|}{1 - \|z\|},$$

where $\alpha = \text{ord } f$. The equality is attained for $g = K_\alpha$ and $z = (z_1, 0, \ldots, 0) \in B^n$. Thus we get (2.4). The equality is attained for $g = K_\alpha, z = (z_1, 0, \ldots, 0) \in B^n$.

Conversely, suppose that a holomorphic function $h$ satisfies inequality (2.4). Now, let $f(z) = (z_1, \ldots, z_{n-1}, \int_0^x \exp[h(z_1, \ldots, z_{n-1}, s) - h(\mathcal{O})] ds)$. Note that $f$ belongs to $\mathcal{L}S_n$ and $J_f(z) = \exp[h(z) - h(\mathcal{O})]$. Thus for an automorphism $\phi \in \mathcal{A}$ we get

$$\exp[h(\phi(z)) - h(\phi(\mathcal{O}))] = \frac{J_f[\phi(z)]}{J_f[\phi(\mathcal{O})]}.$$ 

As in the first part the proof, for $g(z) = \Lambda_{\phi}(f)(z)$ we have

$$J_g(z) = \frac{J_f[\phi(z)] \cdot J_{\phi}(z)}{J_{\phi}(\mathcal{O}) \cdot J_f[\phi(\mathcal{O})]}.$$
Observe that

\[
\log |J_g(z)| = \log \left| \frac{J_f[\phi(z)]}{J_f[\phi(\mathbb{D})]} \right| + \log \left| \frac{J_\phi(z)}{J_\phi(\mathbb{D})} \right| = \Re [h(\phi(z)) - h(\phi(\mathbb{D}))] + \log \left| \frac{J_\phi(z)}{J_\phi(\mathbb{D})} \right|.
\]

Thus by (2.4) we obtain

\[
\left| \log |J_g(z)| + \frac{n+1}{2} \log(1 - \|z\|^2) \right| \leq C \log \frac{1 + \|z\|}{1 - \|z\|}, \quad z \in \mathbb{B}^n.
\]

Hence for \( z = \rho w, \rho \in [0, 1), w \in \partial \mathbb{B}^n, \)

\[-C \log \frac{1 + \rho}{1 - \rho} \leq \Re \left[ \log J_g(\rho w) + \frac{n+1}{2} \log(1 - \rho^2) \right] \leq C \log \frac{1 + \rho}{1 - \rho}.\]

For \( \rho = 0 \) the equality holds in the above inequalities. Therefore, after differentiation with respect to \( \rho \) at \( \rho = 0 \) we get (using (1.4))

\[-2C \leq \Re [\text{tr}(D g(\mathbb{D}))^{-1} D^2 g(\mathbb{D})(w, \cdot)] \leq 2C.\]

Since \( D g(\mathbb{D}) = I \), we have

\[|\Re [\text{tr}\{D^2 g(\mathbb{D})(w, \cdot)\}]| \leq 2C.\]

For fixed \( u \in \mathbb{C}^n \) we have

\[\|u\| \leq \sup_{\|w\|=1} \Re \langle w, u \rangle \leq \sup_{\|w\|=1} |\langle w, u \rangle| \leq \|u\|.\]

Therefore

\[\sup_{\|w\|=1} |\langle w, u \rangle| = \sup_{\|w\|=1} \Re \langle w, u \rangle.\]

Note that \( \text{tr}\{D^2 g(\mathbb{D})(w, \cdot)\} = \langle w, u \rangle \) for some \( u \in \mathbb{C}^n \). Then

\[\max_{\|w\|=1} |\Re [\text{tr}\{D^2 g(\mathbb{D})(w, \cdot)\}]| = \max_{\|w\|=1} |\text{tr}\{D^2 g(\mathbb{D})(w, \cdot)\}| \leq 2C.\]

Thus \( f \in \mathcal{U}_C \) and (by Theorem 2.1) \( h \in \mathcal{B} \).

Now let us observe that from the proof it follows that \( \alpha = \text{ord} f \leq C \).

Thus from the first part of the proof we get that \( C = \text{ord} f = \alpha \) is the best constant in (2.4). □
Remark 2.1. ([GLS]) From Theorem A and the fact that

\[ J_{A_\phi(f)}(z) = \frac{J_f(\phi(z))J_\phi(z)}{J_f(\phi(0))J_\phi(0)}, \]

it follows that for \( f_1, f_2 \in \mathcal{L}S_n \) with \( J_{f_1}(z) = J_{f_2}(z) \) we have \( \text{ord} f_1 = \text{ord} f_2. \)

3. Bloch mappings. In this section we will consider Bloch mappings from the unit ball \( B^n \) into \( \mathbb{C}^n \) and their connections with linearly invariant families of mappings. Now we give a definition of Bloch mappings (see [L]).

Definition 3.1. A holomorphic mapping \( h : B^n \to \mathbb{C}^n \) is called a Bloch mapping if it has a finite Bloch norm

\[ \|h\|_{\mathcal{B}(n)} = \|h(0)\| + \sup_{\phi \in \mathcal{A}} \|D(h \circ \phi)(0)\|, \]

where \( \|D h(z)\| \) denotes the norm of linear operator \( D h(z) \).

The family of all such mappings will be denoted by \( \mathcal{B}(n) \). Let functions \( f_k \) belong to \( U_{\alpha_k} \), for \( k = 1, \ldots, n \). Then by (1.2) we have

\[ \log |J_{f_k}(z)| \leq \left( \alpha - \frac{n + 1}{2} \right) \log(1 + \|z\|) - \left( \alpha + \frac{n + 1}{2} \right) \log(1 - \|z\|), \]

\( k = 1, \ldots, n \). The next theorem gives a relationship between \( \mathcal{B}(n) \) and \( U_{\alpha} \).

Theorem 3.1. A holomorphic mapping \( h : B^n \to \mathbb{C}^n \) belongs to \( \mathcal{B}(n) \) if and only if there exist mappings \( f_1, \ldots, f_n \in \cup_{\alpha < \infty} U_{\alpha} \) such that

\[ h(z) - h(0) = (\log J_{f_1}(z), \ldots, \log J_{f_n}(z)). \]

Moreover, if \( \alpha_k = \text{ord} f_k, k = 1, \ldots, n \) then

\[ 2 \sqrt{\sum_{k=1}^{n} \left( \alpha_k - \frac{n + 1}{2} \right)^2} \leq \|h - h(0)\|_{\mathcal{B}(n)} \leq 2 \sqrt{\sum_{k=1}^{n} \left( \alpha_k + \frac{n + 1}{2} \right)^2}; \]

and both inequalities are best possible.

Proof. Let \( h = (h^1, \ldots, h^n) = (\log J_{f_1}, \ldots, \log J_{f_n}) \) and let for every \( k = 1, \ldots, n \) \( \text{ord} f_k = \alpha_k < \infty \). Then by Theorem 2.1

\[ \|h^k\|_{\mathcal{B}} = |h^k(0)| + \sup_{\phi \in \mathcal{A}} \|\nabla (h^k \circ \phi)(0)\| \leq 2\alpha_k + n + 1, \]
for every $k = 1, \ldots, n$ and $h^k \in \mathcal{B}$. Because $D(h \circ \phi)(\Omega) = \left(\frac{\partial(h \circ \phi)}{\partial z_k}(\Omega)\right)_{j,k=1}^n$, then for every $\phi \in \mathcal{A}$, we have
\[
\|D(h \circ \phi)(\Omega)\| = \sup_{\|w\|=1} \|D(h \circ \phi)(\Omega)w\| = \sup_{\|w\|=1} \|(\nabla(h^1 \circ \phi)(\Omega), \ldots, \nabla(h^n \circ \phi)(\Omega), \bar{w})\| \leq \sqrt{\sum_{k=1}^n \|\nabla(h^k \circ \phi)(\Omega)\|^2} \leq \sqrt{\sum_{k=1}^n (2\alpha_k + n + 1)^2}.
\]

By the above we get that $h \in \mathcal{B}(n)$ and
\[
\|h - h(\Omega)\|_{\mathcal{B}(n)} \leq \sqrt{\sum_{k=1}^n (2\alpha_k + n + 1)^2}.
\]

From the proof of Theorem 1 exactness of the last inequality follows. The equality is attained for the mapping $h = (h_{\alpha_1}, \ldots, h_{\alpha_n})$, where $h_{\alpha_k}$ were defined in Theorem 2.1.

Conversely, let $h \in \mathcal{B}(n)$, $h = (h^1, \ldots, h^n) = (\log J_{f_1}, \ldots, \log J_{f_n})$, where (similarly as in the proof of Theorem 2.1)
\[
f_k(z) = \left(z_1, \ldots, z_{n-1}, \int_0^{z_n} \exp \left[h^k(z_1, \ldots, z_{n-1}, s) - h^k(\Omega)\right] \, ds\right) \in \mathcal{L}S_n, \quad k = 1, \ldots, n.
\]

Then by Definition 3.1 there is a constant $C = C(h)$ such that for every automorphism $\phi \in \mathcal{A}$ holds $\|D(h \circ \phi)(\Omega)\| \leq C$, which is equivalent to
\[
\sup_{\|w\|=1, \phi \in \mathcal{A}} \left\|\langle \nabla(h^1 \circ \phi)(\Omega), \bar{w}\rangle, \ldots, \langle \nabla(h^n \circ \phi)(\Omega), \bar{w}\rangle\right\| \leq C.
\]

Thus for every $k = 1, \ldots, n$ sup$_{\phi \in \mathcal{A}} \|\nabla(h^k \circ \phi)(\Omega)\| \leq C$, or equivalently $h^k \in \mathcal{B}$ by Definition 2.1. By Theorem 2.1 ord $f_k = \alpha_k < \infty$, which means that $f_1, \ldots, f_n \in \cup_{\alpha < \infty} U_\alpha$. Then we obtain
\[
2\alpha_k - n - 1 \leq \sup_{\phi \in \mathcal{A}} \|\nabla(h^k \circ \phi)(\Omega)\| = \sup_{\phi \in \mathcal{A}, \|w\|=1} \|\langle \nabla(h^k \circ \phi)(\Omega), \bar{w}\rangle\|,
\]
and therefore
\[
\|h - h(\Omega)\|_{\mathcal{B}(n)} = \sup_{\phi \in \mathcal{A}, \|w\|=1} \|D(h \circ \phi)(\Omega)w\| = \sup_{\|w\|=1, \phi \in \mathcal{A}} \|(\langle \nabla(h^1 \circ \phi)(\Omega), \bar{w}\rangle, \ldots, \langle \nabla(h^n \circ \phi)(\Omega), \bar{w}\rangle)\| \geq \sqrt{\sum_{k=1}^n (2\alpha_k - n - 1)^2}.
\]

The equality holds for $h(z) \equiv \Omega$. □
Remark 3.1. A holomorphic mapping $h = (h_1, \ldots, h_n)$ belongs to $B(n)$ if and only if for every $k = 1, \ldots, n$ a function $h_k$ belongs to $B$.

REFERENCES


[R] Rudin, W., Function Theory in the Unit Ball of $\mathbb{C}^n$, Springer-Verlag, New York, 1980.


