Liftings of horizontal 1-forms
to some vector bundle functors
on fibered fibered manifolds

Abstract. Let \( F : \mathcal{F}^2 \mathcal{M} \to \mathcal{VB} \) be a vector bundle functor on fibered fibered manifolds. We classify all natural operators

\[
T \mathcal{F}^2 \mathcal{M}^{proj} \mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2} \to T^{(0,0)}(F|_{\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}})^* 
\]

transforming \( \mathcal{F}^2 \mathcal{M} \)-projectable vector fields on \( Y \) to functions on the dual bundle \( (FY)^* \) for any \((m_1, m_2, n_1, n_2)\)-dimensional fibered fibered manifold \( Y \). Next, under some assumption on \( F \) we study natural operators

\[
T \mathcal{F}^2 \mathcal{M}^{hor} \mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2} \to T^*(F|_{\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}})^* 
\]

lifting \( \mathcal{F}^2 \mathcal{M} \)-horizontal 1-forms on \( Y \) to 1-forms on \((FY)^* \) for any \( Y \) as above. As an application we classify natural operators

\[
T^* \mathcal{F}^2 \mathcal{M}^{hor} \mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2} \to T^*(F|_{\mathcal{F}^2 \mathcal{M}_{m_1, m_2, n_1, n_2}})^* 
\]

for a particular vector bundle functor \( F \) on fibered fibered manifolds.

0. Introduction. The concept of fibered fibered manifolds was introduced in [16]. Fibered fibered manifolds are fibered surjective submersions between

\[2000\ Mathematics\ Subject\ Classification.\ 58A20.\]

\[Key\ words\ and\ phrases.\ \text{Fibered\ fibered\ manifold, bundle\ functor, natural\ operator.}\]
fibered manifolds. They appear naturally in differential geometry if we consider transverse natural bundles in the sense of R. Wolak [18]. Product preserving bundle functors on fibered fibered manifolds are studied in [17].

In this paper we consider the following categories over manifolds: the category \( \mathcal{M}f \) of manifolds and maps, the category \( \mathcal{M}f_m \) of \( m \)-dimensional manifolds and embeddings, the category \( \mathcal{F}M \) of fibered manifolds and fibered maps, the category \( \mathcal{F}M_{m,n} \) of fibered manifolds of dimension \((m, n)\) (i.e. with \( m \)-dimensional bases and \( n \)-dimensional fibers) and fibered embeddings, the category \( \mathcal{F}^2M \) of fibered fibered manifolds and their fibered fibered maps, the category \( \mathcal{F}^2M_{m_1,m_2,n_1,n_2} \) of fibered fibered manifolds of dimension \((m_1, m_2, n_1, n_2)\) and fibered fibered embeddings, the category \( \mathcal{VB} \) of vector bundles and vector bundle maps.

The notions of bundle functors and natural operators can be found in the fundamental monograph [4].

In [7], given a vector bundle functor \( F : \mathcal{M}f \to \mathcal{VB} \) we classified all natural operators \( A : T_{\mathcal{M}f_m} \to T^{(0,0)}(F|_{\mathcal{M}f_m})^* \) transforming vector fields \( Z \) on \( m \)-dimensional manifolds \( M \) into functions \( A(Z) : (FM)^* \to \mathbb{R} \) on the dual vector bundle \( (FM)^* \) and proved that every natural operator \( B : T_{\mathcal{M}f_m}^* \to T^*(F|_{\mathcal{M}f_m})^* \) transforming 1-forms \( \omega \) from \( m \)-manifolds \( M \) into 1-forms \( B(\omega) \) on \( (FM)^* \) is of the form \( B(\omega) = a\omega^V + \lambda \) for some uniquely determined canonical map \( a : (FM)^* \to \mathbb{R} \) and some canonical 1-form \( \lambda \) on \( (FM)^* \). These results were generalizations of [1],[6].

In [8], we studied similar problems for a vector bundle functor \( F : \mathcal{F}M \to \mathcal{VB} \) on fibered manifolds instead of on manifolds. For natural numbers \( m, n \) we classified all natural operators \( A : T_{\mathcal{F}M_{m,n}}^{\proj} \to T^{(0,0)}(F|_{\mathcal{F}M_{m,n}})^* \) transforming projectable vector fields \( Z \) on \( (m, n) \)-dimensional fibered manifolds \( Y \) into functions \( A(Z) : (FY)^* \to \mathbb{R} \) on the dual vector bundle \( (FY)^* \) and proved (under some assumption on \( F \)) that every natural operator \( B : T_{\mathcal{F}M_{m,n}}^{\hor} \to T^*(F|_{\mathcal{F}M_{m,n}})^* \) transforming horizontal 1-forms \( \omega \) from \( (m, n) \)-dimensional fibered manifolds \( Y \) into 1-forms \( B(\omega) \) on \( (FY)^* \) is of the form \( B(\omega) = a\omega^V + \lambda \) for some uniquely determined canonical map \( a : (FY)^* \to \mathbb{R} \) and some canonical 1-form \( \lambda \) on \( (FY)^* \).

In the present paper we study similar problems for a vector bundle functor \( F : \mathcal{F}^2M \to \mathcal{VB} \) on fibered fibered manifolds instead of on manifolds or on fibered manifolds. For natural numbers \( m_1, m_2, n_1 \) and \( n_2 \) we classify all natural operators \( A : T_{\mathcal{F}^2M_{m_1,m_2,n_1,n_2}}^{\proj} \to T^{(0,0)}(F|_{\mathcal{F}^2M_{m_1,m_2,n_1,n_2}})^* \) transforming \( \mathcal{F}^2M \)-projectable vector fields \( Z \) on \( (m_1, m_2, n_1, n_2) \)-dimensional fibered fibered manifolds \( Y \) into functions \( A(Z) : (FY)^* \to \mathbb{R} \) on the dual vector bundle \( (FY)^* \) and prove (under an assumption on \( F \)) that every natural operator
transferring \( \mathcal{F}_2^1 \)-horizontal 1-forms \( \omega \) from \((m_1, m_2, n_1, n_2)\)-dimensional fibered fibered manifolds \( Y \) into a 1-form \( B(\omega) \) on \((FY)^*\) is of the form \( B(\omega) = a\omega^V + \lambda \) for some uniquely determined canonical map \( a : (FY)^* \to \mathbb{R} \) and some canonical 1-form \( \lambda \) on \((FY)^*\). As an application we classify all natural operators transforming fibered fibered manifolds into fibered fibered manifolds of dimension \((m_1, m_2, n_1, n_2)\) for a particular vector bundle functor \( F \) on fibered fibered manifolds.

Natural operators lifting functions, vector fields and 1-form to some bundle functors were used practically in all papers in which problem of prolongations of geometric structures was studied, e.g. [19]. That is why such natural operators have been classified, see [1], [3]—[14], etc.

From now on the usual coordinates on \( \mathbb{R}^{m_1, m_2, n_1, n_2} = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) will be denoted by \( x^1, \ldots, x^{m_1}, y^1, \ldots, y^{m_2}, w^1, \ldots, w^{n_1}, v^1, \ldots, v^{n_2} \).

All manifolds are assumed to be finite dimensional and smooth, i.e. of class \( C^\infty \). Maps between manifolds are assumed to be smooth.

1. Fibered fibered manifolds. The concept of fibered fibered manifolds was introduced in [16]. A fibered fibered manifold is a fibered surjective submersion \( \pi : Y \to X \) between fibered manifolds, i.e. a surjective submersion which sends fibers into fibers such that the restricted and corestricted maps are submersions. (We will write \( Y \) instead of \( \pi \) if \( \pi \) is clear.) If \( \pi : Y \to X \) is another fibered fibered manifold, a morphism \( \pi : Y \to X \) is a fibered map \( f : Y \to X \) such that there is a fibered map \( f_o : X \to X \) with \( \pi \circ f = f_o \circ \pi \).

Thus all fibered fibered manifolds form a category which will be denoted by \( \mathcal{F}_2^1 \). This category is over manifolds, local and admissible in the sense of [4].

Fibered fibered manifolds appear naturally in differential geometry. To see this, we consider a fibered manifold \( p : X \to M \). Then \( X \) has the foliated structure \( \mathcal{F} \) by fibres. Its normal bundle \( Y = N(X, \mathcal{F}) = TX/T\mathcal{F} \) has the induced foliation, [18]. This foliation is by the fibered manifold \( [Tp] : Y \to TM \), the quotient map of the differential \( Tp : TX \to TM \). Clearly, the projection \( \pi : Y \to X \) of the normal bundle is a fibered fibered manifold. Considering other transverse natural bundles in the sense of [18] instead of \( N(X, \mathcal{F}) \), we can produce many fibered fibered manifolds.

A fibered fibered manifold \( \pi : Y \to X \) of dimension \((m_1, m_2, n_1, n_2)\) if fibered manifold \( Y \) has dimension \((m_1 + n_1, m_2 + n_2)\) and fibered manifold \( X \) has dimension \((m_1, m_2)\). All fibered fibered manifolds of dimension \((m_1, m_2, n_1, n_2)\) and their local isomorphisms form a subcategory \( \mathcal{F}_2^1 \mathcal{M}_{m_1, m_2, n_1, n_2} \subset \mathcal{F}_2^1 \). Every \( \mathcal{F}_2^1 \mathcal{M}_{m_1, m_2, n_1, n_2} \)-object is locally isomorphic to \( \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \), the projection, where \( \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) (or \( \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \)) is over \( \mathbb{R}^{m_1} \times \mathbb{R}^{n_1} \) (or \( \mathbb{R}^{m_1} \)).
2. A classification of natural operators $T_{\mathcal{F}^2\mathcal{M} \twoheadrightarrow \mathcal{F}^2\mathcal{M}_{m_1,n_1,n_2}}$. Let $F : \mathcal{F}^2\mathcal{M} \to \mathcal{Y}\mathcal{B}$ be a vector bundle functor. Let $m_1, m_2, n_1, n_2 \in \mathbb{N}$. In this section we classify natural operators $A : T_{\mathcal{F}^2\mathcal{M} \twoheadrightarrow \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^{(0,0)}(F|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})^*$ transforming $\mathcal{F}^2\mathcal{M}$-projectable vector fields $Z$ on $(m_1, m_2, n_1, n_2)$-dimensional fibered manifolds $Y$ into functions $A(Z) : (FY)^* \to \mathbb{R}$ on the dual vector bundle $(FY)^*$.

We recall (see [16]) that a $\mathcal{F}^2\mathcal{M}$-projectable vector field on a fibered manifold $\pi : Y \to X$ is a projectable vector field on fibered manifold $Y$ such that there exists a $\pi$-related (with $Z$) projectable vector field $Z_0$ on fibered manifold $X$. If $Z$ is $\mathcal{F}^2\mathcal{M}$-projectable then its flow is formed by local $\mathcal{F}^2\mathcal{M}$-isomorphisms.

Example 1. Let $v \in F_0(\mathbb{R}^{1,0,0})$. Consider a $\mathcal{F}^2\mathcal{M}$-projectable vector field $Z$ on an $(m_1, m_2, n_1, n_2)$-dimensional fibered manifold $\pi : Y \to X$. We define $A^v(Z) : (FY)^* \to \mathbb{R}$, $A^v(Z)_\eta = \langle\eta, F(\Phi^y_v)(v)\rangle$, $\eta \in (F_0Y)^*$, $y \in Y_x$, $x \in X$. Here $\Phi^y_v : (\epsilon, x, t) \to (\epsilon, \Phi^y_v(t) = \text{Exp}(tZ)_y, t \in (-\epsilon, \epsilon)$, $\epsilon > 0$.

We consider $\Phi^X_y$ as fibered fibered map $\mathbb{R}^{1,0,0} \to Y$. The correspondence $A^v : T_{\mathcal{F}^2\mathcal{M} \twoheadrightarrow \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^{(0,0)}(F|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})^*$ is a natural operator.

**Proposition 1.** Let $v_1, \ldots, v_L \in F_0\mathbb{R}^{1,0,0}$ be a basis. Every natural operator $A : T_{\mathcal{F}^2\mathcal{M} \twoheadrightarrow \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^{(0,0)}(F|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})^*$ is of the form

$$A = H(A^{v_1}, \ldots, A^{v_L})$$

for some uniquely determined smooth map $H \in C^\infty(\mathbb{R}^L)$.

**Proof.** Let $v_1^*, \ldots, v_L^* \in (F_0\mathbb{R}^{1,0,0})^*$ be the dual basis. Let $q = x^1 : \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}$ be the projection onto the first factor. It is a fibered fibered map $\mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{1,0,0}$. For $A$ as above we define $H : \mathbb{R}^L \to \mathbb{R}$,

$$H(t_1, \ldots, t_L) = A\left(\frac{\partial}{\partial x^1}\right)(F_0^q\sum_{s=1}^L t_s v_s^*)$$

We prove that $A = H(A^{v_1}, \ldots, A^{v_L})$. Since any $\mathcal{F}^2\mathcal{M}$-projectable vector field $Z$ on an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object $Y$ such that its underlying projectable vector field has non-vanishing underlying vector field is locally $\frac{\partial}{\partial x^1}$ in some local fibered fibered coordinates on $Y$, it is sufficient to show that $A(\frac{\partial}{\partial x^1})_\eta = H(A^{v_1}(\frac{\partial}{\partial x^1})_\eta, \ldots, A^{v_L}(\frac{\partial}{\partial x^1})_\eta)$ for any $\eta \in (F_0\mathbb{R}^{m_1,m_2,n_1,n_2})^*$. By the invariance of $A$ and $A^{v_s}$ with respect to $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-morphisms $(x^1, \frac{1}{2}x^2, \ldots, \frac{1}{2}x^m, y^1, \ldots, y^m, w^1, \ldots, w^m, \frac{1}{2}w^1, \frac{1}{2}w^1, \frac{1}{2}v^1, \ldots, \frac{1}{2}v^1) : \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ for $t \neq 0$ and next putting $t \to 0$, we can assume that
\( \eta = (F_0 q)^* (\sum_{s=1}^L t_s v_s^\gamma) \). Now, it remains to observe that \( A^{v_s}(\frac{\partial}{\partial x^s}) \eta = t_s \) for \( s = 1, \ldots, L \).

The uniqueness of \( H \) is clear as \( (A^{v_s}(\frac{\partial}{\partial x^s}))_{s=1}^L \) is a surjection onto \( \mathbb{R}^L \).

\( \square \)

We have functors \( i_\alpha : \mathcal{M} f \to \mathcal{F} \mathcal{M} \), \( i_1(M) = (\text{id}_M : M \to M), i_2(M) = (M \to pt) \), \( i_\alpha(f) = f : i_\alpha(M) \to i_\alpha(N), \alpha = 1, 2, \) \( M \in \text{obj}(\mathcal{M} f) \), \( f : M \to N \) is a map, \( pt \) is one point manifold. We have also a functor \( j : \mathcal{M} f \to \mathcal{F}^2 \mathcal{M} \), \( j(M) = (\text{id}_M : i_1(M) \to i_2(M)), j(f) = f : j(M) \to j(N), M \in \text{obj}(\mathcal{M} f) \), \( f : M \to N \) a map.

Thus we have a vector bundle functor \( F \circ j : \mathcal{M} f \to \mathcal{V} \mathcal{B} \). So, by [2], we can choose a basis \( v_1, \ldots, v_L \in F_0 \mathbb{R}^{1,0,0} = (F \circ j)_0 \mathbb{R} \) such that \( v_s \) is homogeneous of weight \( n_s \in \mathbb{N} \cup \{0\} \), i.e. \( F(\tau \text{id})(v_s) = \tau^{n_s} v_s \) for any \( \tau \in \mathbb{R} \).

\( (*) \) By a permutation we assume that \( v_1, \ldots, v_{k_1} \) are of weight 0, \( v_{k_1+1}, \ldots, v_{k_2} \) are of weight 1, etc.

Then \( A^{v_1}(Z), \ldots, A^{v_{k_1}}(Z) \) do not depend on \( Z \), i.e. \( A^{v_1}, \ldots, A^{v_{k_1}} \) are natural functions on \( (FY)^* \). Moreover \( A^{v_{k_1+1}}(Z), \ldots, A^{v_{k_2}}(Z) \) depend linearly on \( Z \), i.e. \( A^{v_{k_1+1}}, \ldots, A^{v_{k_2}} \) are linear operators.

**Corollary 1.** Every natural (canonical) function \( G \) on \( (F_1 \mathcal{F}^2 \mathcal{M}_{m_1,m_2,n_1,n_2})^* \) is of the form

\[
G = K(A^{v_1}, \ldots, A^{v_{k_1}})
\]

for some uniquely determined \( K \in \mathcal{C}^\infty(\mathbb{R}^{k_1}) \). If \( F \circ j \) has the point property, i.e. \( F \circ j(pt) = pt \), then \( G = \text{const} \).

**Corollary 2.** Let \( A : T \mathcal{F}^2 \mathcal{M} \to \text{proj}_! \mathcal{F} \mathcal{M}_{m_1,m_2,n_1,n_2} \to T^{(0,0)}(F_1 \mathcal{F}^2 \mathcal{M}_{m_1,m_2,n_1,n_2})^* \) be a natural linear operator. Then

\[
A = \sum_{s=k_1+1}^{k_2} K_s(A^{v_1}, \ldots, A^{v_{k_1}}) A^{v_s}
\]

for some uniquely determined \( K_s \in \mathcal{C}^\infty(\mathbb{R}^{k_1}) \).

**Proof.** The corollaries are consequences of Proposition 1 and the homogeneous function theorem, [4]. \( \square \)

**3. A decomposition proposition.** Let \( F \) and \( v_1, \ldots, v_L \) be as in Section 1 with the assumption \( (*) \). Let \( j : \mathcal{M} f \to \mathcal{F}^2 \mathcal{M} \) be the functor as in Section 2.

Let \( \pi : Y \to X \) be a fibered fibered manifold. A 1-form \( \omega : TY \to \mathbb{R} \) on \( Y \) is called \( \mathcal{F}^2 \mathcal{M} \)-horizontal if \( \omega|VY = 0 \) and \( \omega|VY = 0 \), where \( VY \) is the
vertical bundle of the fibered manifold $Y$ and $\hat{V}Y$ is the vertical bundle of fibered manifold $\pi : Y \to X$.

In this section we study natural operators $B : T^*_{\mathcal{F}^2\mathcal{M}\to\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*(F|_{\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})$ transforming $\mathcal{F}^2\mathcal{M}$-horizontal 1-forms $\omega$ on fibered manifolds $Y$ of dimension $(m_1,m_2,n_1,n_2)$ into 1-forms $B(\omega)$ on the dual vector bundle $(FY)^*$.

**Example 2.** If $\omega : TY \to \mathbb{R}$ is a $\mathcal{F}^2\mathcal{M}$-horizontal 1-form on a fibered manifold $\pi : Y \to X$, we have its vertical lifting $B^V(\omega) = \omega \circ T\pi^F : (FY)^* \to \mathbb{R}$, where $\pi^F : (FY)^* \to Y$ is the bundle projection. The correspondence $B^V : T^*_{\mathcal{F}^2\mathcal{M}\to\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*(F|_{\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})$ is a natural operator.

**Assumption 1.** From now on we assume that there exists a basis $w_1, \ldots, w_K \in F_0\mathbb{R}^{m_1,m_2,n_1,n_2}$ such that $w_s$ is homogeneous of weight $n_s \in \mathbb{N} \cup \{0\}$. It means that $F(\tau id)(w_s) = \tau^{n_s}w_s$ for any $\tau \in \mathbb{R}$.

**Remark 1.** It seems that every vector bundle functor $F : \mathcal{F}^2\mathcal{M} \to VB$ satisfies Assumption 1.

**Proposition 2 (Decomposition Proposition).** Consider a natural operator $B : T^*_{\mathcal{F}^2\mathcal{M}\to\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \to T^*(F|_{\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})$. Under Assumption 1 there exists the uniquely determined natural function $a$ on $(F|_{\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$ such that

$$B = aB^V + \lambda$$

for some canonical 1-form $\lambda$ on $(F|_{\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$.

**Lemma 1.**

(a) We have $(B(\omega) - B(0))(V(F\mathbb{R}^{m_1,m_2,n_1,n_2})^*)_0 = 0$ for any $\mathcal{F}^2\mathcal{M}$-horizontal 1-form $\omega$ on $\mathbb{R}^{m_1,m_2,n_1,n_2}$, where $(V(F\mathbb{R}^{m_1,m_2,n_1,n_2})^*)_0$ is the fiber over $0 \in \mathbb{R}^{m_1,m_2,n_1,n_2}$ of the $\pi^F$-vertical subbundle in $T(F\mathbb{R}^{m_1,m_2,n_1,n_2})$.

(b) If $F \circ j$ has the point property then $B(\omega)(V(F\mathbb{R}^{m_1,m_2,n_1,n_2})^*)_0 = 0$ for any $\mathcal{F}^2\mathcal{M}$-horizontal 1-form $\omega$ on $\mathbb{R}^{m_1,m_2,n_1,n_2}$.

**Proof.**

ad (a) We use the invariance of $(B(\omega) - B(0))(V(F\mathbb{R}^{m_1,m_2,n_1,n_2})^*)_0$ with respect to the homotheties $\frac{1}{t}id_{\mathbb{R}^{m_1,m_2,n_1,n_2}}$ for $t \neq 0$ and apply the homogeneous function theorem. We obtain that $(B(\omega) - B(0))(V(F\mathbb{R}^{m_1,m_2,n_1,n_2})^*)_0$ is independent of $\omega$. This ends the proof of the part (a).

ad (b) We observe that if $F \circ j$ has the point property then $(F_0\mathbb{R}^{m_1,m_2,n_1,n_2})^*$ has no non-zero homogeneous elements of weight 0. Next, we use the invariance of $B(\omega)(V(F\mathbb{R}^{m_1,m_2,n_1,n_2})^*)_0$ with respect to the homotheties $\frac{1}{t}id_{\mathbb{R}^{m_1,m_2,n_1,n_2}}$ for $t \neq 0$ and put $t \to 0$. □
**Proof of Proposition 2.** Clearly, $B(0)$ is a canonical 1-form. Then replacing $B$ by $B - B(0)$ we have $B(0) = 0$ and $B(0)_i(V(F R^{m_1,m_2,n_1,n_2})_0) = 0$. Then $B$ is determined by the values $B(\omega)_\eta, F^*(\omega)(\partial / \partial x^1) \eta$ for all $F^2 \mathcal{M}$-horizontal 1-forms $\omega = \sum_{t=1}^{m_1} \omega_t dx^1$ on $R^{m_1,m_2,n_1,n_2}$ and $\eta \in (F_0 R^{m_1,m_2,n_1,n_2})^*$, where $F^*(\partial / \partial x^1)$ is the complete lifting (flow prolongation) of $\partial / \partial x^1$ to $(F R^{m_1,m_2,n_1,n_2})^*$.

Using the invariance of $B$ with respect to the homotheties $\frac{1}{t} \text{id} R^{m_1,m_2,n_1,n_2}$ for $t \neq 0$ we get the homogeneity condition

$$t \langle B(\omega)_\eta, F^* \left( \frac{\partial}{\partial x^1} \right)_\eta \rangle = \langle B((t \text{id} R^{m_1,m_2,n_1,n_2})^* \omega), F((\frac{1}{t} \text{id} R^{m_1,m_2,n_1,n_2})^*)^*(\eta) \rangle$$

Then by the non-linear Petree theorem [4], the homogeneous function theorem and $B(0) = 0$ we deduce that $\langle B(\omega)_\eta, F^*(\omega)(\partial / \partial x^1)_\eta \rangle$ is a linear combination of $\omega_1(0), \ldots, \omega_{m_1}(0)$ with coefficients being smooth maps in homogeneous coordinates of $\eta$ of weight 0.

Then using the invariance of $B$ with respect to $F^2 \mathcal{M}_{m_1,m_2,n_1,n_2}$-morphisms $(x^1, \frac{1}{t} x^2, \ldots, \frac{1}{t} \omega m_1, \frac{1}{t} y^1, \ldots, \frac{1}{t} y m_2, \frac{1}{t} w^1, \ldots, \frac{1}{t} w_{m_1}, \frac{1}{t} v^1, \ldots, \frac{1}{t} v_{m_2} : R^{m_1,m_2,n_1,n_2} \rightarrow R^{m_1,m_2,n_1,n_2}$ for $t \neq 0$ and put $t \rightarrow 0$ we end the proof. $\square$

4. On canonical 1-forms on $(F[F^2 \mathcal{M}_{m_1,m_2,n_1,n_2}])^*$.

**Proposition 3.** Every canonical 1-form $\lambda$ on $(F[F^2 \mathcal{M}_{m_1,m_2,n_1,n_2}])^*$ induces a linear natural operator

$$A^{(\lambda)} : T_{F^2 \mathcal{M} - \text{proj}} F^2 \mathcal{M}_{m_1,m_2,n_1,n_2} \rightarrow T^{(0,0)} (F[F^2 \mathcal{M}_{m_1,m_2,n_1,n_2}])^*$$

such that $A^{(\lambda)}(Z)_\eta = \langle \lambda_\eta, F^*(Z) \eta \rangle$, $\eta \in (FY)^*$, $Z$ is a $F^2 \mathcal{M}$-projectable vector field on $Y$, where $F^*(Z)$ is the complete lifting (flow operator) of $Z$ to $(FY)^*$. If $F \circ j$ has the point property, then (under Assumption 1) the correspondence $\langle \lambda \rightarrow A^{(\lambda)} \rangle$ is a linear injection.

**Proof.** The injectivity is a consequence of Lemma 1 (b). $\square$

5. A corollary. Let $j : M f \rightarrow F^2 \mathcal{M}$ be the functor as in Section 2.

**Corollary 3.** Assume that $F \circ j$ has the point property and there are no non-zero elements from $F_0 R^{1,0,0,0}$ of weight 1. (For example, let $F = F_1 \otimes F_2 : F^2 \mathcal{M} \rightarrow \mathcal{VB}$ be the tensor product of two vector bundle functors $F_1, F_2$ :
\( F^2 \mathcal{M} \to \mathcal{VB} \) such that \( F_1 \circ j, F_2 \circ j \) have the point property.) Then (under Assumption 1) every natural operator \( B : T^* F^2 \mathcal{M} - \text{hor}|F^2 \mathcal{M}_m, n, 1 \to T^* (F|F^2 \mathcal{M}_m, n, 1)^* \) is a constant multiple of the vertical lifting.

**Proof.** Since there are no non-zero elements from \( F_0 \mathbb{R}^{1,0,0,0} \) of weight 1, we see that every canonical 1-form on \( (F|F^2 \mathcal{M}_m, n, 1)^* \) is zero because of Corollary 2 and Proposition 3. Then Proposition 2 together with Corollary 1 ends the proof. \( \square \)

### 6. An application.

Let \( r_1, r_2, \ldots, r_8 \in \mathbb{N} \) be such that \( r_8 \geq r_4 \leq r_5 \geq r_3 \) and \( r_8 \geq r_6 \leq r_7 \geq r_2 \) and \( r_1 \leq r_i \) for \( i = 2, 3, \ldots, 8 \).

The concept of \( r \)-jets and \((r, s, q)\)-jets can be generalized as follows. Let \( \pi: Y \to X \) be a fibered fibered manifold being surjective fibered submersion between fibered manifolds \( p^Y : Y \to Y' \) and \( p^X: X \to X' \). Let \( \pi': Y' \to X' \) be another fibered fibered manifold being surjective fibered submersion between \( p'^Y: Y' \to Y' \) and \( p'^X: X' \to X' \). Let \( y \in Y \) be a point and \( y = p^Y(y) \in Y' \) and \( x = p^X(x) \in X' \) be its underlying points. Let \( f, g : Y \to Y' \) be two fibered fibered maps and \( f, g : Y \to Y' \), \( f_0, g_0 : X \to X' \) and \( f_0, g_0 : X \to X' \) be their underlying maps. We say that \( f, g \) determine the same \((r_1, \ldots, r_8)\)-jet \( j_y^{(r_1, \ldots, r_8)} f = j_y^{(r_1, \ldots, r_8)} g \) at \( y \in Y \) if \( j_y^{r_1} f = j_y^{r_1} g \), \( j_y^{r_2} (f|Y_2) = j_y^{r_2} (g|Y_2) \), \( j_y^{r_3} (f|Y_2) = j_y^{r_3} (g|Y_2) \), \( j_y^{r_4} (f_0) = j_y^{r_4} (g_0) \), \( j_y^{r_5} (f_0|X_2) = j_y^{r_5} (g_0|X_2) \), \( j_y^{r_6} (f) = j_y^{r_6} (g) \), \( j_y^{r_7} (f|Y_2) = j_y^{r_7} (g|Y_2) \) and \( j_y^{r_8} (f_0) = j_y^{r_8} (g_0) \). The space of all \((r_1, r_2, \ldots, r_8)\)-jets of \( Y \) into \( Y' \) is denoted by \( J^{(r_1, \ldots, r_8)}(Y, Y') \). The composition of fibered fibered maps induces the composition of \((r_1, \ldots, r_8)\)-jets.

The (described in [4] and [5], [15]) vector bundle functors \( T^{(r)} = (J^r(\cdot, \mathbb{R})_0^* : \mathcal{M} \to \mathcal{VB} \) and \( T^{(r, s, q)} = (J^{(r, s, q)}(\cdot, \mathbb{R}^{1,1})_0^* : F^2 \mathcal{M} \to \mathcal{VB} \) can be generalized as follows. The space \( J^{(r_1, \ldots, r_8)}(Y, \mathbb{R}^{1,1,1,1})_0 \), \( 0 \in \mathbb{R}^4 \), has an induced structure of a vector bundle over \( Y \). Every fibered fibered map \( f : Y \to Y' \), \( f(y) = y' \), induces a linear map \( \lambda(j_y^{(r_1, \ldots, r_8)} f) : j_y^{(r_1, \ldots, r_8)}(Y', \mathbb{R}^{1,1,1,1})_0 \to j_y^{(r_1, \ldots, r_8)}(Y, \mathbb{R}^{1,1,1,1})_0 \) by means of the jet composition. If we denote by \( T^{(r_1, \ldots, r_8)} Y \) the dual vector bundle of \( J^{(r_1, \ldots, r_8)}(Y, \mathbb{R}^{1,1,1,1})_0 \) and define \( T^{(r_1, \ldots, r_8)} f : T^{(r_1, \ldots, r_8)} Y \to T^{(r_1, \ldots, r_8)} Y' \) by using the dual maps to \( \lambda(j_y^{(r_1, \ldots, r_8)} f) \), we obtain a vector bundle functor \( T^{(r_1, \ldots, r_8)} : F^2 \mathcal{M} \to \mathcal{VB} \).

**Example 3.** We have 1-forms \( \lambda^{(r_1, \ldots, r_8)} \) : \( T J^{(r_1, \ldots, r_8)}(Y, \mathbb{R}^{1,1,1,1})_0 \to \mathbb{R} \) on \( J^{(r_1, \ldots, r_8)}(Y, \mathbb{R}^{1,1,1,1})_0 \), \( \alpha = 1, 2, 3, 4 \), \( \lambda^{(r_1, \ldots, r_8)}(v) = d\gamma_\alpha(T\tilde{\pi}(v)) \), \( v \in T_{\tilde{\pi}y} J^{(r_1, \ldots, r_8)}(Y, \mathbb{R}^{1,1,1,1})_0 \), \( w = j_y^{(r_1, \ldots, r_8)}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \), \( y \in Y \), \( \tilde{\pi} : J^{(r_1, \ldots, r_8)}(Y, \mathbb{R}^{1,1,1,1})_0 \to Y \) is the bundle projection.
Corollary 4. Every natural operator

\[ B : T^*_{F_2 M} \to (J^{r_1 \ldots r_s}(\cdot, \mathbb{R}^{1,1,1,1})) \]

is a linear combination of the vertical lifting \( B^V \) and the canonical 1-forms \( \lambda^{(r_1 \ldots r_s)}_{\alpha} \) for \( \alpha = 1, 2, 3, 4 \) with real coefficients.

**Proof.** The vector bundle functor \( T^{(r_1 \ldots r_s)} \) satisfies Assumption 1. Moreover, \( T^{(r_1 \ldots r_s)} \circ j \) has the point property and the subspace of elements from \( T^{(r_1 \ldots r_s)}_0 \mathbb{R}^{1,0,0,0} \) of weight 1 is 4-dimensional. Then by Proposition 3 together with Corollaries 1 and 2, the space of canonical 1-forms on \( J^{(r_1 \ldots r_s)}(\cdot, \mathbb{R}^{1,1,1,1})_0 \) is at most 4-dimensional. Now, Proposition 2 ends the proof. \( \Box \)

**References**


