WŁODZIMIERZ M. MIKULSKI

The natural affinors on dual r-jet prolongations of bundles of 2-forms

Abstract. Let $J^r(\Lambda^2 T^*)M$ be the r-jet prolongation of $\Lambda^2 T^*M$ of an n-dimensional manifold $M$. For natural numbers $r$ and $n \geq 3$, all natural affinors on $(J^r(\Lambda^2 T^*)M)^*$ are the constant multiples of the identity affinor only.

0. Let us recall the following definitions (see e.g. [4]).

Let $F : \mathcal{M}f_n \rightarrow \mathcal{F} \mathcal{M}$ be a functor from the category $\mathcal{M}f_n$ of all $n$-dimensional manifolds and their local diffeomorphisms into the category $\mathcal{F} \mathcal{M}$ of fibered manifolds. Let $B$ be the base functor from the category of fibered manifolds to the category of manifolds.

A natural bundle over $n$-manifolds is a functor $F$ satisfying $B \circ F = \text{id}$ and the localization condition: for every inclusion of an open subset $i_U : U \rightarrow M$, $FU$ is the restriction $p_M^{-1}(U)$ of $p_M : FM \rightarrow M$ over $U$ and $Fi_U$ is the inclusion $p_M^{-1}(U) \rightarrow FM$.

An affinor $Q$ on a manifold $M$ is a tensor type $(1, 1)$, i.e. a linear morphism $Q : TM \rightarrow TM$ over $\text{id}_M$.

1991 Mathematics Subject Classification. 58A20, 53A55.

Key words and phrases. natural bundles, natural transformations, natural affinors.
A natural affinor on a natural bundle $F$ is a system of affinors $Q : TFM \to TFM$ on $FM$ for every $n$-manifold $M$ satisfying $TFf \circ Q = Q \circ Tf$ for every local diffeomorphism $f : M \to N$.

A connection on a fibre bundle $Y$ is an affinor $\Gamma : TY \to TY$ on $Y$ such that $\Gamma \circ \Gamma = \Gamma$ and $\text{im}(\Gamma) = VY$, the vertical bundle of $Y$.

A natural connection on a natural bundle $F$ is a system of connections $\Gamma : TFM \to TFM$ on $FM$ for every $n$-manifold $M$ which is (additionally) a natural affinor on $F$.

In [5] it was shown how natural affinors $Q$ on some natural bundles $FM$ can be used to study the torsion $\tau = [\Gamma, Q]$ of connections $\Gamma$ on the same bundles $FM$. That is why natural affinors have been classified in many papers, [1]-[3], [6]-[11].

In this paper one considers the natural bundle $F = (J^r(\Lambda^2T^*))^*M$ which associates to every $n$-manifold $M$ the vector bundle $(J^r(\Lambda^2T^*))^*M = (J^r(\Lambda^2T^*)M)^*$, where $J^r(\Lambda^2T^*)M = \{j^r_\omega \mid \omega \text{ is a 2-form on } M, x \in M\}$, and to every embedding $\phi : M \to N$ of $n$-manifolds the induced vector bundle mapping $(J^r(\Lambda^2T^*))^*\phi = (J^r(\Lambda^2T^*)\phi^{-1})^* : (J^r(\Lambda^2T^*)M)^* \to (J^r(\Lambda^2T^*)N)^*$, where the map $J^r(\Lambda^2T^*)\phi : J^r(\Lambda^2T^*)M \to J^r(\Lambda^2T^*)N$ is given by $j^r_\omega \to j^r_{\phi(x)}(\phi_*\omega)$.

For integers $r \geq 1$ and $n \geq 3$ we classify all natural affinors on $(J^r(\Lambda^2T^*))^*M$. We prove that every natural affinor $Q$ on $(J^r(\Lambda^2T^*))^*M$ is proportional to the identity affinor.

We note that the classification of natural affinors on $(J^rT^*M)^*$ is different. In [9] we proved that for $n \geq 2$ the vector space of all natural affinors on $(J^rT^*M)^*$ is 2-dimensional.

The above result shows that “torsion” of a connection $\Gamma$ on $(J^r(\Lambda^2T^*))^*M$ makes no sense because of $[\Gamma, \text{id}] = 0$.

The above result also shows that for integers $r \geq 1$ and $n \geq 3$ there are no natural connections on $(J^r(\Lambda^2T^*))^*$ over $n$-manifolds.

The usual coordinates on $\mathbb{R}^n$ are denoted by $x^i$ and $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$.

All manifolds and maps are assumed to be of class $C^\infty$.

1. We start with the classification of all linear natural transformations $A : T(J^r(\Lambda^2T^*))^*M \to (J^r(\Lambda^2T^*))^*M$ in the sense of [4] over $n$-manifolds $M$.

A natural transformation $T(J^r(\Lambda^2T^*))^* \to (J^r(\Lambda^2T^*))^*$ over $n$-manifolds is a system of fibered maps $A : T(J^r(\Lambda^2T^*))^*M \to (J^r(\Lambda^2T^*))^*M$ over $\text{id}_M$, for every $n$-manifold $M$ satisfying $(J^r(\Lambda^2T^*))^*f \circ A = A \circ T(J^r(\Lambda^2T^*))^*f$ for every local diffeo. $f : M \to N$. The linearity means that $A$ gives a linear map $T_y(J^r(\Lambda^2T^*))^*M \to (J^r(\Lambda^2T^*))^*M$ for any $y \in (J^r(\Lambda^2T^*))^*M$, $x \in M$. 


Proposition 1. If \( n \geq 3 \) and \( r \) are natural numbers then every linear natural transformation \( A : T(J'((\Lambda^2T^*)^*)) \rightarrow (J'((\Lambda^2T^*)^*))^* \) over \( n \)-manifolds is 0.

Proof. Every element from the fibre \((J'((\Lambda^2T^*)^*))_0^R^n\) is a linear combination of the \((j_0^R)(x^\alpha dx^i \wedge dx^j))^*\) for all \( \alpha \in (\mathbb{N} \cup \{0\})^n \) with \( |\alpha| \leq r \) and \( i, j = 1, \ldots, n, i < j \), where the \((j_0^R)(x^\alpha dx^i \wedge dx^j))^*\) form the basis dual to the \((j_0^R)(x^\alpha dx^i \wedge dx^j) \in (J'((\Lambda^2T^*)^*))_0^R^n\) for \( \alpha \) and \( i, j \) as beside.

Consider a linear natural transformation \( A : T(J'((\Lambda^2T^*)^*)) \rightarrow (J'((\Lambda^2T^*)^*))^* \) over \( n \)-manifolds.

Clearly, \( A \) is uniquely determined by the values \( \langle A(u), J_0^R(x^\alpha dx^i \wedge dx^j) \rangle \in \mathbb{R} \) for \( u \in (T(J'((\Lambda^2T^*)^*))^R_n)_0 = \mathbb{R}^n \times (V(J'((\Lambda^2T^*)^*))^R_n)_0 = \mathbb{R}^n \times (J'((\Lambda^2T^*)^*))^R_n \mathbb{R}^n \times (J'((\Lambda^2T^*)^*))^R_n \) for \( \alpha \in (\mathbb{N} \cup \{0\})^n \) with \( |\alpha| \leq r \) and \( i, j = 1, \ldots, n, i < j \), where \( \mathbb{R}^n \) is the standard trivialization and the canonical identification.

Since \( A \) is invariant with respect to the coordinate permutations, \( A \) is uniquely determined by the values \( \langle A(u), j_0^R(x^\alpha dx^i \wedge dx^j) \rangle \), where \( u \) and \( \alpha \) are as above.

If \( |\alpha| \geq 1 \), then the local diffeomorphisms \( \varphi_\alpha = (x^1, x^2, x^3, x^4, \ldots, x^n)^{-1} \) sends \( j_0^R(x^\alpha dx^i \wedge dx^j) \) into \( j_0^R(x^\alpha dx^i \wedge dx^j) + j_0^R(x^\alpha dx^i \wedge dx^j) \). Then by the invariance of \( A \) with respect to the \( \varphi_\alpha \)'s, \( A \) is uniquely determined by the values \( \langle A(u), j_0^R(x^\alpha dx^i \wedge dx^j) \rangle \in \mathbb{R} \) and \( \langle A(u), j_0^R(x^\alpha dx^i \wedge dx^j) \rangle \in \mathbb{R} \), where \( u \in (T(J'((\Lambda^2T^*)^*))^R_n)_0 = \mathbb{R}^n \times (J'((\Lambda^2T^*)^*))^R_n \times (J'((\Lambda^2T^*)^*))^R_n \).

The proof of Proposition 1 will be complete after proving that \( \langle A(u), j_0^R(x^\alpha dx^i \wedge dx^j) \rangle = 0 \) and \( \langle A(u), j_0^R(x^\alpha dx^i \wedge dx^j) \rangle = 0 \) for any \( u \in (T(J'((\Lambda^2T^*)^*))^R_n)_0 = \mathbb{R}^n \times (J'((\Lambda^2T^*)^*))^R_n \times (J'((\Lambda^2T^*)^*))^R_n \). We will prove these conditions in Lemmas 1 — 6.

At first we study the values \( \langle A(u), j_0^R(x^\alpha dx^i \wedge dx^j) \rangle \).

**Lemma 1.** There exist the numbers \( \lambda, \mu, \nu \in \mathbb{R} \) such that

\[
\langle A(u), j_0^R(x^\alpha dx^i \wedge dx^j) \rangle = \lambda u_1^1 u_2^2 + \mu u_2(0,1,2) + \nu u_3(0,1,2)
\]

for every \( u = (u_1, u_2, u_3) \in \mathbb{R}^n \times (J'((\Lambda^2T^*)^*))^R_n \times (J'((\Lambda^2T^*)^*))^R_n \), where \( u_1 = (u_1^1, \ldots, u_n^1) \in \mathbb{R}^n \), \( u_\tau, \alpha, i, j \) is the coefficient of \( u_\tau \in (J'((\Lambda^2T^*)^*))^R_n \) on \( (j_0^R(x^\alpha dx^i \wedge dx^j))^\tau, \gamma = 2, 3, \) \( (0) = (0, \ldots, 0) \in (\mathbb{N} \cup \{0\})^n \).

**Proof of Lemma 1.** By the naturality of \( A \) with respect to the homotheties \( a_t = (t^1 x^1, \ldots, t^n x^n) \) for \( t = (t^1, \ldots, t^n) \in \mathbb{R}^n_+ \),

\[
\langle A(T(J'((\Lambda^2T^*)^*))^*(a_t)(u), j_0^R(x^\alpha dx^i \wedge dx^j) \rangle = t^1 t^2 \langle A(u), j_0^R(x^\alpha dx^i \wedge dx^j) \rangle
\]

for any \( t = (t^1, \ldots, t^n) \in \mathbb{R}^n_+ \). For \( t \in \mathbb{R}^n_+ \), \( i, j = 1, \ldots, n, i < j \) and \( \alpha \in (\mathbb{N} \cup \{0\})^n \) we have \( T(J'((\Lambda^2T^*)^*))^*(a_t)((j_0^R(x^\alpha dx^i \wedge dx^j))^\alpha = t^{\alpha + i + j} (j_0^R(x^\alpha dx^i \wedge dx^j))^\alpha \). Then the lemma follows from the homogeneous function theorem, [4]. \( \square \)
Lemma 2. We have $\lambda = \mu = \nu = 0$.

Proof of Lemma 2. Since $\langle A(u_1, u_2, u_3), j_0^\nu(dx^1 \wedge dx^2) \rangle$ is linear in $(u_1, u_3)$ for $u_2$, we have $\lambda = \mu = 0$. Then (in particular) we have

\[
(2) \quad \langle A(\partial^C_1 |_w), j_0^\nu(dx^1 \wedge dx^2) \rangle = \langle A(e_1, w, 0), j_0^\nu(dx^1 \wedge dx^2) \rangle = 0
\]

for $w \in (J^*(A^2T^*))_0^c \mathbb{R}^n$, where $(\cdot)^C$ is the complete lift.

To prove $\nu = 0$ it is sufficient to show that

\[
\langle A(0, 0, (j_0^\nu(dx^1 \wedge dx^2))^*), j_0^\nu(dx^1 \wedge dx^2) \rangle = 0.
\]

But we have

\[
0 = \langle A((x^1)^{r+1} \partial_1)^C_0^c |_w, j_0^\nu(dx^1 \wedge dx^2) \rangle
\]

\[
(3) \quad = (r + 1) \langle A(0, w, (j_0^\nu(dx^1 \wedge dx^2))^* + \ldots, j_0^\nu(dx^1 \wedge dx^2) \rangle
\]

\[
= (r + 1) \langle A(0, 0, (j_0^\nu(dx^1 \wedge dx^2))^*), j_0^\nu(dx^1 \wedge dx^2) \rangle,
\]

where $w = (j_0^\nu((x^1)^r dx^1 \wedge dx^2))^*$ and the dots mean the linear combination of the $(j_0^\nu(x^\alpha dx^1 \wedge dx^2))^*$ with $(j_0^\nu(x^\alpha dx^1 \wedge dx^2))^* \neq (j_0^\nu(dx^1 \wedge dx^2))^*$.

Let us explain (3).

Let $\varphi_t$ be the flow of $(x^1)^{r+1} \partial_1$. We have

\[
\langle ((x^1)^{r+1} \partial_1)^C_0^c |_w, j_0^\nu(dx^1 \wedge dx^2) \rangle
\]

\[
= \langle \frac{d}{dt} \big|_{t=0} (J^*(A^2T^*))_0^c(\varphi_t)(w), j_0^\nu(dx^1 \wedge dx^2) \rangle
\]

\[
= \frac{d}{dt} \big|_{t=0} \langle (J^*(A^2T^*))_0^c(\varphi_t)(w), j_0^\nu(dx^1 \wedge dx^2) \rangle
\]

\[
= \frac{d}{dt} \big|_{t=0} \langle w, j_0^\nu((\varphi_{-t})^*(dx^1 \wedge dx^2)) \rangle
\]

\[
= \langle w, j_0^\nu(L_{(x^1)^{r+1} \partial_1} dx^1 \wedge dx^2) \rangle
\]

\[
= (r + 1) \langle w, j_0^\nu((x^1)^r dx^1 \wedge dx^2) \rangle = r + 1.
\]

Then $((x^1)^{r+1} \partial_1)^C_0^c |_w = (r + 1)(j_0^\nu(dx^1 \wedge dx^2))^* + \ldots$ under the canonical isomorphism $V_w((J^*(A^2T^*))^* \mathbb{R}^n) \cong (J^*(A^2T^*))_0^c \mathbb{R}^n$, i.e. $\langle A((x^1)^{r+1} \partial_1)^C_0^c |_w, j_0^\nu(dx^1 \wedge dx^2) \rangle = (r + 1) \langle A(0, w, (j_0^\nu(dx^1 \wedge dx^2))^* + \ldots, j_0^\nu(dx^1 \wedge dx^2) \rangle$.

The equality $(r + 1) \langle A(0, w, (j_0^\nu(dx^1 \wedge dx^2))^* + \ldots, j_0^\nu(dx^1 \wedge dx^2) \rangle = (r + 1) \langle A(0, 0, (j_0^\nu(dx^1 \wedge dx^2))^*), j_0^\nu(dx^1 \wedge dx^2) \rangle$ is clear because of (1) and $\mu = 0$. 

We can prove the equality \( 0 = \langle A((x^1)^{r+1}\partial_1)^C_{|w}), j_0^r(dx^1 \wedge dx^2) \rangle \) as follows. Vector fields \( \partial_1 + (x^1)^{r+1}\partial_1 \) and \( \partial_1 \) have the same \( r \)-jets at 0. Then by [11], there exists a diffeomorphism \( \varphi : \mathbb{R}^n \to \mathbb{R}^n \) such that \( j_0^{r+1}\varphi = \text{id} \) and \( \varphi, \partial_1 = \partial_1 + (x^1)^{r+1}\partial_1 \) near 0. Clearly, \( \varphi \) preserves \( j_0^r(dx^1 \wedge dx^2) \) because of the jet argument. Then, by the naturality of \( A \) with respect to \( \varphi \), it follows from (2) that

\[
\langle A((\partial_1 + (x^1)^{r+1}\partial_1)^C_{|w}), j_0^r(dx^1 \wedge dx^2) \rangle = 0
\]

for any \( w \in (J^r(\Lambda^2T^*))_0^R \mathbb{R}^n \). Now, applying the linearity of \( A \), we end the proof of the equality. \( \square \)

Now, we study the values \( \langle A(u), j_0^r(x^3dx^1 \wedge dx^2) \rangle = 0 \).

**Lemma 3.** There exist the numbers \( a, b, c, e, f, g \in \mathbb{R} \) such that

\[
\langle A(u), j_0^r(x^3dx^1 \wedge dx^2) \rangle = au_1^1u_2,(0),2,3 + bu_2^2u_2,(0),1,3 + cu_1^3u_2,(0),1,2 + eu_3,e_1,2,3 + fu_3,e_2,1,3 + gu_3,e_3,1,2
\]

for any \( u = (u_1, u_2, u_3) \), where \( u_1 = (u_1^1, ..., u_1^n) \in \mathbb{R}^n \), \( u_2, u_3 \in (J^r(T^* \wedge T^*))_0^R \mathbb{R}^n \), \( u_{\tau, o, i, j} \) is as in Lemma 1 and \( e_i = (0, ..., 1, 0, ..., 0) \in (\mathbb{N} \cup \{0\})^n \), 1 in \( i \)-position.

**Proof of Lemma 3.** The proof is similar to the proof of Lemma 1. We apply the naturality of \( A \) with respect to the homotheties \( a_t = (t^1x^1, ..., t^n x^n) \) for \( t = (t^1, ..., t^n) \in \mathbb{R}_+^n \), the homogeneous function theorem and the linearity of \( A \). \( \square \)

To prove \( g = f = e = a = b = c = 0 \) we shall use the following

**Lemma 4.** For every \( u \in (T(J^r(\Lambda^2T^*))_0^R \mathbb{R}^n) \) we have

\[
\langle A(u), j_0^r(x^3dx^1 \wedge dx^2) \rangle = \langle A(u'), j_0^r(x^3dx^1 \wedge dx^2) \rangle
\]

where \( u' \) is the image of \( u \) by \( (x^2, x^3, x^1) \times \text{id}_{\mathbb{R}^{n-3}} \).

**Proof of Lemma 4.** We consider \( u \in (T(J^r(\Lambda^2T^*))_0^R \mathbb{R}^n) \). Let \( u' \) be the image of \( u \) by \( (x^1 + x^1 x^3, x^2, ..., x^n) \). By Lemma 2 we have \( \lambda = \mu = \nu = 0 \), i.e. \( \langle A(u), j_0^r(dx^1 \wedge dx^2) \rangle = \langle A(u), j_0^r(dx^1 \wedge dx^2) \rangle = 0 \). Then by the invariance of \( A \) with respect to \( (x^1 + x^1 x^3, x^2, ..., x^n)^{−1} \) we get

\[
0 = \langle A(u), j_0^r(dx^1 \wedge dx^2) \rangle + \langle A(u), j_0^r(x^3dx^1 \wedge dx^2) \rangle - \langle A(u), j_0^r(x^1dx^2 \wedge dx^3) \rangle
\]

as \( (x^1 + x^1 x^3, x^2, ..., x^n)^{−1} \) sends \( j_0^r(dx^1 \wedge dx^2) \) into \( j_0^r(dx^1 \wedge dx^2) + j_0^r(x^3dx^1 \wedge dx^2) - j_0^r(x^1dx^2 \wedge dx^3) \). Hence \( \langle A(u), j_0^r(x^3dx^1 \wedge dx^2) \rangle = \langle A(u), j_0^r(x^1dx^2 \wedge dx^3) \rangle \). Therefore we have (5) because \( (x^2, x^3, x^1) \times \text{id}_{\mathbb{R}^{n-3}} \) sends \( j_0^r(x^1dx^2 \wedge dx^3) \) into \( j_0^r(x^3dx^1 \wedge dx^2) \). \( \square \)
Lemma 5. We have $g = f = e = 0$.

Proof of Lemma 5. We have to show

$$\langle A(0, 0, (j_0^w(x^3 dx^1 \wedge dx^2))^*), j_0^w(x^3 dx^1 \wedge dx^2) \rangle$$

$$= \langle A(0, 0, -(j_0^w(x^2 dx^1 \wedge dx^3))^*), j_0^w(x^3 dx^1 \wedge dx^2) \rangle$$

$$= \langle A(0, 0, (j_0^w(x^1 dx^2 \wedge dx^3))^*), j_0^w(x^3 dx^1 \wedge dx^2) \rangle = 0.$$

We see that $(x^2, x^3, x^1) \times \text{id}_{R^n} \rightarrow R^n$ sends $(j_0^w(x^3 dx^1 \wedge dx^2))^*$ into $-(j_0^w(x^2 dx^1 \wedge dx^3))^*$ and $-(j_0^w(x^1 dx^2 \wedge dx^3))^*$ into $(j_0^w(x^1 dx^2 \wedge dx^3))^*$. Then due to (5) it suffices to verify that $\langle A(0, 0, (j_0^w(x^3 dx^1 \wedge dx^2))^*), j_0^w(x^3 dx^1 \wedge dx^2) \rangle = 0$.

But we have

$$0 = \langle A(((x^1)^r \partial_1)_w), j_0^w(x^3 dx^1 \wedge dx^2) \rangle$$

$$= r(\langle A(0, w, (j_0^w(x^3 dx^1 \wedge dx^2))^*), j_0^w(x^3 dx^1 \wedge dx^2) \rangle)$$

$$= r(\langle A(0, 0, (j_0^w(x^3 dx^1 \wedge dx^2))^*), j_0^w(x^3 dx^1 \wedge dx^2) \rangle),$$

where $w = (j_0^w(x^3(x^1)^{-1} dx^1 \wedge dx^2))^* \in (J^r(A^2 T^*))^n R^n$.

Let us explain (6).

That $\langle A(0, w, (j_0^w(x^3 dx^1 \wedge dx^2))^*), j_0^w(x^3 dx^1 \wedge dx^2) \rangle = \langle A(0, 0, (j_0^w(x^3 dx^1 \wedge dx^2))^*), j_0^w(x^3 dx^1 \wedge dx^2) \rangle$ is clear, see (4).

We can prove $0 = \langle A(((x^1)^r \partial_1)_w), j_0^w(x^3 dx^1 \wedge dx^2) \rangle$ as follows. Vector fields $\partial_1 + (x^1)^r \partial_1$ and $\partial_1$ have the same $r-1$-jets at 0. Then by [11] there exists a diffeomorphism $\varphi = \varphi_1 \times \text{id}_{R^n-1} : R^n \rightarrow R \times R^{n-1}$ such that $\varphi : R \rightarrow R$, $j_0^w \varphi = \text{id}$ and $\varphi_1 \partial_1 = \partial_1 + (x^1)^r \partial_1$ near 0. Let $\varphi^{-1}$ send $w$ into $\tilde{w}$. Then $\tilde{w}$ is the linear combination of the $(j_0^w(x^a dx^i \wedge dx^j))^* \in (J^r(A^2 T^*))^n R^n$ for $|a| \geq 1$ and $i, j = 1, ..., n$ with $i < j$. (For, $\langle \tilde{w}, j_0^w(dx^i \wedge dx^j) \rangle = \langle w, j_0^w(dx^i \circ \varphi^{-1} \wedge dx^j \circ \varphi^{-1}) \rangle = 0$.) Then, by (4), $\langle A(\partial_1^w)_w, j_0^w(x^3 dx^1 \wedge dx^2) \rangle = \langle A(\partial_1, w, 0), j_0^w(x^3 dx^1 \wedge dx^2) \rangle = 0$. Clearly, $\varphi$ preserves $j_0^w(x^3 dx^1 \wedge dx^2)$. Then, using the naturality of $A$ with respect to $\varphi$ we get $\langle A((\partial_1 + (x^1)^r \partial_1)_w), j_0^w(x^3 dx^1 \wedge dx^2) \rangle = 0$. Now, applying the linearity of $A$, we end the proof of equality.

Using the flow argument one can prove $\langle A(((x^1)^r \partial_1)_w), j_0^w(x^3 dx^1 \wedge dx^2) \rangle = r(\langle A(0, w, (j_0^w(x^3 dx^1 \wedge dx^2))^*), j_0^w(x^3 dx^1 \wedge dx^2) \rangle)$ as follows. For any $\alpha \in (\mathbb{N} \cup \{0\})^n$ with $|\alpha| \leq r$ and any $i, j = 1, ..., n$ with $i < j$ we have

$$\langle ((x^1)^r \partial_1)_w, j_0^w(x^\alpha dx^i \wedge dx^j) \rangle = \langle w, j_0^w(L_{x^1} \partial_1 x^\alpha dx^i \wedge dx^j) \rangle = \langle w, \alpha_1 j_0^w((x^1)^{-1} x^\alpha dx^i \wedge dx^j) \rangle + \langle w, j_0^w(x^\alpha \delta_1^w r(x^1)^{-1} dx^i \wedge dx^j) \rangle.$$

Since $w = (j_0^w(x^3(x^1)^{-1} dx^1 \wedge dx^2))^*$, the sum is equal to $r$ if $\alpha = e_3$ and $(i, j) = (1, 2)$ and equal to 0 in the other cases. Hence $((x^1)^r \partial_1)_w = r(\langle A(0, w, (j_0^w(x^3 dx^1 \wedge dx^2))^* \rangle) \in V w$. This ends the proof of $\langle A(((x^1)^r \partial_1)_w), j_0^w(x^3 dx^1 \wedge dx^2) \rangle = r(\langle A(0, w, (j_0^w(x^3 dx^1 \wedge dx^2))^*), j_0^w(x^3 dx^1 \wedge dx^2) \rangle).$
Lemma 6. We have \(a = b = c = 0\).

**Proof of Lemma 6.** By (5), similarly as for \(e = f = g = 0\), it is sufficient to prove that \(c = 0\), i.e. \(A(\partial^C_3 |_{(j^C_0(dx^1 \wedge dx^2))^r})_r j^C_0(x^3 dx^1 \wedge dx^2) = 0\). But we have

\[
0 = \langle A(\partial^C_3 |_{(j^C_0(dx^1 \wedge dx^2))^r})_r j^C_0(x^3 dx^1 \wedge dx^2) \rangle_r = \langle A(\partial^C_3 |_{(j^C_0(dx^1 \wedge dx^2))^r})_r j^C_0(x^3 dx^1 \wedge dx^2) \rangle_r = \langle A(\partial^C_3 |_{(j^C_0(dx^1 \wedge dx^2))^r})_r j^C_0(x^3 dx^1 \wedge dx^2) \rangle_r,
\]

where the dots denote the linear combination of the \((j^C_0(x^a dx^1 \wedge dx^2))^r \neq (j^C_0(dx^1 \wedge dx^2))^r\) for \(|a| \leq r\) and \(i, j = 1, \ldots, n\), \(i < j\).

Let us explain (7).

The equality \(0 = \langle A(\partial^C_3 |_{(j^C_0(dx^1 \wedge dx^2))^r})_r j^C_0(x^3 dx^1 \wedge dx^2) \rangle_r = \langle A(\partial^C_3 |_{(j^C_0(dx^1 \wedge dx^2))^r})_r j^C_0(x^3 dx^1 \wedge dx^2) \rangle_r\) follows from (4). Similarly, from (4) we obtain \(\langle A(\partial^C_3 |_{(j^C_0(dx^1 \wedge dx^2))^r})_r j^C_0(x^3 dx^1 \wedge dx^2) \rangle_r = \langle A(\partial^C_3 |_{(j^C_0(dx^1 \wedge dx^2))^r})_r j^C_0(x^3 dx^1 \wedge dx^2) \rangle_r\).

We consider the local diffeomorphism \(\varphi = (x^1 + \frac{1}{r+1}(x^1)^{r+1}, x^2, \ldots, x^n)^{-1}\). We see that \(\varphi^{-1}\) preserves \((j^C_0((x^1)^r dx^1 \wedge dx^2))^r \) and \(\partial_3\). Moreover, we see that \(\varphi^{-1}\) sends \((j^C_0((x^1)^r dx^1 \wedge dx^2))^r \) into \((j^C_0(dx^1 \wedge dx^2))^r \), where the dots are as above, because of \(|(j^C_0((x^1)^r dx^1 \wedge dx^2))^r, j^C_0(\varphi_*(dx^1 \wedge dx^2))| = 1\). Now, by the invariance of \(A\) with respect to \(\varphi^{-1}\) we get \(\langle A(\partial^C_3 |_{(j^C_0(dx^1 \wedge dx^2))^r})_r j^C_0(x^3 dx^1 \wedge dx^2) \rangle_r = \langle A(\partial^C_3 |_{(j^C_0(dx^1 \wedge dx^2))^r})_r j^C_0(x^3 dx^1 \wedge dx^2) \rangle_r\) \(\Box\).  

The proof of Proposition 1 is complete. \(\Box\)

2. The tangent map \(T_\pi: T(J^r(\Lambda^2T^*))^*M \to TM\) of the bundle projection \(\pi: (J^r(\Lambda^2T^*))^*M \to M\) defines a linear natural transformation \(T_\pi: T(J^r(\Lambda^2T^*))^* \to T\) over \(n\)-manifolds. (The definition of linear natural transformations \(T(J^r(\Lambda^2T^*))^* \to T\) over \(n\)-manifolds is similar to the one of Section 1.)

**Proposition 2.** If \(r\) and \(n \geq 2\) are natural numbers, then every linear natural transformation \(B: T(J^r(\Lambda^2T^*))^* \to T\) over \(n\)-manifolds is proportional to \(T_\pi\).

**Proof.** Due to similar arguments as in the proof of Proposition 1, \(B\) is uniquely determined by the values \(\langle B(u), d_0 x^1 \rangle\) for \(u \in (T(J^r(\Lambda^2T^*))^* \mathbb{R}^n)_0 \equiv \mathbb{R}^n \times (J^r(\Lambda^2T^*))_0 \mathbb{R}^n \times (J^r(\Lambda^2T^*))_0 \mathbb{R}^n\).

By the naturality of \(B\) with respect to the homotheties \((t^1 x^1, \ldots, t^n x^n)\) for \(t \in \mathbb{R}_+^n\) and the homogeneous function theorem we deduce that \(\langle B(\cdot), dx^1 \rangle = x^1 \circ p_1\), where \(p_1: \mathbb{R}^n \times (J^r(\Lambda^2T^*))_0 \mathbb{R}^n \times (J^r(\Lambda^2T^*))_0 \mathbb{R}^n \to \mathbb{R}^n\) is the canonical projection.

Then the vector space of all \(B\) as above is 1-dimensional. \(\Box\)
3. The main result of this paper is the following theorem.

**Theorem 1.** If \( n \geq 3 \) and \( r \) are natural numbers, then every natural affinor \( Q \) on \( (J^r(\Lambda^2T^*))^* \) over \( n \)-manifolds is a constant multiple of \( \text{id} \).

**Proof.** Let \( Q : T(J^r(\Lambda^2T^*))^*M \to T(J^r(\Lambda^2T^*))^*M \) be a natural affinor on \( (J^r(\Lambda^2T^*))^* \) over \( n \)-manifolds. Then \( B = T\pi \circ Q : T(J^r(\Lambda^2T^*))^* \to T \) is a linear natural transformation. By Proposition 2, \( B = T\pi \circ Q = \lambda T\pi \) for some \( \lambda \). Clearly, \( T\pi \circ \text{id} = T\pi \). Then \( Q - \lambda \text{id} \) is an affinor of vertical type. Now, applying Proposition 1 we deduce that \( Q - \lambda \text{id} \) is the zero affinor. \( \Box \)

From Theorem 1 we obtain immediately

**Corollary 1.** If \( n \geq 3 \) and \( r \) are natural numbers, then there is no natural connection on \( (J^r(\Lambda^2T^*))^* \) over \( n \)-manifolds.

**References**


Institute of Mathematics  
Kraków, Reymonta 4, Poland  
e-mail: mikulski@im.uj.edu.pl

W.M. Mikulski  
Jagiellonian University  
Institute of Mathematics received February 7, 2001