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Plane convex sets via distributions

Abstract. We will establish the correspondence between convex compact subsets of $\mathbb{R}^2$ and $2\pi$–periodic distributions in $\mathbb{R}$. We also give the necessary and sufficient condition for the positively homogeneous extension $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ of $u : S^{n-1} \to \mathbb{R}$ to be a convex function.

1. Introduction. We say that a $2\pi$–periodic function $p : \mathbb{R} \to \mathbb{R}$ is a support function if there exists a convex compact set $C \subset \mathbb{R}^2$ such that

$$p(t) = \max_{x \in C} \langle x, e(t) \rangle, \quad t \in \mathbb{R},$$

where $e(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$ and $\langle x, y \rangle$ stands for the scalar product of vectors $x, y \in \mathbb{R}^2$.

We refer to Rademacher’s test for convexity (see [7], and [1, p. 28]) as a necessary and sufficient condition for $p$ to be a support function. There are also other tests, one of them was proposed by Gelfond ([5, p. 132]), and another one by Firey ([3, p. 239, Lemma]).

2000 Mathematics Subject Classification. 52A10, 46F99.

Key words and phrases. Support functions, distributions, plane convex sets.
Gelfond’s test. A $2\pi$-periodic function $p : \mathbb{R} \to \mathbb{R}$ is a support function iff
\[
\det\begin{bmatrix}
cos t_1 & \sin t_1 & p(t_1) \\
cos t_2 & \sin t_2 & p(t_2) \\
cos t_3 & \sin t_3 & p(t_3)
\end{bmatrix} \geq 0
\]
for all $0 \leq t_1 \leq t_2 \leq t_3 \leq 2\pi$, such that $t_2 - t_1 \leq \pi$ and $t_3 - t_2 \leq \pi$.

Let
\[ S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \} . \]
We say that $p : S^{n-1} \to \mathbb{R}$ is a support function if there exists a convex compact set $C \subset \mathbb{R}^n$ such that
\[ p(u) = \max_{x \in C} \langle x, u \rangle , \quad u \in S^{n-1} . \]

Firey’s test. Let $\{a_1, a_2, \ldots, a_n\}$ be a fixed orthonormal basis in $\mathbb{R}^n$. A function $p : S^{n-1} \to \mathbb{R}$ is a support function iff
\[
\det\begin{bmatrix}
\langle u_1, a_1 \rangle & \ldots & \langle u_1, a_n \rangle & p(u_1) \\
\ldots & \ldots & \ldots & \ldots \\
\langle u_n, a_1 \rangle & \ldots & \langle u_n, a_n \rangle & p(u_n) \\
\langle u_{n+1}, a_1 \rangle & \ldots & \langle u_{n+1}, a_n \rangle & p(u_{n+1})
\end{bmatrix}
\times \det\begin{bmatrix}
\langle u_1, a_1 \rangle & \ldots & \langle u_1, a_n \rangle \\
\ldots & \ldots & \ldots & \ldots \\
\langle u_n, a_1 \rangle & \ldots & \langle u_n, a_n \rangle
\end{bmatrix} \leq 0
\]
for all $u_1, \ldots, u_{n+1} \in S^{n-1}$, such that $u_{n+1} = \sum_{i=1}^{n} t_i u_i$, $t_i \geq 0$, $i = 1, 2, \ldots, n$.

In this paper we propose another test for convexity involving distributional derivatives of the function $p$.

2. Main result. In this section we will present the main result of the paper.

The symbol $D'(\mathbb{R})$ will stand for the space of all distributions in $\mathbb{R}$ and $L^1$ will denote the Lebesgue measure in $\mathbb{R}$. Distribution theory will be the main tool used in the sequel.

Theorem 1. Let $C \subset \mathbb{R}^2$ be a nonempty convex compact subset of $\mathbb{R}^2$. Define $p_C : \mathbb{R} \to \mathbb{R}$,
\[ p_C(t) = \max_{x \in C} \langle x, e(t) \rangle , \]
where $e(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$. Under these assumptions, the distribution $p_C + p_C''$ is a $2\pi$-periodic non-negative Radon measure in $\mathbb{R}$.
Theorem 2. Given a $2\pi$-periodic non-negative Radon measure $\varrho$ in $\mathbb{R}$, satisfying the condition
\[
\int_0^{2\pi} e(t) \varrho(dt) = 0.
\]
Let $p \in D'(\mathbb{R})$ be a distributional solution of the differential equation
\[
(1) \quad p + p'' = \varrho.
\]
Under these assumptions
(a) $p$ is a $2\pi$-periodic Lipschitz function,
(b) for each $t \in \mathbb{R}$,
\[
p(t) = \max_{x \in C_p} \langle x, e(t) \rangle
\]
where $C_p$ is the closure of the convex hull of all points of the form
\[
p(t)e(t) + p'(t)e'(t),
\]
(c) if $q \in D'(\mathbb{R})$ is another solution of (1) then
\[
C_q = C_p + w
\]
for some $w \in \mathbb{R}^2$.

Theorems 1 and 2 establish a “local” version of the Rademacher–Gelfond’s test for convexity. Proofs of Theorem 1 and Theorem 2 will be presented in sections 3 and 4.

3. From set to measure.

A. Let $C \subset \mathbb{R}^2$ be a nonempty convex compact set. Define
\[
u(y) = \max_{x \in C} \langle x, y \rangle, \; y \in \mathbb{R}^2.
\]
Clearly
\[
p_C(t) = \nu(e(t)), \; t \in \mathbb{R}.
\]
Since $\nu$ is Lipschitz and positively homogeneous, there exists a set $E \subset \mathbb{R}$ such that $L^1(\mathbb{R} \setminus E) = 0$ and for each $t \in E$, $\nu$ has a usual derivative $\nu'$ at $e(t)$ and $e(t)$ is a Lebesgue point of $\nu'$. Indeed, if $\nu'(e(t))$ does not exist then $\nu'(\lambda e(t))$ does not exist for all $\lambda > 0$. Therefore, if the measurable set $\{t \in \mathbb{R} : \nu'(e(t)) \text{ does not exist}\}$ has a positive measure, then the set $\{x \in \mathbb{R}^2 : \nu'(x) \text{ does not exist}\}$ has a positive measure which contradicts Rademacher’s theorem.
Moreover,
\begin{equation}
\langle u'(e(t)), e(t) \rangle = u(e(t)), \quad t \in E.
\end{equation}

\textbf{B.} Let us fix \( \psi \in C_0^\infty(\mathbb{R}^2) \) with the following properties
\begin{align*}
\psi & \geq 0, \\
\text{supp } \psi & \subset B[0, 1] = \{ x \in \mathbb{R}^2 : \| x \| \leq 1 \}, \\
\int_{\mathbb{R}^2} \psi(x) \, dx & = 1.
\end{align*}
Next, for each \( \varepsilon > 0, \ x \in \mathbb{R}^2 \) and \( t \in \mathbb{R} \), define
\begin{align*}
\psi_\varepsilon(x) & = \frac{1}{\varepsilon^2} \psi\left(\frac{x}{\varepsilon}\right) \\
u_\varepsilon(x) & = \int_{\mathbb{R}^2} u(x - y) \psi_\varepsilon(y) \, dy \\
p_\varepsilon(t) & = u_\varepsilon(e(t)).
\end{align*}
Obviously, \( u_\varepsilon \) is convex, both \( u_\varepsilon \) and \( p_\varepsilon \) are \( C^\infty \) functions and \( p_\varepsilon \to p_C \) uniformly in \( \mathbb{R} \). Since \( e'' = -e \) we have
\begin{align*}
p''_\varepsilon(t) & = \langle u''_\varepsilon(e(t)) \, e'(t), e'(t) \rangle - \langle u'_\varepsilon(e(t)), e(t) \rangle, \quad t \in \mathbb{R}.
\end{align*}
Consequently, for each \( \varphi \in C_0^\infty(\mathbb{R}) \)
\begin{align*}
\langle p_\varepsilon + p''_\varepsilon, \varphi \rangle_{L^2} & = \int_\mathbb{R} (p_\varepsilon(t) + p''_\varepsilon(t)) \varphi(t) \, dt \\
& = \int_\mathbb{R} \langle u''_\varepsilon(e(t)) \, e'(t), e'(t) \rangle \varphi(t) \, dt \\
& \quad + \int_\mathbb{R} (p_\varepsilon(t) - \langle u'_\varepsilon(e(t)), e(t) \rangle) \varphi(t) \, dt.
\end{align*}
By (2), see e.g. [2, Theorem 1 (iv), (v), p. 123],
\begin{align*}
\lim_{\varepsilon \downarrow 0} \int_\mathbb{R} (p_\varepsilon(t) - \langle u'_\varepsilon(e(t)), e(t) \rangle) \varphi(t) \, dt = 0.
\end{align*}
Thus, when \( \varphi \geq 0 \),
\begin{equation}
\langle p_C + p''_C, \varphi \rangle_{L^2} = \lim_{\varepsilon \downarrow 0} \langle p_\varepsilon + p''_\varepsilon, \varphi \rangle_{L^2} \geq 0.
\end{equation}
C. Clearly, $p_C + p_C''$ is $2\pi$–periodic. It follows from (3), see e.g. [6, Theorems 2.1.7, 2.1.8, 2.1.9], that $p_C + p_C''$ is a non-negative Radon measure in $\mathbb{R}$.

4. From measure to set.

D. Every solution to (1) has the form (see e.g. [4, p. 28])

$$p(t) = a \cos t + b \sin t + S(t),$$

where $a, b \in \mathbb{R}$ and

$$S(t) = \int_0^t \sin (t - s) \varrho(ds), \ t \in \mathbb{R}.$$ It is easy to verify that

$$\langle S', \varphi \rangle_{L^2} = \langle C, \varphi \rangle_{L^2}, \ \varphi \in C_0^\infty(\mathbb{R}),$$

where

$$C(t) = \int_0^t \cos (t - s) \varrho(ds), \ t \in \mathbb{R}.$$ Therefore, see [2, Theorem 5, p. 131], $S$ is Lipschitz.

E. Let $p$ be a solution to (1). Denote by $E$ the set of all $t \in \mathbb{R}$ for which the usual derivative $p'$ exists. Let

$$z(t) \overset{\text{def}}{=} p(t)e(t) + p'(t)e'(t), \ t \in E,$$

$$Z = \{z(t) : t \in E\}.$$ We claim that

$$p(\tau) = \sup_{t \in E} \langle z(t), e(\tau) \rangle, \ \tau \in E.$$ Indeed, for $t \in E$, we have

$$\langle z(t), e(\tau) \rangle = \langle p(t)e(t) + p'(t)e'(t), e(\tau) \rangle$$

and

$$\lim_{t \to \tau} \langle z(t), e(\tau) \rangle = p(\tau).$$ On the other hand, in the sense of distribution theory,

$$\frac{d}{dt} \langle z(t), e(\tau) \rangle = \langle p'e + pe' + p''e + p'e'', e(\tau) \rangle$$

$$= (p + p'') \langle e'(t), e(\tau) \rangle = q \sin (\tau - t).$$
It follows from [6, Theorem 4.1.6], that \( \langle z(t), e(\tau) \rangle \) is nondecreasing in \((\tau - \pi, \tau)\) and nonincreasing in \((\tau, \tau + \pi)\). Consequently, since \( p \) is \(2\pi\)-periodic, we have
\[
p(t) = \sup_{t \in E} \langle z(t), e(\tau) \rangle, \quad \tau \in E,
\]
as claimed.

**F.** Let \( C_p \) be the closure of the convex hull of \( Z \). Obviously,
\[
p(\tau) = \max_{x \in C_p} \langle x, e(\tau) \rangle, \quad \tau \in E.
\]
Since \( p \) and \( e \) are continuous and \( E \) is dense in \( \mathbb{R} \), we have,
\[
p(t) = \max_{x \in C_p} \langle x, e(t) \rangle, \quad t \in \mathbb{R}.
\]

### 5. Convex extension
In this section a simple application of Theorem 1 and Theorem 2 will be given. We will prove the necessary and sufficient condition for the positively homogeneous extension \( \tilde{u} : \mathbb{R}^n \to \mathbb{R} \) of \( u : S^{n-1} \to \mathbb{R} \) to be a convex function.

Let \( S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \} \) and let \( u : S^{n-1} \to \mathbb{R} \) be a function. For each \( a, b \in S^{n-1} \) satisfying \( \langle a, b \rangle = 0 \), define \( e_{a,b} : \mathbb{R} \to S^{n-1}, u_{a,b} : \mathbb{R} \to \mathbb{R} \) and \( \tilde{u} : \mathbb{R}^n \to \mathbb{R} \)
\[
e_{a,b}(t) = a \cos t + b \sin t,
\]
\[
u(a,b)(t) = u(e_{a,b}(t)),
\]
\[
\tilde{u}(x) = \begin{cases} \|x\| \cdot u \left( \frac{x}{\|x\|} \right), & x \neq 0 \\ 0, & x=0. \end{cases}
\]
Recall that \( u : \mathbb{R}^n \to \mathbb{R} \) is positively homogeneous if
\[
u(\alpha x) = \alpha \cdot u(x)
\]
for all \( x \in \mathbb{R}^n \) and \( \alpha > 0 \).

**Theorem 3.** If \( u : \mathbb{R}^n \to \mathbb{R} \) is convex and positively homogeneous then \( u_{a,b} + u''_{a,b} \) is a \(2\pi\)-periodic, non-negative Radon measure on \( \mathbb{R} \) for all \( a, b \in S^{n-1} \), where \( \langle a, b \rangle = 0 \).

**Proof.** Fix \( a, b \in S^{n-1}, \langle a, b \rangle = 0 \). Let \( v : \mathbb{R}^2 \to \mathbb{R}, \)
\[
v(x_1, x_2) = u(x_1 a + x_2 b)\]
be a restriction of $u$ to $\text{lin}\{a, b\}$. Obviously, $v$ is convex. The set

$$C = \{x \in \mathbb{R}^2 : \forall y \in \mathbb{R}^2 \langle x, y \rangle \leq v(y)\}$$

is a convex compact subset of $\mathbb{R}^2$ and

$$v(y) = \max_{x \in C} \langle x, y \rangle$$

for all $y \in \mathbb{R}^2$, see e.g. [8, Corollary 13.2.1]. Consider

$$u_{a,b}(t) = u(e_{a,b}(t)) = v(e(t))$$

and apply Theorem 1 to show that $u_{a,b} + u''_{a,b}$ is a $2\pi$–periodic, non-negative Radon measure on $\mathbb{R}$. □

**Theorem 4.** If $u : S^{n-1} \to \mathbb{R}$ is continuous and $u_{a,b} + u''_{a,b}$ is a $2\pi$–periodic, non-negative Radon measure on $\mathbb{R}$, satisfying

$$\int_0^{2\pi} e_{a,b}(t) \left(u_{a,b} + u''_{a,b}\right)(dt) = 0$$

for all $a, b \in S^{n-1}$, where $\langle a, b \rangle = 0$, then $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ is convex.

**Proof.** Let $z, y \in \mathbb{R}^n$ be fixed. There exist $a, b \in S^{n-1}$, $\langle a, b \rangle = 0$, such that $z, y \in \text{lin}\{a, b\}$. Applying Theorem 2 to the function $u_{a,b}$, we have

$$\tilde{u}(z + y) = \|z + y\| \max_{x \in C} \left\langle x, \frac{z + y}{\|z + y\|} \right\rangle \\
\leq \|z\| \max_{x \in C} \left\langle x, \frac{z}{\|z\|} \right\rangle + \|y\| \max_{x \in C} \left\langle x, \frac{y}{\|y\|} \right\rangle \\
= \tilde{u}(z) + \tilde{u}(y).$$

for some convex compact set $C \subset \text{lin}\{a, b\}$. Therefore $\tilde{u}$ is convex. □

**References**


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Received April 14, 2003