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On a theorem of Lindelöf

Dedicated to the memory of Professor Jan G. Krzyż

Abstract. We give a quasiconformal version of the proof for the classical Lindelöf theorem: Let $f$ map the unit disk $D$ conformally onto the inner domain of a Jordan curve $C$. Then $C$ is smooth if and only if $\arg f'(z)$ has a continuous extension to $D$. Our proof does not use the Poisson integral representation of harmonic functions in the unit disk.

1. Introduction. Let $f : \mathbb{D} \to \mathbb{C}$ be a conformal mapping of the unit disk $\mathbb{D}$ onto $f(\mathbb{D})$. The smoothness of $\partial f(\mathbb{D})$ yields the smoothness of $f$ on $\partial \mathbb{D}$. The classical Lindelöf theorem [7] as well as Warschawski’s theorem [9] on differentiability of $f$ at the boundary $\partial \mathbb{D}$ are the basic results of this kind of behavior.

In this paper we adopt a different point of view. Assuming that the boundary curve is smooth, i.e. it has a continuously turning tangent, we extend $f$ over the unit disk to a quasiconformal mapping and apply some results from the infinitesimal geometry of quasiconformal mappings developed in [5], see also [4]. In order to illustrate our approach, we give a quasiconformal version of the proof for the aforementioned Lindelöf theorem. Recall that the standard proof of the Lindelöf theorem is based on the
Poisson formula, see, e.g. [8], p. 44. Our version of the proof does not use the Poisson integral representation of harmonic functions in the unit disk $\mathbb{D}$. In order to make our method easily understandable, we have collected in Chapter 3 basic notations and auxiliary lemmas from the geometric theory of plane quasiconformal mappings.

2. The Lindelöf Theorem. Let $f$ map $\mathbb{D}$ conformally onto the inner domain of a smooth Jordan curve $C$. Since the characterization of smoothness in terms of tangent does not depend on the parametrization, we may choose the conformal parametrization

$$C : w(t) = f(e^{it}), \quad 0 \leq t \leq 2\pi.$$  

An analytic characterization of the smoothness is given by the classical Lindelöf [7] theorem:

**Theorem 1.** Let $f$ map $\mathbb{D}$ conformally onto the inner domain of a Jordan curve $C$. Then $C$ is smooth if and only if $\arg f'(z)$ has a continuous extension to $\mathbb{D}$. If $C$ is smooth, then

$$\arg f'(e^{it}) = \beta(t) - t - \frac{\pi}{2},$$  

where $\beta(t)$ stands for the tangent angle of the curve $f(e^{it})$ at the point $t$.

**Proof.** Let $C$ be a closed smooth Jordan curve in the complex plane $\mathbb{C}$ and $f$ be a conformal mapping of the disk $\mathbb{D}$ onto the inner domain of $C$. The smoothness of $C$ implies the existence of a continuous function $\beta(t)$ on the segment $[0, 2\pi]$ such that

$$\arg \left[ f(e^{i\theta}) - f(e^{it}) \right] \to \begin{cases} \beta(t), & \text{as } \theta \to t + 0, \\ \beta(t) + \pi, & \text{as } \theta \to t - 0. \end{cases}$$

Since each smooth curve $C$ is asymptotically conformal, see [8], p. 246, the mapping $f$ can be extended to a quasiconformal mapping of the complex plane $\mathbb{C}$ in such a way that the corresponding complex dilatation $\mu(z)$ will satisfy the condition $\mu(z) \to 0$ as $|z| \to 1+$. On the other hand, the standard rescaling arguments and convergence and compactness theory imply, see Lemma 1, that for the extended mapping

$$\lim_{z,\zeta \to 0} \left\{ \frac{f(z + \eta) - f(\eta)}{f(\zeta + \eta) - f(\eta)} \frac{z}{\zeta} \right\} = 0$$

uniformly with respect to $\eta \in \partial \mathbb{D}$, provided that $|z/\zeta| \leq \delta$ for each fixed $\delta > 0$. If we replace $z$ by $z\zeta$, then we get that

$$\lim_{\zeta \to 0} \frac{f(\zeta z + \eta) - f(\eta)}{f(\zeta + \eta) - f(\eta)} = z$$
locally uniformly in $z \in \mathbb{C}$ and uniformly in $\eta \in \partial \mathbb{D}$. In particular, setting 
$\zeta = re^{i\theta_1}$ and $z = re^{i(\theta_2 - \theta_1)}$, we obtain

$$\lim_{r \to 0} \left[ \arg \frac{f(\eta + re^{i\theta_1}) - f(\eta)}{re^{i\theta_2}} - \arg \frac{f(\eta + re^{i\theta_1}) - f(\eta)}{re^{i\theta_1}} \right] = 0$$

(2.4) for an appropriate branch of the argument uniformly in $\theta_1, \theta_2 \in [0, 2\pi]$ and $\eta \in \partial \mathbb{D}$. Let $\Gamma$ be an arc of the unit circle $\partial \mathbb{D}$ ending at the point $\eta = e^{it}$.

Since

$$\lim_{z \to e^{it}} \left[ \arg \frac{f(z) - f(e^{it})}{z - e^{it}} \right] = \beta(t) - t - \frac{\pi}{2},$$

we see that the relation (2.4) implies the existence of the limit

$$\arg f'(e^{it}) = \lim_{z \to e^{it}} \arg \frac{f(z) - f(e^{it})}{z - e^{it}} = \beta(t) - t - \frac{\pi}{2}$$

(2.5) which is uniform with respect to the parameter $t$.

In order to prove that $arg f'(z)$ has a continuous extension to the closed unit disk we proceed as follows.

For $z = 1 + \rho e^{i\theta}$ in the disk $|z - 1| < 1$, i.e. $\rho < 1$, we have $|(r - 1)z + 1| = |r - 1 + \rho e^{i\theta}(r - 1) + 1| < r + (1 - r)\rho < 1$, i.e. $\eta(r - 1)z + \eta \in \mathbb{D}$ for $\eta \in \partial \mathbb{D}$.

Since $f$ is analytic in $\mathbb{D}$, the functions of the family

$$F_r(z) = \frac{f(\eta(r - 1)z + \eta) - f(\eta)}{f(r\eta) - f(\eta)}$$

are analytic at the point $z = 1$ for each $0 < r < 1$. Since $F_r(z) \to z$ as $r \to 1 - 0$ locally uniformly in $z \in \mathbb{D}$, the Weierstrass theorem yields that $F_r'(1) \to 1$, i.e.

$$\lim_{r \to 1} \frac{f'(r\eta)(r\eta - \eta)}{f(r\eta) - f(\eta)} = 1$$

(2.6) uniformly in $\eta \in \partial \mathbb{D}$. Formula (2.6) is the well-known Visser–Ostrowski condition, see, e.g. [8], p. 252.

Thus,

$$\lim_{r \to 1} \left( \arg f'(re^{it}) - \arg \frac{f(re^{it}) - f(e^{it})}{re^{it} - e^{it}} \right) = 0$$

uniformly in $t$. Hence, by (2.5), there exists the limit

$$\lim_{r \to 1} \arg f'(re^{it}) = \arg f'(e^{it}) = \beta(t) - t - \frac{\pi}{2}$$

which is uniform in $t \in [0, 2\pi]$. The latter formula and the continuity of $\arg f'(e^{it})$ on $\partial \mathbb{D}$ implies the required continuous extension of $\arg f'(z)$ to $\mathbb{D}$. Thus, we complete the proof of the first part of the theorem.

The converse part of the theorem is elementary and we refer the reader to the standard text given in [8], p. 44.
3. On the infinitesimal geometry of QC-maps. This chapter contains some basic notions and auxiliary lemmas from geometric theory of plane quasiconformal mappings. These were used in our proof of the Lindelöf theorem.

Let $G$ be a domain in the complex plane $\mathbb{C}$ and $\mu : G \to \mathbb{C}$ be a measurable function satisfying

$$\|\mu\|_{\infty} = \text{ess sup}_{G} |\mu(z)| < 1.$$  

An orientation preserving homeomorphism $f : G \to \mathbb{C}$ of the Sobolev class $W^{1,2}_{\text{loc}}$ is called quasiconformal with complex dilatation $\mu$, if it satisfies the Beltrami equation

$$f_{\bar{z}} = \mu(z)f_{z} \; \text{a.e.}$$

A Jordan curve $\Gamma \subset \mathbb{C}$ is called a quasiconformal curve or quasicircle if it is the image of the unit circle under a quasiconformal mapping of $\mathbb{C}$, see, e.g. [8], p. 107. In 1963 L. Ahlfors [1] gave a simple geometric characterization of quasicircles. He proved that the curve $\Gamma$ is a quasicircle if the quantity

$$\gamma \equiv \gamma(w_1, w_2, w) = \frac{|w_1 - w| + |w - w_2|}{|w_1 - w_2|}$$

is bounded for all $w_1, w_2 \in \Gamma$ and $w \in \Gamma(w_1, w_2)$, where $\Gamma(w_1, w_2)$ denotes the sub-arc of $\Gamma$ corresponding to $w_1, w_2 \in \Gamma$ with smaller diameter.

Let $\Gamma \subset \mathbb{C}$ be a quasicircle in the complex plane and let $f$ denote a conformal mapping of the unit disk $D = \{z : |z| < 1\}$ onto the interior of $\Gamma$. By a result of L. Ahlfors, see [2], p. 71, $f$ admits a quasiconformal extension over the unit circle $\partial D$. If there exists a quasiconformal extension with complex dilatation $\mu(z)$ such that

$$\text{ess sup}_{1 \leq |z| \leq t} |\mu(z)| \to 0, \quad t \to 1 + 0,$$

then the curve $\Gamma$ is called asymptotically conformal, see [8], p. 246.

Ch. Pommerenke and J. Becker proved, see [8], p. 247, that (3.4) is equivalent to the condition

$$\lim_{|w_1 - w_2| \to 0} \frac{|w_1 - w| + |w - w_2|}{|w_1 - w_2|} = 1$$

uniformly with respect to $w \in \Gamma(w_1, w_2)$.

It is easy to see that every smooth closed Jordan curve $\Gamma \subset \mathbb{C}$ is asymptotically conformal.

The following result is a key lemma on infinitesimal behavior on the boundary for quasiconformal extensions of conformal mappings. Its proof has been given in [3], see also [4].

**Lemma 1.** Let $f$ be a conformal mapping of $D$ onto the interior of a Jordan domain $G \subset \mathbb{C}$ bounded by an asymptotically conformal (in particular,
smooth) curve $\Gamma = \partial G$. Then $f$ can be extended quasiconformally to $\mathbb{C}$ in such a way that

\begin{equation}
\lim_{z, \zeta \to 0} \left\{ \frac{f(z + \eta) - f(\eta)}{f(\zeta + \eta) - f(\eta)} - \frac{z}{\zeta} \right\} = 0
\end{equation}

uniformly with respect to $\eta \in \partial \mathbb{D}$, provided that $|z/\zeta| \leq \delta$ for each fixed $\delta > 0$.

**Proof.** Since $\partial G = f(\partial \mathbb{D})$ is asymptotically conformal, there exists a quasiconformal extension of $f$ over the unit disk $\mathbb{D}$ to $\mathbb{C}$ with complex dilatation $\mu(z)$ such that

\begin{equation}
\text{ess sup}_{1 < |z| \leq 1 + t} |\mu(z)| \to 0, \quad t \to +0.
\end{equation}

For the extended mapping $f$, let us consider the following approximating family of $f$ at $\eta \in \partial \mathbb{D}$, see [5],

$$F_{t, \eta}(z) = \frac{f(tz + \eta) - f(\eta)}{f(t + \eta) - f(\eta)}, \quad t > 0.$$ 

We shall consider the class $\mathbb{F}_Q$ of all $Q$-quasiconformal self-mappings of the extended complex plane normalized with the conditions $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$. Note that this space of quasiconformal mappings is sequentially compact with respect to the locally uniform convergence, see [6], p. 73.

Now we see that all the mappings $F_{t, \eta}$ are in the class $\mathbb{F}_Q$ at $\eta \in \partial \mathbb{D}$. Since $\mathbb{F}_Q$ is sequentially compact, every convergent subsequence $F_{t_n, \eta_n}$ as $n \to \infty$ has a limit mapping $F_0$ which is in the class $\mathbb{F}_Q$.

Suppose that (3.6) does not hold. Then we can find $\varepsilon > 0$ and sequences $z_n \to 0$, $\zeta_n \to 0$ as $n \to \infty$, satisfying the inequality $|z_n/\zeta_n| \leq \lambda$ for some $\lambda > 0$, and $\eta_n \in \partial \mathbb{D}$ such that

\begin{equation}
\left| \frac{f(z_n + \eta_n) - f(\eta_n)}{f(\zeta_n + \eta_n) - f(\eta_n)} - \frac{z_n}{\zeta_n} \right| > \varepsilon.
\end{equation}

We write $F_n = F_{|\zeta_n|, \eta_n}$. All the functions $F_n$, $n = 1, 2, \ldots$, belong to the space $\mathbb{F}_Q$ and have complex dilatations $\mu_{F_n}(z) = \mu(|\zeta_n|z + \eta_n)$. From (3.7) it follows that $\mu(|\zeta_n|z + \eta_n) \to 0$ as $n \to \infty$ almost everywhere in $\mathbb{C}$. Without loss of generality we may assume that $F_n$ converges locally uniformly in $\mathbb{C}$ to a quasiconformal mapping $F_0 \in \mathbb{F}_Q$ and simultaneously that the sequence of their complex dilatations $\mu_{F_n}$ converges to 0 almost everywhere in $\mathbb{C}$ as $n \to \infty$. Otherwise, one can pass to an appropriate subsequence.

Next, we apply the well-known Bers–Bojarski convergence theorem. This theorem states that if $f_n$ is a sequence of $K$-quasiconformal mappings of $G$ which converges locally uniformly to a quasiconformal mapping $f$ with complex dilatation $\mu_f$ and if their complex dilatations $\mu_n$ tend to a limit function $\mu$ a.e. in $G$, then $\mu = \mu_f$ a.e. in $G$, see [6], p. 187–188. Thus,
the limit function $F_0$ must have the complex dilatation $\mu_0 \equiv 0$. Applying the measurable Riemann mapping theorem, see [6], p. 194, we see that $F_0(z) = z$.

Let now the sequences $z_n$ and $\zeta_n$ be chosen in such a way that $z_n/|\zeta_n| \to z_0 \in \mathbb{C}$. Since the unit circle is compact one can also assume that $\zeta_n/|\zeta_n| \to \zeta_0$, $|\zeta_0| = 1$. Hence

$$
\lim_{n \to \infty} \left| \frac{f(z_n + \eta_n) - f(\eta_n)}{f(\zeta_n + \eta_n) - f(\eta_n)} - \frac{z_n}{\zeta_n} \right| = \lim_{n \to \infty} \frac{|F_n(z_n/|\zeta_n|) - z_n|}{|F_n(\zeta/|\zeta_n|) - \zeta_n|} = \frac{|z_0 - \zeta_0|}{|\zeta_0 - \zeta_0|} = 0
$$

which contradicts (3.8). □

References

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