On the construction of a stable sequence with given density

Abstract. The notion of a stable sequence of events generalizes the notion of mixing sequence and was introduced by A. Rényi. A sequence of random elements $X_n$ is said to be stable if for every $B \in \mathcal{A}$ with $P(B) > 0$ there exists a probability measure $\mu_B$ on $(S, \mathcal{B})$ such that $\lim_{n \to \infty} P(X_n \in A \mid B) = \mu_B(A)$ for every $A \in \mathcal{A}$ with $\mu_B(\delta A) = 0$. Given a density function, the aim of this note is to give a martingale construction of a stable sequence of random elements having the given density function. The problem was solved in the special case $\Omega = <0, 1>$ by the second named author and S. Gutkowska.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. By $(S, \rho)$ we denote a metric space and $\mathcal{B}$ stands for the $\sigma$–field generated by open sets of $S$.

Let $\mathcal{X}$ be the set of all random elements (r.e.):

$$\mathcal{X} = \{ X : \Omega \to S : X^{-1}(A) \in \mathcal{A}, A \in \mathcal{B} \}$$

Definition 1. An infinite sequence of events $A_1, A_2, \ldots, A_n, \ldots$ ($A_i \in \mathcal{A}, i \geq 1$) will be called a stable sequence if the limit

$$\lim_{n \to \infty} P(A_n B) = Q(B)$$

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exists for every $B \in \mathcal{A}$.

Thus $Q$ is a bounded measure on $\mathcal{A}$ which is absolutely continuous with respect to the measure $P$ and consequently

$$Q(B) = \int_B \alpha \, dP$$

for every $B \in \mathcal{A}$, where $\alpha = \alpha(\omega)$ is a measurable function on $\Omega$ such that $0 \leq \alpha(\omega) \leq 1$ almost surely (a.s.).

In the case when the local density is constant, the sequence $\{A_n, n \geq 1\}$ will be called a mixing sequence of events with density $\alpha$.

In the special case when $\Omega = \langle 0, 1 \rangle$ a construction of a stable sequence with given continuous density function $\alpha$ is described, cf. [7]. In this paper we give a construction in a more general situation.

It is well known [6] that any sample space $\Omega$ can be represented as

$$\Omega = B \cup \bigcup_{k=1}^{\infty} B_k, B_m \cap B_n = \emptyset \text{ for } m \neq n, B \cap B_n = \emptyset, n = 1, 2, \ldots$$

where each $B_k$ is an atom or an empty set and $B$ has the property that for any given $A \in \mathcal{A}$ such that $A \subset B$ and any $\varepsilon, 0 < \varepsilon < P(A)$, there exists $C \in \mathcal{A}, C \subset A$, such that $P(C) = \varepsilon$. Random elements are constant a.s. on atoms.

**Theorem 1.** Assume that $(\Omega, \mathcal{A}, P)$ is an atomless probability space. Then for every measurable real function $\alpha$ ($0 \leq \alpha \leq 1$ a.s.) there exists a stable sequence of events $\{A_n, n \geq 1\}$ such that

$$\lim_{n \to \infty} P(A_n B) = \int_B \alpha \, dP = Q(B).$$

**Proof.** Let $\mathcal{A}' \subset \mathcal{A}$ be the $\sigma$-field generated by the sets $\alpha^{-1}(B(x_i, r_j))$, where $x_i$ and $r_j$ are rational numbers ($0 \leq x_i \leq 1, r_j > 0$) and

$$B(x_i, r_j) = \{x : |x - x_i| < r_j\}.$$

We can assume that $\mathcal{A}'$ is generated by $B_1, B_2, \ldots, B_n, \ldots$ with $B_i \in \mathcal{A}, i \geq 1$. We denote by $\mathcal{C}_n = \sigma(B_1, B_2, \ldots, B_n)$ the $\sigma$-field generated by the set $B_1, B_2, \ldots, B_n$. $\mathcal{C}_n$ is generated by the measurable partition $\{C^{1}_{n}, C^{2}_{n}, \ldots, C^{k_{n}}_{n}\}$.

By the martingale convergence theorem, we have

$$\alpha(\omega) = \lim_{n \to \infty} E^\mathcal{C}_n \alpha(\omega) \text{ a.s.},$$
where $E^{C_n}$ denotes the conditional expectation with respect to the $\sigma$-field $C_n$.

Since $(\Omega, \mathcal{A}', P)$ is atomless, for every $n$ and $1 \leq i \leq k_n$ there exists in $\mathcal{A}'$ a set $A^i_n \subset C^i_n$ such that

$$P(A^i_n) = \int_{C^i_n} \alpha(\omega) dP.$$

We put $A_n = \bigcup_{i=1}^{k_n} A^i_n$. For $\omega \in C^i_n$, we have

$$E^{C_n}(I_{A_n})(\omega) = \frac{P(A_n \cap C^i_n)}{P(C^i_n)} = \frac{P(A^i_n)}{P(C^i_n)} = \frac{\int_{C^i_n} \alpha(\omega) dP}{P(C^i_n)} = E^{C_n} \alpha(\omega).$$

If $B \in C_n$ for some $n \geq 1$ then we have

$$\lim_{n \to \infty} E(I_{A_n} I_B) = \lim_{n \to \infty} E(E^{C_n}(I_{A_n} I_B)) = \lim_{n \to \infty} EI_B E^{C_n}(I_{A_n}) = \lim_{n \to \infty} EI_B E^{C_n} \alpha = EI_B \alpha.$$

Let now $\mathcal{K} = \{ B \in \mathcal{A}': \lim_{n \to \infty} E(I_{A_n} I_B) = EI_B \alpha \}$. The set $\mathcal{K}$ contains $\emptyset$ and $\bigcup_{n=1}^{\infty} C_n \subset \mathcal{K}$. We prove that $\mathcal{K}$ is a $\sigma$-field. It is easy to see that if $B \in \mathcal{K}$ then $B^c \in \mathcal{K}$. Let now $B_n \in \mathcal{K}$, $n \geq 1$, be an increasing sequence and $B = \bigcup_{n=1}^{\infty} B_n$. For any $\varepsilon > 0$ there exists $n_0$ such that $P(B) \leq P(B_{n_0}) + \varepsilon$. Then we have $\liminf_{n \to \infty} E(I_{A_n} I_B) \geq \lim_{n \to \infty} E(I_{A_n} I_{B_{n_0}}) = E\alpha I_{B_{n_0}} \geq E\alpha I_B - \varepsilon$ and $\limsup_{n \to \infty} E(I_{A_n} I_B) \leq \lim_{n \to \infty} E(I_{A_n} I_{B_{n_0}}) + \varepsilon = E\alpha I_{B_{n_0}} + \varepsilon \leq E\alpha I_B + \varepsilon$ which implies

$$\lim_{n \to \infty} E(I_{A_n} I_B) = E\alpha I_B$$

and this proves that $\mathcal{K}$ is a $\sigma$-field and $\mathcal{K}$ contains $\mathcal{A}'$.

Next, we show that equality $\lim_{n \to \infty} E(I_{A_n} I_B) = E(\alpha I_B)$ remains true for each $B \in \mathcal{A}$. If $g : \Omega \to <0, 1>$ is some $\mathcal{A}'$-measurable function, we can find for each $\varepsilon > 0$ a step function $f : \Omega \to <0, 1>$ which is $\mathcal{A}'$-measurable and such that $|f - g| < \varepsilon$ on a set $\Omega'$ with $P(\Omega') > 1 - \varepsilon$. Then
as \( f = \sum_{s=1}^{m} \lambda_s I_{D_s} \) where \( D_s \in \mathcal{A}' \) and \( \lambda_s \in \mathcal{R} \) for \( s = 1, 2, \ldots, m \), we have

\[
\lim_{n \to \infty} \int f I_{A_n} \, dP = \lim_{n \to \infty} \int \left( \sum_{s=1}^{m} \lambda_s I_{D_s} \right) I_{A_n} \, dP \\
= \lim_{n \to \infty} \sum_{s=1}^{m} \lambda_s \int I_{D_s} I_{A_n} \, dP \\
= \lim_{n \to \infty} \sum_{s=1}^{m} \lambda_s \int I_{D_s} \alpha \, dP \\
= \int f \alpha \, dP
\]

Thus

\[
\liminf_{n \to \infty} E(g I_{A_n}) \geq \liminf_{n \to \infty} E(g I_{A_n} I_{\Omega'}) \\
\geq \lim_{n \to \infty} E(f I_{A_n} I_{\Omega'}) - \varepsilon \\
= E(f \alpha I_{\Omega'}) - \varepsilon \\
\geq E(g \alpha I_{\Omega'}) - 2\varepsilon \\
\geq E(g \alpha) - 3\varepsilon
\]

and

\[
\limsup_{n \to \infty} E(g I_{A_n}) \leq \limsup_{n \to \infty} E(g I_{A_n} I_{\Omega'}) + \varepsilon \\
\leq \lim_{n \to \infty} E(f I_{A_n} I_{\Omega'}) + 2\varepsilon \\
= E(f \alpha I_{\Omega'}) + 2\varepsilon \\
\leq E(g \alpha I_{\Omega'}) + 3\varepsilon \\
\leq E(g \alpha) + 4\varepsilon.
\]

Since \( \varepsilon \) is arbitrary, we have

\[
(1) \quad \lim_{n \to \infty} E(g I_{A_n}) = E(g \alpha)
\]

for each \( \mathcal{A}' \)-measurable function \( g \) such that \( 0 \leq g \leq 1 \).

Now, let \( B \in \mathcal{A} \). We have

\[
\lim_{n \to \infty} E(I_{A_n} I_B) = \lim_{n \to \infty} E(E^{A'}(I_{A_n} I_B)) = \lim_{n \to \infty} E(I_{A_n} E^{A'} I_B)
\]

because \( A_n \in \mathcal{A}' \), \( n \geq 1 \), and by (1) we have

\[
\lim_{n \to \infty} E(I_{A_n} I_B) = \lim_{n \to \infty} E(I_{A_n} E^{A'} I_B) = E(\alpha E^{A'} I_B) = E(E^{A'} \alpha I_B) \\
= E(\alpha I_B),
\]
which completes the proof. □

By this construction we see that if \( \alpha', \alpha \) are measurable real functions such that \( 0 \leq \alpha' \leq \alpha \leq 1 \), then there exist stable sequences \( \{A'_n, n \geq 1\} \) and \( \{A_n, n \geq 1\} \) with density \( \alpha' \) and \( \alpha \), respectively, such that \( A'_n \subset A_n, n \geq 1 \).

It is obvious that the sequence \( \{A_n \setminus A'_n, n \geq 1\} \) is stable with density \( \alpha - \alpha' \).

If \( \alpha', \alpha \) are nonnegative measurable real functions such that \( 0 \leq \alpha' + \alpha \leq 1 \), then there exist stable sequences \( \{A'_n, n \geq 1\} \) and \( \{A_n, n \geq 1\} \) with density \( \alpha' \) and \( \alpha \) respectively, such that \( A_n \cap A'_n = \emptyset, n \geq 1 \).

**Definition 2.** A sequence \( \{X_n, n \geq 1\} \) of r.e. is said to be stable if for every \( A \in \mathcal{A}_+ = \{A \in \mathcal{A} : P(A) > 0\} \) there exists a probability measure \( \mu_A \), defined on \((S, \mathcal{B})\), such that

\[
(2) \quad \lim_{n \to \infty} P([X_n \in B] | A) = \mu_A(B)
\]

for every \( B \in \mathcal{C}_{\mu_A} = \{B \in \mathcal{B} : \mu_A(\partial B) = 0\} \) where \( \partial B \) denotes the boundary of \( B \).

If \( \mu_A(B) = \mu(B) \) for every \( A \in \mathcal{A}_+ \) and \( B \in \mathcal{B} \) then the sequence \( \{X_n, n \geq 1\} \) of r.e. is said to be \( \mu \)-mixing.

Let \( Q_B(A) = \mu_A(B)P(A) \). Obviously \( Q_B \) is an absolutely continuous measure with respect to \( P \). By the Radon-Nikodym Theorem there exists a nonnegative function \( \alpha_B : \Omega \to R^+ \), such that

\[
Q_B(A) = \int_A \alpha_B dP.
\]

The function \( \alpha_B \) is called the density of the stable sequence \( \{X_n, n \geq 1\} \).

The set \( \mathcal{P}_A(S) = \{\mu_A : A \in \mathcal{A}_+\} \) of all probability measures defined by (2) satisfies the following condition:

\[
(3) \quad P\left(\bigcup_{i=1}^n A_i\right)\mu^{-n}_{\bigcup_{i=1}^n A_i}(B) = \sum_{i=1}^n \mu_{A_i}(B)P(A_i)
\]

for every \( A_i \in \mathcal{A}_+, i = 1, 2, \ldots, n, n \geq 1, A_i \cap A_j = \emptyset, i \neq j \).

Moreover, it is known [10] that a sequence \( \{X_n, n \geq 1\} \) of r.e. converges in probability to a r.e. \( X \) iff \( \{X_n, n \geq 1\} \) is a stable sequence and \( \mathcal{P}_A(S) \) satisfies the following condition:

\[
(4) \quad \text{If } \mu_A(B) > 0 \text{ then there exists a set } A' \in \mathcal{A}_+, A' \subset A \text{ such that } \mu_{A'}(B) = 1.
\]
Theorem 2. Assume that \((\Omega, A, P)\) is an atomless probability space. If the set \(P_B(S) = \{\mu_A : A \in A_+\}\) of probability measures on \((S, B)\) satisfies Condition (3) then there exists a stable sequence \(\{X_n, n \geq 1\}\) such that
\[
\lim_{n \to \infty} P([X_n \in B], A) = \mu_A(B)P(A), \quad B \in B, \ A \in A_+ .
\]

Remark. It is easy to check that Condition (3) expresses the fact that the set function \(\tilde{\mu}(A \times B) = \mu_A(B)P(A)\) can be extended to a probability measure on the \(\sigma\)-algebra \(A \otimes B\), whereas Condition (3) means that the measure \(\tilde{\mu}\) is supported by the graph of a r.e.

Proof of Theorem 2. Let \(Q_B(A) = \mu_A(B)P(A), \ B \in B, A \in A_+\) and \(Q_B(A) = 0\) for \(P(A) = 0\). Obviously \(Q_B\) is an absolutely continuous measure with respect to \(P\) and there exists a measurable function \(\alpha_B\) such that
\[
Q_B(A) = \int_A \alpha_B \, dP, \quad 0 \leq \alpha_B \leq 1 \text{ a.e..}
\]

Now, there exists a variant \(\lambda(B, \cdot)\) of \(\alpha(B, \cdot)\) such that with probability 1 \(\lambda(\cdot, \omega)\) is a probability measure on \((S, B)\) \((P\{\omega : \lambda(B, \omega) \neq \alpha(B, \omega)\} = 0\) for every \(B \in B\) [9].

Let us choose a sequence of Borel subsets \(S_{i_1, i_2, \ldots, i_k} \in C_{\mu_0}\) satisfying the following conditions [8]:

(a) \(S_{i_1, i_2, \ldots, i_k} \cap S_{i_1, i_2', \ldots, i_k'} = \emptyset\) if \(i_s \neq i_s'\) for some \(1 \leq s \leq k\),

(b) \(\bigcup_{i_k = 1}^{\infty} S_{i_1, i_2, \ldots, i_{k-1}, i_k} = S_{i_1, i_2, \ldots, i_{k-1}}, \bigcup_{i_1 = 1}^{\infty} S_{i_1} = S\),

(c) \(d(S_{i_1, i_2, \ldots, i_k}) < \frac{1}{2^k}\), where \(d(B)\) denotes the diameter of the set \(B \subset S\).

By Theorem 1, for every \(S_{i_1, i_2, \ldots, i_k}\) there exists a stable sequence \(\{A^n_{i_1, i_2, \ldots, i_k}, n \geq 1\}\) with density \(\alpha(S_{i_1, i_2, \ldots, i_k}, \cdot)\) such that

(a') \(A^n_{i_1, i_2, \ldots, i_k} \cap A^n_{i_1, i_2', \ldots, i_k'} = \emptyset\) if \(i_s \neq i_s'\) for some \(1 \leq s \leq k\) and

(b') \(A^n_{i_1, i_2, \ldots, i_{k-1}} \subset A^n_{i_1, i_2, \ldots, i_k}, n \geq 1, k \geq 1\) and
\[
\bigcup_{i_{k+1} = 1}^{\infty} A^n_{i_1, i_2, \ldots, i_k, i_{k+1}} = A^n_{i_1, i_2, \ldots, i_k}, \quad \bigcup_{i_1 = 1}^{\infty} A^n_{i_1} = \Omega, \ n \geq 1.
\]

If \(z_{i_1, i_2, \ldots, i_k} \in S_{i_1, i_2, \ldots, i_k}\) we can define
\[
X_n^k(\omega) = z_{i_1, i_2, \ldots, i_k} \quad \text{for} \quad \omega \in A^n_{i_1, i_2, \ldots, i_k}, \ n \geq 1.
\]

Then for every \(\omega\) the sequence \(\{X_n^k, k \geq 1\}\) satisfies the Cauchy condition and therefore converges to some r.e. \(X_n\).
Moreover, for every $k$, the sequence $\{X^k_n, n \geq 1\}$ is stable. Let $A \in \mathcal{A}$ and $\varepsilon > 0$. We can choose $\delta > 0$ such that

$$\int_A \alpha(S_{i_1,i_2,\ldots,i_l}^{2\delta}, \cdot) \, dP \leq \int_A \alpha(S_{i_1,i_2,\ldots,i_l}^{\delta}, \cdot) \, dP + \varepsilon,$$

where $B^\delta = \{x : \inf_{y \in B} \rho(x, y) < \delta\}$.

Hence, if we set

$$S'(\delta) = \bigcup_{\{i_1,i_2,\ldots,i_s : s > \log_2 \frac{1}{\delta}, S_{i_1,i_2,\ldots,i_s} \cap S_{i_1,i_2,\ldots,i_l}^\delta \neq \emptyset\}} S_{i_1,i_2,\ldots,i_s},$$

we have

$$P([X_n \in S_{i_1,i_2,\ldots,i_l}] \cap A) \leq P([X_n^k \in S_{i_1,i_2,\ldots,i_l}^\delta] \cap A) \leq P([X_n^k \in S'(\delta)] \cap A) \xrightarrow{n \to \infty} \int_A \alpha(S'(\delta), \cdot) \, dP \leq \int_A \alpha(S_{i_1,i_2,\ldots,i_l}^{2\delta}, \cdot) \, dP \leq \int_B \alpha(S_{i_1,i_2,\ldots,i_l}, \cdot) \, dP + \varepsilon.$$

Similarly,

$$\lim_{n \to \infty} P([X_n \in S_{i_1,i_2,\ldots,i_l}] \cap A) \geq \int_A \alpha(S_{i_1,i_2,\ldots,i_l}, \cdot) \, dP - \varepsilon,$$

which proves that

$$\lim_{n \to \infty} P([X_n \in S_{i_1,i_2,\ldots,i_l}] \cap A) = \int_A \alpha(S_{i_1,i_2,\ldots,i_l}, \cdot) \, dP.$$

This completes the proof, since the sets $S_{i_1,i_2,\ldots,i_l}$ form a convergence-determining class. □

References


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