S. P. GOYAL and PRANAY GOSWAMI

Majorization for certain classes of meromorphic functions defined by integral operator

Abstract. Here we investigate a majorization problem involving starlike meromorphic functions of complex order belonging to a certain subclass of meromorphic univalent functions defined by an integral operator introduced recently by Lashin.

1. Introduction and preliminaries. Let $f(z)$ and $g(z)$ be analytic in the open unit disk

$\Delta = \{ z \in \mathbb{C} \text{ and } |z| < 1 \}$. (1.1)

For analytic functions $f(z)$ and $g(z)$ in $\Delta$, we say that $f(z)$ is majorized by $g(z)$ in $\Delta$ (see [9]) and write

$f(z) \ll g(z) \ (z \in \Delta), \quad (1.2)$

if there exists a function $\phi(z)$, analytic in $\Delta$ such that $|\phi(z)| \leq 1$, and

$f(z) = \phi(z)g(z) \ (z \in \Delta). \quad (1.3)$

Let $\Sigma$ denote the class of meromorphic functions of the form

$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \quad (1.4)$

2000 Mathematics Subject Classification. Primary 30C45; Secondary 30C80. Key words and phrases. Meromorphic univalent functions, majorization property, starlike functions, integral operators.
which are analytic and univalent in the punctured unit disk
(1.5) \( \Delta^* := \{ z \in \mathbb{C} : 0 < |z| < 1 \} := \Delta \setminus \{0\} \)
with a simple pole at the origin.

For functions \( f_j \in \Sigma \) given by
(1.6) \( f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k \quad (j = 1, 2; z \in \Delta^*) \),
we define the Hadamard product (or convolution) of \( f_1 \) and \( f_2 \) by
(1.7) \( (f_1 \ast f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 \ast f_1)(z) \).

Analogously to the operators defined by Jung, Kim and Srivastava [7] on the normalized analytic functions, Lashin [8] introduced the following integral operators
\( \mathcal{P}_\alpha^\beta : \Sigma \longrightarrow \Sigma \)
defined by
(1.8) \( \mathcal{P}_\alpha^\beta f(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{z^{\beta+1}} \int_0^z t^{\beta}(\log \frac{z}{t})^{\alpha-1} f(t) dt \)
\((\alpha > 0, \beta > 0; z \in \Delta^*)\), where \( \Gamma(\alpha) \) is the familiar Gamma function.

Using the integral representation of the Gamma function and (1.4), it can be easily shown that
(1.9) \( \mathcal{P}_\beta^\alpha f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left( \frac{\beta}{\beta + k + 1} \right)^\alpha a_k z^k \), \( (\alpha > 0, \beta > 0; z \in \Delta^*) \).

Obviously
(1.10) \( \mathcal{P}_{\beta}^{1} f(z) := J_\beta. \)

The operator
\( J_\beta : \Sigma \longrightarrow \Sigma \)

has also been studied by Lashin [8].

It is easy to verify that (see [8]),
(1.11) \( z(\mathcal{P}_{\beta}^\alpha f(z))' = \beta \mathcal{P}_{\beta}^{\alpha-1} f(z) - (\beta + 1) \mathcal{P}_{\beta}^\alpha f(z). \)

**Definition 1.1.** A function \( f(z) \in \Sigma \) is said to be in the class \( \mathcal{S}_{\beta}^{\alpha,j}(\gamma) \) of meromorphic functions of complex order \( \gamma \neq 0 \) in \( \Delta \) if and only if
(1.12) \( R \left\{ 1 - \frac{1}{\gamma} \left( \frac{z(\mathcal{P}_{\beta}^{\alpha} f(z))^{(j+1)}}{(\mathcal{P}_{\beta}^{\alpha} f(z))^{(j)}} + j + 1 \right) \right\} > 0 \)
\((z \in \Delta, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \alpha > 0, \beta > 0, \gamma \in \mathbb{C} \setminus \{0\}).\)
Clearly, we have the following relationships:

(i) \( S_{\beta}^{0,0}(\gamma) = S(\gamma) \) (\( \gamma \in \mathbb{C} \setminus \{0\} \)),

(ii) \( S_{\beta}^{0,0}(1 - \eta) = S^*(\eta) \) (0 \( \leq \eta < 1 \)).

The classes \( S(\gamma) \) and \( S^*(\eta) \) are said to be classes of meromorphic star-like univalent functions of complex order \( \gamma \neq 0 \) and meromorphic starlike univalent functions of order \( \eta \) (\( \eta \in \mathbb{R} \) such that 0 \( \leq \eta < 1 \)) in \( \Delta^* \).

A majorization problem for the normalized classes of starlike functions has been investigated by Altinbas et al. [1] and MacGregor [9]. In the recent paper Goyal and Goswami [2] generalized these results for the class of multivalent functions, using fractional derivatives operators. Further, Goyal et al. [3], Goswami and Wang [4], Goswami [5], Goswami et al. [6] studied majorization property for different classes. In this paper, we will study majorization properties for the class of meromorphic functions using integral operator \( P^{\alpha}_{\beta} \).

2. Majorization problems for the class \( S_{\beta}^{\alpha,j}(\gamma) \).

**Theorem 2.1.** Let the function \( f \in \Sigma \) and suppose that \( g \in S_{\beta}^{\alpha,j}(\gamma) \). If \( (P^{\alpha}_{\beta}f(z))^{(j)} \) is majorized by \( (P^{\alpha}_{\beta}g(z))^{(j)} \) in \( \Delta^* \), then

\[
| (P^{\alpha-1}_{\beta}f(z))^{(j)} | \leq | (P^{\alpha-1}_{\beta}g(z))^{(j)} | \quad \text{for} \quad |z| \leq r_1(\beta, \gamma),
\]

where

\[
r_1(\beta, \gamma) = \frac{k_1 - \sqrt{k_1^2 - 4\beta |\beta + 2\gamma|}}{2|\beta + 2\gamma|}
\]

and

\[
k_1 = \beta + 2 + |\beta + 2\gamma|, \quad (\beta > 0, j \in \mathbb{N}_0, \gamma \in \mathbb{C} \setminus \{0\}).
\]

**Proof.** Since \( g \in S_{\beta}^{\alpha,j}(\gamma) \), we find from (2.1) that if

\[
h_1(z) = 1 - \frac{1}{\gamma} \left( \frac{z(P^{\alpha}_{\beta}g(z))^{(j+1)}}{(P^{\alpha}_{\beta}g(z))^{(j)}} + j + 1 \right)
\]

(\( \alpha, \beta > 0, \gamma \in \mathbb{C} \setminus \{0\}, j \in \mathbb{N}_0 \)), then \( \Re\{h_1(z)\} > 0 \) (\( z \in \Delta \)) and

\[
h_1(z) = \frac{1 + w(z)}{1 - w(z)} \quad (w \in \mathcal{P}),
\]

where \( \mathcal{P} \) denotes the well-known class of bounded analytic functions in \( \Delta \) and \( w(z) = c_1z + c_2z^2 + \ldots \) satisfies the conditions

\[ w(0) = 0 \quad \text{and} \quad |w(z)| \leq |z| \quad (z \in \Delta). \]
Differentiating with respect to \( z \), we get

\[
\frac{z(P_\beta^\alpha g(z))^{(j+1)}}{(P_\beta^\alpha g(z))^{(j)}} = \frac{(1 + j - 2\gamma)w(z) - (1 + j)}{1 - w(z)}.
\]

By the principle of mathematical induction, and (1.11), we easily get

\[
z(P_\beta^\alpha g(z))^{(j+1)} = \beta(P_\beta^{\alpha-1} g(z))^{(j)} - (\beta + j + 1)(P_\beta^\alpha g(z))^{(j)},
\]

\((\alpha > 1, \beta > 0; z \in \Delta^*)\). Now using (2.6) in (2.5), we find that

\[
\frac{\beta(P_\beta^{\alpha-1} g(z))^{(j)}}{(P_\beta^\alpha g(z))^{(j)}} = \frac{(\beta + j + 1) + (1 + j - 2\gamma)w(z) - (1 + j)}{1 - w(z)}
\]

or

\[
(P_\beta^\alpha g(z))^{(j)} = \frac{\beta(1 - w(z))}{\beta - (\beta + 2\gamma)w(z)}(P_\beta^{\alpha-1} g(z))^{(j)}.
\]

Since \(|w(z)| \leq |z| \ (z \in \Delta)\), the formula (2.6) yields

\[
\left| (P_\beta^\alpha g(z))^{(j)} \right| \leq \frac{\beta[1 + |z|]}{\beta - |\beta + 2\gamma||z|} \left| (P_\beta^{\alpha-1} g(z))^{(j)} \right|.
\]

Next since \((P_\beta^\alpha f(z))^{(j)}\) is majorized by \((P_\beta^\alpha g(z))^{(j)}\) in the unit disk \(\Delta^*\), from (1.3), we have

\[
(P_\beta^\alpha f(z))^{(j)} = \phi(z)(P_\beta^\alpha g(z))^{(j)}.
\]

Differentiating it with respect to \( z \) and multiplying by \( z \), we get

\[
z(P_\beta^\alpha f(z))^{(j+1)} = z\varphi'(z)(P_\beta^\alpha g(z))^{(j)} + z\varphi(z)(P_\beta^\alpha g(z))^{(j+1)}.
\]

Using (2.7), in the above equality, it yields

\[
(P_\beta^{\alpha-1} f(z))^{(j)} = \frac{z\varphi'(z)}{\beta}(P_\beta^\alpha g(z))^{(j)} + \varphi(z)(P_\beta^{\alpha-1} g(z))^{(j)}.
\]

Thus, nothing that \( \varphi \in \mathcal{P} \) satisfies the inequality (see, e.g. Nehari [6])

\[
|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2}
\]

and making use of (2.8) and (2.10) in (2.9), we get

\[
\left| (P_\beta^{\alpha-1} f(z))^{(j)} \right| \leq \left( |\varphi(z)| + \frac{1 - |\varphi(z)|^2}{1 - |z|} \frac{|z|}{|\beta - 2\gamma + |\beta||z|} \right) \left| (P_\beta^{\alpha-1} g(z))^{(j)} \right|,
\]

which upon setting

\[|z| = r \text{ and } |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1),\]
leads us to the inequality
\[
\left| \left( (P^\alpha - 1 \beta f(z))^{(j)} \right) \right| \leq \frac{\Theta(\rho)}{(1 - r)(\beta - |2\gamma + \beta|r)} \left| (P^\alpha - 1 \beta g(z))^{(j)} \right|
\]
where
\[
\Theta(\rho) = -r\rho^2 + (1 - r)(\beta - |2\gamma + \beta|r)\rho + r
\]
takes its maximum value at \( \rho = 1 \), with \( r_2 = r_2(\beta, \gamma) \), where \( r_2(\beta, \gamma) \) is given by equation (2.2). Furthermore, if \( 0 \leq \rho \leq r_2(\beta, \gamma) \), then the function \( \theta(\rho) \) defined by
\[
\theta(\rho) = -\sigma\rho^2 + (1 - \sigma)(\beta - |2\gamma + \beta|\sigma)\rho + \sigma
\]
is an increasing function on the interval \( 0 \leq \rho \leq 1 \), so that
\[
\theta(\rho) \leq \theta(1) = (1 - \sigma)(\beta - |2\gamma + \beta|\sigma),
\]
\( 0 \leq \rho \leq 1; 0 \leq \sigma \leq r_1(\beta, \gamma) \). Hence upon setting \( \rho = 1 \) in (2.14), we conclude that (2.1) of Theorem 2.1 holds true for \( |z| \leq r_1(\beta, \gamma) \), where \( r_1(\beta, \gamma) \) is given by (2.2). This completes the proof of Theorem 2.1. \( \square \)

Setting \( \alpha = 1 \) in Theorem 2.1, we get

**Corollary 2.1.** Let the function \( f \in \Sigma \) and suppose that \( g \in S^1_{\beta} \). If \( (J_\beta f(z))^{(j)} \) is majorized by \( (J_\beta g(z))^{(j)} \) in \( \Delta^* \), then
\[
|f(z)| \leq |g(z)| \text{ for } |z| \leq r_2(\beta, \gamma),
\]
where
\[
r_2(\beta, \gamma) = \frac{k_2 - \sqrt{k_2^2 - 4\beta|\beta + 2\gamma|}}{2|\beta + 2\gamma|}
\]
and
\[
k_2 = \beta + 2 + |\beta + 2\gamma|, \quad (\beta > 0, j \in \mathbb{N}_0, \gamma \in \mathbb{C}\setminus\{0\}).
\]

Further putting \( \beta = 1 \) and \( \gamma = 1 - \eta, j = 0 \) in Corollary 2.1, we get

**Corollary 2.2.** Let the function \( f \in \Sigma \) and suppose that \( g \in S^1_{1} \). If \( (J_1 f(z)) \) is majorized by \( (J_1 g(z)) \) in \( \Delta^* \), then
\[
|f(z)| \leq |g(z)| \text{ for } |z| \leq r_3,
\]
where
\[
r_3 = \frac{3 - \eta - \sqrt{\eta^2 - 4\eta + 6}}{3 - \eta}.
\]
For \( \eta = 0 \), the above corollary reduces to the following result:

**Corollary 2.3.** Let the function \( f(z) \in \Sigma \) and suppose that \( g \in S^1_{1,0} \). If \( (J_1 f(z)) \) is majorized by \( (J_1 g(z)) \) in \( \Delta^* \), then
\[
|f(z)| \leq |g(z)| \text{ for } |z| \leq \frac{3 - \sqrt{6}}{3}.
\]
Acknowledgments. The first author is thankful to CSIR, New Delhi, India for awarding Emeritus Scientist under scheme No. 21(084)/10/EMR-II.

REFERENCES


S. P. Goyal Pranay Goswami
Department of Mathematics Department of Mathematics
University of Rajasthan AMITY University Rajasthan
Jaipur-302055 Jaipur-302002
India India
e-mail: somprg@gmail.com e-mail: pranaygoswami83@gmail.com

Received March 7, 2011