On maximum modulus of polynomials

Abstract. For a polynomial \( p(z) \) of degree \( n \), it is known that
\[
|p(Re^{i\theta})| + |q(Re^{i\theta})| \leq (R^n + 1) \max_{|z|=1} |p(z)|,
\]
for \( R \geq 1 \) and \( 0 \leq \theta \leq 2\pi \), where
\[
q(z) = z^n p(1/z).
\]

We obtain a refinement, as well as a generalization, of this inequality.

1. Introduction and statement of results. For an arbitrary entire function \( f(z) \), let \( M(f, r) = \max_{|z|=r} |f(z)| \). For a polynomial \( p(z) \) of degree \( n \), it is known ([4, section 5], [1, Lemma]) that
\[
|p(Re^{i\theta})| + |q(Re^{i\theta})| \leq (R^n + 1) M(p, 1), R \geq 1 \text{ and } 0 \leq \theta \leq 2\pi,
\]
where
\[
q(z) = z^n p(1/z).
\]

In this note, we obtain a refinement, as well as a generalization, of inequality (1.1). More precisely, we prove

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Theorem 1. If \( p(z) \) is a polynomial of degree \( n \), \( n \geq 3 \), then for every positive integer \( s \), we have
\[
|p(\Re e^{i\theta})|^s + |q(\Re e^{i\theta})|^s \leq (R^{ns} + 1)\{M(p, 1)\}^s
\]
(1.3)
\[- \left( \frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns - 2} \right) ||p'(0)| - |q'(0)|| s\{M(p, 1)\}^{s-1},
\]
\( R \geq 1 \) and \( 0 \leq \theta \leq 2\pi \).

Remark. For \( s = 1 \), inequality (1.3) becomes
\[
|p(\Re e^{i\theta})| + |q(\Re e^{i\theta})| \leq (R^n + 1)M(p, 1)
\]
\[- \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2} \right) ||p'(0)| - |q'(0)||,
\]
and is therefore a refinement of inequality (1.1), as
\[
\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2} \geq 0.
\]
Further, by (1.3), we obviously have
\[
|p(\Re e^{i\theta})|^s + |q(\Re e^{i\theta})|^s \leq (R^{ns} + 1)\{M(p, 1)\}^s,
\]
suggesting a generalization of inequality (1.1).

2. Lemmas. For the proof of Theorem 1, we require the following lemmas.

Lemma 1. If \( p(z) \) is a polynomial of degree at most \( n \), \( n \geq 2 \), then for \( R > 1 \)
\[
M(p, R) \leq R^n M(p, 1) - (R^n - R^{n-2})|p(0)|.
\]
The coefficient of \( |p(0)| \) is best possible for each \( R \).

This lemma is due to Frappier, Rahman and Ruscheweyh, cf. [2, Theorem 2].

Lemma 2. If \( p(z) \) is a polynomial of degree \( n \), then for \( |z| = 1 \)
\[
|p'(z)| + |q'(z)| \leq nM(p, 1).
\]

This lemma is due to Malik [3, inequality 17].

3. Proof of Theorem 1. The polynomial
\[
G(z) = p'(z) + \alpha q'(z), \ |\alpha| = 1
\]
is of degree at most \( n - 1 \) (\( \geq 2 \)). Hence, if \(|\alpha| = 1\), \( t \geq 1\) and \( 0 \leq \theta \leq 2\pi\), then applying Lemma 1 followed by Lemma 2, we obtain

\[
|p'(te^{i\theta}) + \alpha q'(te^{i\theta})| \leq t^{n-1} \max_{|z|=1} |p'(z) + \alpha q'(z)| - (t^{n-1} - t^{n-3})|p'(0) + \alpha q'(0)| \\
\leq t^{n-1} nM(p,1) - (t^{n-1} - t^{n-3})|p'(0) + \alpha q'(0)|
\]

and so

\[
|p'(te^{i\theta})| + |q'(te^{i\theta})| \leq nt^{n-1}M(p,1) - (t^{n-1} - t^{n-3})||p'(0)| - |q'(0)||.
\]

Since

\[
\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s = \int_1^R \frac{d}{dt}\{p(te^{i\theta})\}^s dt \\
= \int_1^R s\{p(te^{i\theta})\}^{s-1} p'(te^{i\theta})e^{i\theta} dt,
\]

we see that

\[
|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq s \int_1^R |p'(te^{i\theta})||p(te^{i\theta})|^{s-1} dt,
\]

which, by virtue of Lemma 1, implies

\[
(3.2) \quad |\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq s \int_1^R t^{n(s-1)}|M(p,1)|^{s-1} dt.
\]

Similarly, we have

\[
|\{q(Re^{i\theta})\}^s - \{q(e^{i\theta})\}^s| \leq s \int_1^R t^{n(s-1)}|q'(te^{i\theta})||q(te^{i\theta})|^{s-1} dt \\
= s \int_1^R |q'(te^{i\theta})||M(p,1)|^{s-1}t^{n(s-1)} dt
\]
which together with (3.2) gives

\[
\begin{align*}
\{|p(Re^{i\theta})|^s - |p(e^{i\theta})|^s| + |\{q(Re^{i\theta})\}|^s - |q(e^{i\theta})|^s| & \\
& \leq s\{M(p,1)\}^{s-1} \int_1^R t^{n(s-1)} (|p'(te^{i\theta})| + |q'(te^{i\theta})|) \, dt \\
& \leq sn\{M(p,1)\}^s \int_1^R t^{ns-1} dt \\
& - n(s-1) ||p'(0)| - |q'(0)|| \int_1^R (t^{ns-1} - t^{ns-3}) \, dt,
\end{align*}
\]

where at the last step we have used (3.1). Since \(|p(e^{i\theta})| = |q(e^{i\theta})| \leq M(p,1)|
we obtain

\[
\begin{align*}
|p(Re^{i\theta})|^s + |q(Re^{i\theta})|^s & \leq (R^{ns} + 1) (M(p,1))^s \\
& - s(M(p,1))^{s-1} ||p'(0)| - |q'(0)|| \left( \frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns - 2} \right),
\end{align*}
\]

which is what we wanted to prove. □

References


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