Symmetries in Physics: Guidelines for Theories and for Experiments

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ABSTRACT

The works of Maria Sklodowska-Curie and Pierre Curie, of their few predecessors and of their many followers addressed over the years the studies of the atomic nuclei - the smallest objects in the Universe which are unique in that they are governed simultaneously by the strong, electromagnetic and weak interactions. In this article we focus on the concept and nature of symmetries, their omni-presence in physics and their impact on the behaviour of the physical systems. Beginning with a short historical overview covering quickly the birth of certain concepts in the ancient times and their evolution until the most modern ones we cover, on an introductory level, the question of space-time symmetries, the connection between the intrinsic degrees of freedom and the spatial behaviour of quantum particles as well as the question of symmetry-induced conservation-laws. We discuss shortly examples of continuous and discrete symmetry groups, the constraints imposed on the energy spectra (degeneracy of levels) by the symmetries of the underlying Hamiltonians, to end with the question of transitions and symmetry imposed selection rules. The article terminates with a short discussion of the symmetry breaking phenomena, spontaneous symmetry breaking and phase-transition induced symmetry-changes.

2. HISTORY: A BRIEF OVERVIEW

The fascination by symmetry accompanied human creativity already since ancient times and it seemed to be a natural part of human thinking. However, the name symmetry (\textit{συμμετρία}) is ‘moderately’ old and originates from the ancient
Greek language. It is composed of two words: συμ (‘together’) and μετρον (‘measure’). The implied meaning: ‘measured together’ or - more appropriately - well proportioned, evolved in time into ‘beauty’, ‘unity’ and/or ‘harmony’. In sciences the first meaning of the word symmetry was related to proportions. The meaning closer to our contemporary understanding of symmetry in terms of the equality of elements under some transformations, e.g., translations, rotations, and/or reflections arrived only towards the end of the Renaissance.

It is this latter formulation that, through historical perspective, had a strong impact on the research of many-body systems such as atomic nuclei - one of the most important issues in the research of Maria Skłodowska-Curie, Pierre Curie and many other brilliant physicists who studied those systems through their properties of radiation. The fact that perceiving symmetry through equality of elements under certain transformations is perfectly adapted to the quantum-mechanical formulation of the description of the motion of such many-body systems - the invariant object being in this case just the Hamiltonian of the studied physical system.

For centuries, the so called Platonic figures, cf. fig. 1, right, were symbols of geometrical beauty. The ancient Greek philosopher Plato (428-347 BC) has rediscovered three-dimensional polyhedra whose faces are identical planar regular convex polygons (equilateral triangles, squares or regular pentagons). As it seems, neolithic people from Scotland have developed the five Platonic solids about 1000-3000 years before Plato - the corresponding stone models can be admired in the Ashmolean Museum at Oxford.

Fig. 1. Left: Plato (428-347 BC). Right: The celebrated five Platonic solids: tetrahedron, cube, octahedron, dodecahedron and icosahedron [1].

Nearly at the same time Euclid described the Platonic solids in his Elements [2], the last book of which (Book XIII) is devoted to their properties. In the Proposition 18 he argues that there are no further convex regular polyhedra. Much of the information in his Book XIII is likely derived from the work of
Theaetetus who is thought to be the discoverer of the octahedron and icosahedron. The contemporary proof of the existence of only five Platonic solids is based on the theorem formulated by Euler; this theorem relates the numbers of vertices $V$, faces $F$ and edges $E$ as follows

$$V - F + E = 2. \quad (1)$$

The above famous property was actually demonstrated by Lagrange [3].

**Plato and Kepler’s Geometrical Model of the Planetary System.** Natural scientists and philosophers were fascinated by the symmetries of Platonic solids. One of the first in modern times was Johannes Kepler. The

![Fig. 2. Left: J. Kepler (1571-1630). Right: Kepler’s planetary system [1]; the model is explained in the text.](image)

Kepler’s geometrical ‘theory’ of the planetary system was constructed using a sequence of Platonic solids (for Kelper’s historical portrait cf. fig.(2)). Although false, it has an interesting historical value. We quote here its elements after Kepler’s *Mysterium Cosmographicum* of as early as 1596:

- The Earth orbit is the measure of all orbits.
- Around it we circumscribe the dodecahedron.
- The Mars orbit is circumscribed around the dodecahedron.
- Around the Mars orbit we circumscribe the tetrahedron.
- The Jupiter orbit is circumscribed around the tetrahedron.
- Around of the Jupiter orbit we circumscribe the cube.
- The Saturn orbit lies in the sphere surrounding the cube.
- In the Earth orbit the regular icosahedron is inserted.
- The orbit entered in it is the Venus orbit.
- In the Venus orbit the octahedron is inserted.
- The orbit entered in it is the Mercury orbit.
Hundreds years later we know of course that this naive model does not reflect any dynamics of the planetary system. And yet: In ref. [4] you will find an exercise that aims at quantifying Kepler’s modelling. It begins by considering Saturn and Jupiter (a cube, see above) and circumscribing and inscribing two spheres - then calculating the ratio of the radii of these spheres which is 1.73 compared to the presently known experimental ratio of the trajectory radii for the two planets equals to 1.73; continuing the author of [4] obtains consecutively the following ratios: Jupiter/Mars (3.00 compared to 3.42), Mars/Earth (1.26 compared to 1.52), Earth/Venus (1.26 compared to 1.38) and finally Venus/Mercury (1.73 compared to 1.87). This could have been interpreted as an encouraging attempt of using the symmetry concept to generate numbers (theoretical modelling of those times). Today, the underlying principle of ’symmetry as a guideline’ can be found in many contemporary genuinely dynamical theories as e.g., in the quantum field theory of elementary particles and/or advanced quantum-mechanical theories of atomic- and subatomic-scale systems. Today’s applications of the symmetry concepts turns out not only to be pedagogically instructive but also extremely successful in producing experiment-comparable results.

**The First Publication Describing Symmetries of Physical Objects.** In the modern era, the Kepler’s publication ‘On the Six-Cornered Snowflake’ (the original title ‘*Strena seu de nive sexangula*’, 1611) is very likely the historically first work which describes symmetries of physical objects and attempts their explanation. Until today, the snow-crystal symmetries are an interesting subject of investigations. In fig. (3) some examples of hexagonal symmetry of the snowflakes are shown, whereas in fig. (4), a possible ‘microscopic background’ of the associated hexagonal symmetry is illustrated. Obviously Kepler in his times could not know anything neither about the basic hexagonal symmetry of ice crystals, cf. fig. (4), nor about the fact that the actual symmetry of water crystals produced in the air may vary in function of temperature, humidity and pressure (cf. Nakaya diagram, ref. [5]).

Fig. 3. Examples of snowflakes by Kenneth K. Librecht, Caltech
Modern Times: Growing Knowledge about Symmetries in Nature. It took over two centuries to systematically develop today’s image of the geometrical symmetries encountered in crystals and Kepler’s work was just the beginning. Examples of the historical achievements can be shortly summarised as follows:

- **Kepler (1611)** - Observes hexagonal symmetry of snow crystals.
- **Steno (1669)** - Observes that inclination angles of faces of the quartz crystals are the same and independent of the crystal size.
- **De Lisle (1783)** - Observes the fact that angles between crystal faces can be used as a practical criterion to identify crystals.
- **Haüy (1784)** - Studies the symmetries: rotations & reflections.
- **Wollaston (1809)** - Builds the first goniometer to measure crystal angles, thus paving the way for systematic experimental studies.
- **Hausmann (1821)** - Introduces the spherical trigonometry.
- **Hessel (1830)** - Shows the existence of 32 crystal polyhedral symmetries opening the way for modern crystallography.

The development of understanding of geometrical symmetries played historically a very important role and lead to a more profound and mathematically advanced studies of symmetries.

### 3. TOWARDS CONTEMPORARY NOTION OF SYMMETRY

Probably the most important step in the formal developments related to symmetries was an observation that combining a symmetry operation with another symmetry operations leads to yet another symmetry operation. This property brought up the abstract notion of one of the simplest algebraic structures: the group.

An important element in constructing the modern group theory was the notion of permutation introduced by Joseph-Louis Lagrange (1736-1813) and the discovery that permutations form a group. Indeed this was Evariste Galois (1811-1832) who discovered that combining permutations furnishes a prototype of a
group structure. As it has been shown later, all groups composed of finite
number of elements are equivalent to certain specific sub-sets of permutations
(today we say: sub-groups, see below), or in other words, by studying permuta-
tion groups we may learn about all possible finite groups.

Analysing the properties of ensembles \( S_n \) of permutations of \( n \) elements
allowed Augustin Louis Cauchy (1789-1857) to make an important next step
forward and introduce the abstract definition of a group as a general alge-
braic structure. Accordingly, any ensemble of elements, say \( g \in G \), is called
group if its elements can be ‘combined’ according to a certain prescription
denoted ‘\( \circ \)’, in such a way, that the result of a combination of any two ele-
ments of \( G \) will be yet another element of \( G \). This prescription is sometimes
incorrectly called ‘product’ or ‘multiplication’ - incorrectly, since the num-
ber-multiplication is an important, but only one among infinity of possible
combination laws hidden behind the symbol ‘\( \circ \)’. More precisely, the defini-
tion of a group involves the following four postulates:

1° For all \( g_1, g_2 \in G \), the ‘product’, \( g_1 \circ g_2 = g \in G \).

2° For any \( g_1, g_2, g_3 \in G \) we have \( (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3) \).

3° There exist a special element, \( e \in G \), called neutral, such that for all \( g \in G \): \( e \circ g = g \); it then follows that \( g \circ e = g \).

4° For any \( g \in G \) there exist an inverse \( g^{-1} \in G \) such that \( g \circ g^{-1} = e \).

In physics, as well as in mathematics, studying the symmetries is greatly
advanced thanks to human abstract thinking. Indeed, it is easy to see that
turning a tractor by \( 15^\circ \) and turning a cucumber by \( 15^\circ \) contains the same
element, rotation about \( 15^\circ \), the latter concept being independent of the ob-
ject that has actually been manipulated. The fast progress in natural sciences
had been possible to a far extent due to the human brain capacity to dissolve
the inessential practical realisation that differ from one case to another - from
an abstract essential common feature (e.g. tractors and cucumbers are ines-
tsential realisations and rotations are essential elements).

**Categories, Equivalence Relations, Identical Objects (Particles).** We
usually do not pay much attention to the abstract aspect of the notion of ‘iden-
tical objects’ which allows us, humans, to split the universe into simpler sub-
universe type structures (families of objects such as, snakes, elephants, pro-
tons, neutrons and/or elementary particles). Behind the word ‘neutrons’ hides,
from the mathematics point of view, an equivalence relation that here will be
denoted ”\( \sim \)”: it is, more precisely, a binary relation. Let an ensemble of objects
of interest (snakes or neutrons or ...) be \( X \). Our binary relation is said to be an
equivalence relation if the following three descriptors apply:
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- Symmetric → For all \( x, y \in X \): if \( x \sim y \) then \( y \sim x \);
- Transitive → For all \( x, y, z \in X \): if \( x \sim y \) and \( y \sim z \) then also \( x \sim z \);
- Reflexive → For all \( x \in X \) we have \( x \sim x \).

An equivalence relation splits ensemble \( X \) into distinct subsets called classes, more often: classes of equivalence. An equivalence class of an element \( x \) belonging to \( X \) is a set of all elements which are equivalent to \( x \) according to the relation \( \sim \). It then follows that the objects belonging to the same equivalence class are considered to be identical and we arrive at the following concept of the principle of universal categorising:

*All physical objects in our Universe can be classified according to equivalence relations (they may have hierarchical sub-structures).*

The idea of categorising has been applied by humans not only to physical objects but also, on a more abstract level, to various mathematical concepts and their abstract properties such as e.g. transformations and/or reference frames - we will discuss this latter aspect to some extent in what follows.

**Equivalence Relations, Space, Symmetries and Conservation Laws.**

One of the very important examples of equivalence relations - which not only changed the rules of constructing the physics theories but also drastically influenced our perception of the world is hidden in the principle of relativity as formulated by Galileo Galilei (1564-1642). The principle of relativity is equivalent to the observation that there exist distinct reference frames in which the laws of physics have the same form - these reference frames which are called inertial are equivalent. It is because of this feature that natural sciences like physics could be created: Imagine the challenge to all physicists if it turned out that the physical laws studied in Lublin were different as compared to those studied, say, in Strasbourg.

The notions of geometrical space and of the elementary dimensionless elements of it, points, was another mile-stone in the construction of physical theories - but at the same time posed a great intellectual challenge: Are all the points in our space-time equivalent thus presenting a big class of equivalence? Are there any directions which can be considered privileged (‘better’) with respect to the others?

As it is known today, the fundamental conservation laws as established experimentally in mechanics, the conservation of energy and those of linear- and angular-momentum, can be obtained formally by assuming the equivalence of all points (uniformity of space and time) and directions (isotropy of space) in our four-dimensional space-time. This observation strongly sug-
gests that our space and time can be considered as constituted of equivalent points and, moreover, that there are no distinguished directions in space - whereas at the same time it builds a conceptual bridge between space-time properties and the conservation laws of mechanics.

These observations imply in particular an important philosophical (and even a religious) consequence: Our position in the universe is by no means exceptional.

**Symmetric Group, Permutations and Indistinguishable Particles.** As already mentioned, one of the earliest group structures, considered by mathematicians is the permutation group of \( n \) elements, \( S_n \); it is sometimes called symmetric group. Let us notice the probably most important mathematical property of this group as assured by the celebrated

\[
\text{Cayley Theorem: Every finite group of rank } n \text{ is isomorphic with a rank } n \text{ subgroup of the symmetric group } S_n.
\]

A nontrivial but simple example of symmetric group is \( S_3 \) of rank 6 with elements:

\[
S_3 \overset{df}{=} \{ (1,2,3), (1,2,3), (1,2,3), (1,2,3), (1,2,3), (1,2,3) \}.
\]

The symmetric group plays a fundamental role in physics. It is directly related to the notion of identical particles\(^1\). In classical physics there are no special effects of the symmetry under permutation: The particles are identical but remain macroscopically distinguishable to us, humans. On the other hand, in the physics of the micro world (quantum physics), the elementary particles are indistinguishable to the human instruments and thus remain indistinguishable for our theoretical modelling.

In this context an interesting question arises about effects of permutation of identical particles in the physical system. The first observation is that the Hamiltonian of a system of identical particles has to be invariant under their permutations - as it directly follows from the definition of the word identical, equivalent to indistinguishable. It then follows that the operator of permuting

\(^1\) In the framework of quantum field theory the presence of elementary particles in nature can be seen as a manifestation of the vacuum fluctuations also called vacuum excitations. The strong interactions impose various mechanisms resulting at the end at the presence of a given type of particle: two particles are identical (belong to the same class of equivalence) if they behave in the same way under all possible influence of all interactions.
the particles no. \(i\) and \(j\), say \(\hat{P}_{ij}\) must commute with the Hamiltonian and consequently the eigen-states of the Hamiltonian can be simultaneously labelled with the index associated with the Hamiltonian, \(n\), as well as with the eigenvalues of \(\hat{P}_{ij}\)

\[
\hat{P}_{ij} \Psi_{n,p} = p_{ij} \Psi_{n,p}.
\]  

(3)

Since \(\hat{P}_{ij}^2 = 1\) it follows that \(p_{ij}^2 = 1\) and therefore \(p_{ij} = \pm 1\). In three-dimensional space there are indeed observed only two types of wave functions for identical particles: either totally symmetric \((p_{ij} = +1, \text{ bosons})\) or totally antisymmetric \((p_{ij} = -1, \text{ fermions})\).

**Symmetries Generated by Continuous Matrix-Transformations.** In quantum mechanics as well as in classical and quantum field theories there exist transformations of the underlying Hamiltonians (and/or Lagrangians) that can be parametrized with the help of a number of continuous parameters. The importance of those transformations lies in the fact that they express possible symmetries of the Hamiltonians/Lagrangians and thus the related physical systems. Moreover, because the space-time seems to be continuous - both the non-relativistic as well as relativistic space-time symmetries (such as translations, rotations, Lorentz boosts and translations in time) the transformations of the space-time render themselves expressible in terms of the continuous functions - in particular with the help of matrices. In this context the formalism addressing the properties of the so-called continuous groups introduced by Sophus Lie (1842-1899) became very fruitful. The corresponding structures are known today under the name of Lie groups. The transformations of the space-time (translations and rotations) belong to this category. Typical Lie groups such as \(\text{GL}(n)\), \(\text{SL}(n)\), \(\text{U}(n)\), \(\text{SU}(n)\), \(\text{O}(n)\), \(\text{Sp}(2n)\), … have indeed numerous applications.

Fig. 5. Sophus Lie (1842-1899).
One of the most important features of the Lie groups is that each can be
generated by a finite number of operators (generators) related to infinitesimal
transformations. The generators themselves form a basis of the so-called Lie
algebra which is a vector space $\mathcal{L}$ with an intrinsic product defined as fol-
lows:

$$
\begin{align*}
[aX + \beta Y, Z] &= \alpha[X, Z] + \beta[Y, Z], \quad \alpha, \beta \in \mathbb{C}, \\
[X, Y] &= -[Y, X], \\
[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] &= 0,
\end{align*}
$$

for all $X, Y, Z \in \mathcal{L}$. Let the set of generators of a given group be $\{X_n\}$; these
operators can be used to express the elements $\hat{g}$ of the Lie group $G$ in the
form:

$$
\hat{g}\{\alpha\} = \exp\left\{-i\sum_{n=1}^{\infty} \alpha_n X_n\right\}.
$$

For example: The Lie group of space translations can be expressed as:

$$
T(\vec{a}) = \exp\left\{-i\vec{a} \cdot \hat{p} / \hbar\right\},
$$

where $\hat{p}$ is the quantum mechanical linear momentum operator. Similarly, the
elements of the Lie group of rotations in a 3-dimensional space can be
represented as follows:

$$
R(\omega, \vec{n}) = \exp\left\{-i\omega \vec{n} \cdot \hat{J} / \hbar\right\},
$$

where $\hat{J}$ is the quantum mechanical angular momentum operator whereas $\vec{n}$
is a unit vector defining the position of the rotation axis and $\omega$ the associated
rotation angle. Finally, the group of translations in time can conveniently be
expressed by

$$
U_{\hat{H}}(\Delta t) = \exp\left\{-i\Delta t \hat{H} / \hbar\right\},
$$

where $\hat{H}$ denotes the Hamiltonian of the time-translated physical system.
4. SPACE-TIME SYMMETRIES AND CONSERVATION LAWS

The space-time symmetries have been used by Amalle Emma Noether (1882-1935) to show very strong relations between properties of the space and time and the most important conservation laws such as the energy, linear momentum and angular momentum conservation (Noether Theorem) and we will slow down here to present this issue. She considered, among others, the symmetries of the classical systems described by the Lagrangian $L(q, \dot{q}; t)$.

To formulate the problem of invariance (symmetry) consider, after Noether, a one-parameter continuous mapping

$$q_j(t) \rightarrow Q_j(s,t) \text{ such that } Q_j(s,t)|_{s=0} = q_j(t) \text{ for } j = 1, 2, \ldots f.$$  \hfill (11)

Suppose that transformation (11) leaves the Lagrangian invariant - in other words - independent of $s$:

$$L(Q, \dot{Q}; t) = L(q, \dot{q}; t) \forall s \iff \frac{d}{ds} L\left[Q(s, t), \dot{Q}(s, t); t\right] = 0 \forall t.$$  \hfill (12)

Then the quantity

$$F = \sum_{j=1}^{f} \frac{\partial L}{\partial \dot{q}_j} \left( \frac{\partial Q_j}{\partial s} \right)_{s=0}$$  \hfill (13)

is a constant of motion for the considered physical system.

Let us give an elementary illustration of the Noether Theorem for the case of invariance under the group of translations. Consider an isolated physical system described by the Lagrangian

$$L(\vec{r}, \dot{\vec{r}}, t) \text{ with } \vec{r} = \{\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N\}.$$  

Let $\vec{n}$ be an arbitrary fixed unit vector in space. We introduce a one-parameter family of translations by

$$\vec{r}_i \rightarrow \vec{r}'_i = \vec{r}_i + s \vec{n} \rightarrow \dot{\vec{r}}_i \rightarrow \dot{\vec{r}}'_i = \dot{\vec{r}}_i.$$
Suppose that the Lagrangian is invariant with respect to this one-parameter mapping so that we have

\[ L(\vec{r} + s\vec{n}, \dot{\vec{r}}; t) = L(\vec{r}, \dot{\vec{r}}; t) \quad \forall s. \]

In the considered case the transformations in the formulation of the Noether theorem take the Cartesian form rather than that using generalised coordinates

\[ \vec{r} \leftrightarrow q_j(t) \quad \rightarrow \quad Q_j(s, t) \leftrightarrow \vec{r} + s\vec{n} \]

so that

\[ \left. \frac{\partial Q_j}{\partial s} \right|_{s=0} = \vec{n} \rightarrow F = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{r}_i} \cdot \vec{n} = \sum_{i=1}^{N} \vec{p}_i \cdot \vec{n} = \vec{n} \cdot \sum_{i=1}^{N} \vec{p}_i = \vec{n} \cdot \vec{P} = \text{const.}, \]

where \( \vec{P} \) denotes the total linear momentum of the physical system. Since direction of \( \vec{n} \) has been chosen arbitrarily, the above relation signifies that projection of the total linear momentum on any axis remains conserved and thus the vector \( \vec{P} \) itself must be a constant of the motion. Should it happen that the above conditions are satisfied for one particular direction \( \vec{n} \) only, then only the projection of the total linear momentum on that particular direction will be a constant of motion.

The most important relations between symmetries satisfied by \( L \) and conservation laws, implied by Noether theorem, are:

- Translational symmetry in space (cf. above for pedagogical details) which implies the linear momentum conservation.
- Translational symmetry in time which implies the energy conservation.
- Rotational symmetry in space and angular momentum conservation.

It is important to say that the analogous relations (and conservation laws) apply in field theories which have a very different formalism relating observables and states as compared to the one in classical mechanics.

5. INTRINSIC PROPERTIES OF PARTICLES AND SPACE-TIME PROPERTIES

In nature there are two big families of particles whose associated physical quantum states behave differently under rotation about the angle of \( 2\pi \) (recall
that all classical objects return to their original situation when turned through the equivalent angle of $360^\circ$). Whereas the wave functions of particles belonging to one of these families do not change after such a rotation the wave functions belonging to the other family change their sign: to return to the original situation a next $2\pi$ rotation ($4\pi$ in total) will be necessary for those latter ones.

In the historical today experiment by Stern and Gerlach it has been demonstrated that beams of particles injected into an inhomogeneous magnetic field behave according to different scenarios. Whereas some beams pass through such a magnetic field with no effect, some others split into two, three or more ‘transmitted’ beams, each scenario characteristic of a given type of particles - and this - independently of the speed of particles. For instance the incoming beams of electrons or certain atoms like the Silver ones, split into two beams when quitting the magnetic-field, cf. Fig. 6.

![Fig. 6. The Stern and Gerlach experiment.](image)

The point particles have only one spatial characteristic: their $\{x, y, z\}$-position in space. Since this characteristic is common to all types of point-particles (so is the direction-dependent kinematical quantity, $\{p_x, p_y, p_z\}$, the linear momentum) the point particles must couple similarly to a given external field. Consequently, they are expected to split (or not) their beams - all the same: there is no discrete element of distinction from one type of point particle to another to cause one type of beam splitting or the other. And yet, all the best known and successful quantum equations of motion as e.g. Schrödinger, Dirac, Klein-Gordon etc., operate with the notion of point-particles! To understand the result of the Stern-Gerlach experiment for such particles we are forced to introduce the idea of a direction-dependent intrinsic characteristic of quantum particles - a clearly incomprehensible notion in
terms of classical physics in which, to introduce the direction, at least two points are necessary.

The concept of the direction-dependent intrinsic degrees of freedom of point-particles has been associated with the name: the intrinsic spin. Thanks to its presence the discussed point-particles are able to distinguish a direction in space that leads to a splitting of the beams - under the condition that this new quantity may take only discrete values. These discrete values, say $s$, turn out to be multiples of a certain specific unit (which happens to be equal to the Planck constant). The beams of particles with $s = \frac{1}{2}$ split into two, the ones with $s = 1$ into 3, the ones with $s = \frac{3}{2}$ into four ‘secondary’ beams etc.

In a special chapter of quantum mechanics, usually associated with the title The Problem of Spin and Statistics, it is shown that the particles with the half-integer $s$ form exclusively the anti-symmetric many-body wave-functions and thus must be the fermions whereas the ones with an integer $s$ form the totally symmetric many-body wave-functions and thus they are bosons (cf. preceding discussion in this article). As we can see the mathematical properties of space (and time) and the intrinsic properties of particles are ultimately related: the intrinsic spin decides whether the particle wave-functions change sign under a $2\pi$ spatial rotation (or not) but also about the ‘statistics’ - to be a boson or a fermion - in other words: to couple with the other particles of the same kind through the totally symmetric or totally anti-symmetric wave functions.

Within group theory it can be shown that the wave-functions of the quantum particles with integer spin transform under rotations through the angles $\Omega = \{a, \beta, \gamma\}$ with the help of the famous ‘Wigner D-functions’, $D_{m,m'}^s(\Omega)$, where $s$ remains integer and we find that the wave-function after rotation is given by

$$\psi_{m'}^s = \sum_{m} D_{m,m'}^s(\Omega) \psi_{m}^s. \quad (14)$$

The wave-functions of this type are said to form tensors with respect to the rotation group that can be identified in this case with SO(3).

A similar expression applies to the fermion wave functions characterised therefore by $s$ half-integer; the corresponding wave-functions are said to form spinors with respect to the group identified in this case with the special unitary group in two dimensions, SU(2). It can be shown that all the physical objects in classical physics transform like tensors and, furthermore, that there is no classical object in correspondence to the spinor transformation properties. The latter observation brings us to observing that by going from classic-
al- to quantum-physics we do not merely change the spatial scale of physical phenomena - we enter a genuinely *different* physics.

6. SYMMETRIES AND STRUCTURE OF ENERGY SPECTRA

In quantum mechanics one finds an extremely important connection between symmetries and related symmetry groups of the Hamiltonians and the implied energy spectra. A group of transformations $\mathcal{G}$ is called symmetry group of the Hamiltonian $\hat{H}$ if for each $g \in \mathcal{G}$ the Hamiltonian is invariant under this transformation:

$$\Gamma(g) \hat{H} \Gamma(g^{-1}) = \hat{H},$$

(15)

where $\Gamma(g)$ denotes a representation$^2$ of the group $\mathcal{G}$. Usually each group of interest in the present context is characterised by a number of irreducible representations, say $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$, where $r$ may be finite as in the case of the point-groups or infinite in many other cases. Each of the above irreducible representations is characterised by dimensions$^3$: $d_1, d_2, \ldots, d_r$, respectively. It can be demonstrated as an immediate consequence of relation (15) that there exist a $d_l$-fold degeneracy among the eigen-energies of $\hat{H}$. More precisely, the eigen-energies, $\epsilon_l$, enter extra-relations of the form:

$$H \psi_{v_k} = \epsilon_{v_k} \psi_{v_k},$$

(16)

in which eigen-values satisfy

$$\epsilon_{v_k=1} = \epsilon_{v_k=2} = \ldots = \epsilon_{v_k=d_l} \quad (d_l \text{- fold degeneracy})$$

(17)

whereas all the associated wave functions are non-trivially distinct.

This formal degeneracy has important physical consequences, but the attentive reader is warned at this point: The experimental manifestations of this fun-

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$^2$ The reader unfamiliar with the notions of the group representations is referred to Ref. [6] for details. However, the knowledge of the group representation theory is not necessary to follow the arguments in this Section. It will be sufficient for all practical purposes to keep in mind that irreducible representations can be thought of as sets of matrices of dimensions $d \times d$, where $d$ of interest are usually small integers.
fundamental property are far from evident to demonstrate in ‘practice’; this is because of various polarisation effects that arrive in the real many-body systems.

A Case of Nuclear Mean-Field Theory: A Tetrahedral Symmetry. In a series of recent publications, Ref [8], the authors have formulated a hypothesis that atomic nuclei may exist in a very exotic spatial configurations that resemble a ‘pyramid with the base of an equilateral triangle’ (a tetrahedron). Typical shapes of the nuclonic distributions in those hypothetical configurations are shown in Figs. (7-9).

Within the nuclear mean-field theory the underlying Hamiltonians are invariant under transformations belonging to the so-called point-group \( T_d \) (tetrahedral group). Without entering into details of the formalism of the nuclear structure theory it will be sufficient to mention that the ‘degree of tetrahedrality’ can be parametrized using what is called a tetrahedral deformation parameter which for all practical purposes can be seen to vary between 0 (spherical shape) and \( \approx 0.3 \) – cf. Figs. (7-9). One can solve the nuclear mean-field problem for a series of tetrahedral deformations to obtain an illustration shown in Fig. 10. In the figure one can notice the 2-fold and 4-fold degeneracy due to the fact that the tetrahedral group \( T_d^D \) has two two-dimensional and one four-dimensional irreducible representations. Let us emphasise that degeneracy in single-particle spectra of nucleons in the nucleus results in the presence multiplets - thus areas of an increased level densities - as well as the areas of the lower level-densities usually referred to as single-particle energy gaps whose presence increases the stability of nuclear systems with specific numbers of particles. In the case of our illustration these gaps (thus increased stability) corresponds to proton numbers \( Z=64, 70, 90-94 \) and 100.
Fig. 10. Single particle energy spectrum of protons in the nucleus $^{226}$Th. Full lines correspond to four-dimensional irreducible representations (four-fold degeneracy multiplets). The dashed lines represent two-dimensional irreducible representations. One can observe large energy gaps for the proton numbers $N=64$, 70, 90–94 and 100.

In the example above we have discussed the group $T_d$. This is an important but only one example of the ensemble of point group symmetries related by the sub-group relations. There exist 32 very well known, distinct point groups whose symmetry operations are obtained by combining (zero, one, or more) rotations by discrete angles, plane reflections, rotary reflections and, possibly, the inversion operations. The corresponding scheme is presented in Fig. 11. By adding translations one may construct 230 crystallographic space groups, 14 Bravais lattices and 7 basic lattices for crystals. They describe a large class of minerals and artificial crystals and determine, among others, the forms of the electronic motion within mentioned systems.

Fig. 11. The sub-group chains of 32 crystallographic point groups. The dashed lines denote here invariant subgroups - a mathematical detail that can be ignored by the reader unfamiliar with the terminology used in the group theory.
**Point-Groups as the Basis of the Molecular Symmetry Studies.** The point group theory and the related theory of group representations are the basis tools of advanced quantum-mechanical analysis of the properties of complex molecules. The reader is reminded that the corresponding groups, using the notation proposed by by Schönflies can be ordered as follows: $C_1, C_s, C_i, S_n, C_n, C_{nh}, C_{nv}, D_n, D_{nh}, D_{nd}, T, T_h, T_d, O, O_h, I, I_h$, where $n$ refers to certain small integers. A set of examples ordered from low symmetry to high symmetry molecules is presented in the figures (12–19). These examples illustrate the richness of existing structures and are taken from Ref. [9].

Let us observe in passing that the molecular symmetries and the symmetries in the many-body mean-field theories are mathematically identical. Both classes of objects such as e.g. molecules and atomic nuclei are characterised by the presence of the center of symmetry (a point which gives the name to the ‘point groups’). The principal difference concerns the particles subject to the action of the mean-fields of certain point symmetries: in the molecular case these are first of all the electrons - in the nuclear case the nucleons - the protons and the neutrons.

![Fig. 12. NH₃ has the symmetry C₃ᵥ.](image1)

![Fig. 13. B(OH)₃ has the symmetry C₃h.](image2)

![Fig. 14. C₂H₆ (isomer 1) has the symmetry D₃d.](image3)

![Fig. 15. C₂H₆ (isomer 2) has the symmetry D₃h](image4)

![Fig. 16. CH₄ has the symmetry T_d.](image5)

![Fig. 17. SF₆ has the symmetry O_h](image6)
Let us emphasise that the point-symmetries have also an important impact on the analysis of the properties of the collective excitations of molecules and nuclei - the collective vibrations and rotations. This is a fascinating subject, among others, because it offers a possibility of the experimental verifications of the presence of the discussed symmetries in nature - however the corresponding discussion would exceed the scope of the present article.

7. QUANTUM TRANSITIONS

Quantum physical systems and especially their energy spectra can be studied thanks to the electromagnetic transitions connecting various states of those systems. From the point of view of quantum mechanics and the group theory used to represent their symmetries the quantum states can be labelled with the help of the ‘energy index’ n and an index $\Gamma$ identifying the irreducible representation to which the corresponding solution belongs.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig18.png}
\caption{Cr(C_6H_6)_2 has the symmetry D_{6h}.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig19.png}
\caption{C_{60} has the symmetry I_h}
\end{figure}

Similarly, the transition operators can often be labelled with the help of the irreducible representations and yet another index, say $k$, distinguishing among various components of such operators - should they have e.g. tensor character.

Let us denote the multipole electromagnetic transition operator by $Q_{k}^{\Gamma'}$. The matrix elements of this transition operator can be expressed formally as

$$Q_{i,j,k,n}^{\Gamma\Gamma'} = \langle \psi_{i,n}^{\Gamma} | Q_{k}^{\Gamma} | \psi_{j,n}^{\Gamma'} \rangle$$

where the indices $i$ and $j$ refer to the initial and final states, respectively. The symmetries give generally some constraints on the transitions between quantum levels. These constraints are called selection rules; their forms depend on the irreducible representations to which the discussed states belong and
can be often discussed using general methods of the group representation theory. For instance, if for given initial and final states $Q_{n_f, k, n_i}^{\Gamma' \Gamma} = 0$ for all $k$, the transition is forbidden.

**Geometry Opposed to Dynamics in the Transition Amplitudes.** Suppose the symbols $\Gamma''$, $\Gamma$ and $\Gamma'$ label irreducible representations of the considered symmetry group $G$ of the Hamiltonian $\hat{H}$. It can be shown that transitions from $\psi_{\Gamma_i}^{\Gamma}$ to $\psi_{\Gamma_f}^{\Gamma}$ are forbidden if the Kronecker product of the irreducible representations $[\Gamma''] \times [\Gamma] \times [\Gamma']$ does not contain any scalar representation. More precisely, in the decomposition of the product into a direct sum of irreducible representations which have, qualitatively, the following structure

$$[\Gamma''] \times [\Gamma] \times [\Gamma'] \rightarrow \Gamma_i \oplus \Gamma_2 \oplus \ldots \Gamma_{\text{max}}$$

(19)

at least one of the $\Gamma_k$ representations on the right-hand side is scalar. This property is independent of a particular realisation of the model - in other words: of the form of the Hamiltonian - provided each new form of the Hamiltonian is invariant under the same symmetry group. The consequences of this prediction can be used as one possible (among many that will usually be necessary) test of the presence of symmetries.

In the case of the finite-element symmetry groups, such as point groups, the condition on the forbidden transitions is implied by a very important orthogonality property of irreducible representations. Let the matrix form of the irreducible representations $\Gamma^\mu (g)$ be $D^\mu_{jl} (g)$. Then one can demonstrate that:

$$\sum_{g \in G} D^\mu_{il} (g)^* D^\nu_{jm} (g) = \frac{\text{card}(G)}{\text{dim}(\Gamma^\mu)} \delta^\mu_\nu \delta_{lj} \delta_{im},$$

(20)

where $\text{card}(G)$ and $\text{dim}(\Gamma^\mu)$ denote the number of elements in the group and dimension of the irreducible representation, respectively.

In the case of continuous symmetry groups such as SO(3) a very useful information is obtained from the theorem of Wigner and Eckart. It assures that matrix elements of the multipole transition operators depend on the quantum numbers labelling the wave-functions within a given irreducible representation (fixed $\Gamma$) through the Clebsh-Gordan coefficient only. In the case of the SO(3) group these are the projections $M$ of angular momentum
label in $|JM\rangle$ which enter transition-matrix expression through the Clebsch-Gordan coefficients only (see below).

The Wigner-Eckart theorem for the SO(3) group can be formulated as follows. Let the considered initial and final states be $\psi_{n_f}^I \rightarrow |JM\rangle$ and $\psi_{n_f}^F \rightarrow \langle J'M'|$, respectively, with the index enumerating the irreducible representations in this case, $J$, corresponding to $\Gamma \leftrightarrow J$. We have:

$$
\langle J'M'|Q_{\mu}^i|JM\rangle = \delta_{j,j'} \frac{(JM\lambda\mu|J'M')}{\sqrt{2J' + 1}} \langle J'|Q_{\mu}^i|J\rangle
$$

(21)

where the ‘double barred’ symbols $\langle J'|T^i|J\rangle$ denote the so-called reduced matrix elements. The quantum numbers $J$ and $J'$ denote the angular momentum quantum numbers and $\lambda$ is the multipolarity of the transition operator. From the above expression it follows immediately that the only dynamical (i.e. Hamiltonian dependent) impact on the transition amplitudes is contained in $\langle J'|T^i|J\rangle$ whereas the Clebsch-Gordan coefficients, $(JM\lambda\mu|J'M')$, depend on the structure of the symmetry group but not on the Hamiltonian - we say that the Clebsch-Gordan coefficients represent the ‘geometrical aspect of the symmetry’.

**An Example: The Hypothetical Tetrahedral Symmetry in Nuclei.** As already mentioned earlier, there have been published interesting observations suggesting that in some atomic nuclei the tetrahedral symmetry can be manifested. As it is known, the tetrahedral group has 3 irreducible representations usually labelled $\Gamma = A, E$ and $T$ with dimensions 1, 2 and 3, respectively. The electric-dipole ($\lambda = 1$) transition amplitudes which will be taken here to formulate a possible criterion of the presence (or absence) of tetrahedral symmetry can be calculated with the help of the dipole operator $Q_{\mu}^i \rightarrow \{x, y, z\}$, the latter transforming according to representation $T$ of the tetrahedral group (we say that operator $Q_{\mu}^i$ is a tensor of rank $T$).

Using the standard techniques of the group theory one can calculate the Kronecker product of representations needed in this case:

$$
T \times A = T, \quad T \times E = T \oplus T, \quad T \times T = A \oplus E \oplus T \oplus T.
$$

(22)

By using these relations we arrive at the conclusion that the electric-dipole transitions $\psi^A \leftrightarrow \psi^E$, $\psi^A \leftrightarrow \psi^A$, $\psi^E \leftrightarrow \psi^E$ are forbidden - the electric-
dipole transition operator cannot connect states belonging to listed representations.

8. SYMMETRY BREAKING

Even though we are far from completing the story of symmetries in nature (which is in any case the endless one as you may easily guess) an attentive reader may ask yet another question: There are physical systems manifesting a symmetry and systems not manifesting any given symmetry - Are there any intermediate situations possible?

The long practice of research in many domains of physics brings us to the conclusion that the really ‘exact-exact’ symmetries are seldom present in nature and especially in the complex quantum systems and that many, even among the most important symmetries are only approximate. We say: the symmetries in question are in the best case slightly broken - what in no case makes the problem less interesting and the use of symmetry concept less useful and/or less powerful.

The discussion of the symmetry breaking appears commonly in physics - and this for various reasons, some of them merely subjective ones. To explain what is meant let us emphasise that human theories reflect to an extent the evolution of human perception and understanding of the universe and physical phenomena at a given stage of evolution. With research evolving, say from one series of experiments to another, our information becomes more and more complete - so is the form of the Hamiltonian of the system in question. It then follows that the underlying Hamiltonian takes the evolving forms $\hat{H}_0, \hat{H}_1, \hat{H}_2 \ldots$ - just as a result of increasing of the amount of information what allows us to be more and more precise in the description of the system. Thus it becomes clear that if a theory based on the Hamiltonian, say $\hat{H}_k$, is ‘almost satisfactory’, the next improvement coming with the next generation Hamiltonian $\hat{H}_{k+1}$ cannot bring the results which will be dramatically different: in other words, e.g. the expectation values of the energies should satisfy

$$\langle \hat{H}_k \rangle - \langle \hat{H}_{k+1} \rangle \rightarrow 0 \iff \hat{H}_{k+1} = \hat{H}_k + \delta \hat{H},$$

where $\delta \hat{H}$ can be treated as a small perturbation and the evolution stops at certain step, which we will denote $k \equiv k_{\text{satisfactory}}$.

Suppose that at certain stage, say no. $k$, the corresponding Hamiltonian $\hat{H}_k$, commutes with all the elements of a certain group which becomes in
this way its symmetry group. The symmetry group of the dominating part of
the Hamiltonian $\hat{H}_k$ can be considered as an approximate symmetry of the
Hamiltonian $\hat{H}_{k+1}$. We may also say that $\delta \hat{H}$ breaks the symmetry of the Ha-
miltonian $\hat{H}_k$, or - in other words - that the symmetry is ‘approximate’.

The Symmetry Breaking and Spontaneous Symmetry Breaking. The
idea of symmetry breaking caused in the past a serious philosophical and
even conceptual problems. Indeed, suppose that a given physical system ma-
ifests a symmetry. This very affirmation seems to imply that the system in
question ‘lives’ in a stationary situation - stationary - in the sense of ‘pre-
serving the symmetry while existing’. Why should the system leave the state
of symmetric happiness? In this context the easiest situation to imagine is the
one of the potential maximum rather than that of a potential minimum - cf. a
caricature of the situation in Fig. 20. In other words we tend to consider the
unstable rather than stable equilibrium of the system.

![Fig. 20. Buridan’s donkey, having strictly identical bundles of carrots on both sides has
no reason to select one of the two - and dies of starvation.](image)

But then: Which one among infinitely many non-symmetric circumstances
should the system chose and follow? If there are infinitely many of them -
will one of them be chosen at random? Moreover: If there are infinitely many
possibilities - Can it happen that the probability of any choice is exactly zero
and thus no transition may happen at all? This way of thinking may be seen
as an illustration of logic that lead certain philosophers to formulate the prob-
lem mentioned at the beginning of the paragraph$^3$.

$^3$ In polish literature there exists a poem addressing this situation; it has been written by Alek-
sander Fredro (1793–1876) and is known to all children under the title “Osiołkowi w złoby dano”
- there the only difference with respect to the Burridan’s ass is that on both sides lie not the carrots
but equally attractive delicacies. The result was the same as in the case of Burridan’s donkey
which could not decide which delicacy to choose.
To illustrate the issue, let us consider a macroscopic model: a vertical and ideally uniform metal stick under the action of a vertical force, as it is shown in the left-hand side of Fig. (21). The so defined system is obviously axially symmetric. In the ideal world, the action of the vertical force should not change the axial [i.e. SO(2)] symmetry, independently of the value of the force. However, in the real world, at some ‘critical’ value of the applied force the symmetry is broken, as illustrated in the right-hand part of the figure. Since transitions of this type arrive at the time-instant and/or exact critical force which difficult to predict with exactitude, we select using the term spontaneous symmetry breaking for this kind of situations.

Fig. 21. Axial symmetry breaking as a result of applied external force. Because the direction in which the stick will be bent is unpredictable the phenomenon can be treated as a macroscopic illustration of the symmetry breaking phenomena.

The most often cited example of spontaneous symmetry breaking is traditionally related the expectation value of the energy of a field in the quantum field theory. The vacuum state of a scalar field can be thought of as distinguishing no direction (a point on top of the hat representing the state with spherical symmetry). But if a state exists whose energy is lower whereas the symmetry is not spherical anymore the system will follow the symmetry breaking mechanism, cf. a schematic illustration (22). It may happen under conditions discussed abundantly in the literature that the arrival configuration is continuously degenerate as marked in the figure. When this happens a single point (state) on the brim hat has to be chosen spontaneously by the system - it will become the new ground state of the field.
Phase Transitions and Related Spontaneous Symmetry Breaking. Examples of spontaneous symmetry breaking can be associated in condensed matter physics with phase transitions. For instance, some metals (Iron, Nickel, Cobalt) can acquire magnetic properties under room temperatures. In this state they can have locally axial symmetry. For the temperatures $T > T_c$, where $T_c$ denotes the Curie temperature, magnetisation disappears and we obtain para-magnets which have locally rotational symmetry. When the temperature goes down below the Curie temperature the para-magnet transforms into a ferro-magnet with a lower local symmetry.

Another example of symmetry breaking, conceptually distinct from the spontaneous symmetry breaking discussed so far, can be found in structure of solids. Indeed, an ideal, structureless solid is invariant under translations and rotations belonging to the full Euclidean group. The intrinsic, eg. crystalline structure of the body, breaks this symmetry diminishing the symmetry group to a subgroup of the Euclidean group.

About Parity and Parity Breaking in Fundamental Interactions. In 1956 T. D. Lee and C. N. Yang published the paper about a possibility of the parity violation [7], where they introduce a symmetry breaking term ‘added by hand’. The initial reaction of physicists on the proposal of an appropriate experiment was not enthusiastic. The belief in the left-right symmetry of the fundamental interactions in our Universe was so strong that nobody believed in the validity of their considerations.

However, a year later a successful experiment was performed by C.S. Wu (1912–1997) and her group. It proved that the parity violation is caused by the weak interactions. It showed, more importantly, that even the symmetries considered so far as the most fundamental can be broken in nature. A few years latter it turned out that in $K_0$ meson decay one can find an interaction-component which breaks the $CP$ symmetry ($C$ denotes the charge-conjugation: transformation of a particle into its antiparticle and $P$ is the space inversion operation). On the other hand, it is known that the $CPT = 1$ theorem, i.e. $CPT$ symmetry conservation is fundamental for the standard
model of the elementary-particle interactions. The cases of \( P \) and \( CP \) symmetry braking are examples of challenges for theoretical and experimental physics in the research of symmetries of fundamental interactions and their possible breaking.

Fig. 23. An idea of Wu’s experiment showing possibility of distinguishing between left and right orientation [1].

9. A FEW FINAL REMARKS

The story of symmetries in nature is endless as much as is the richness and variety of physical systems and phenomena in the Universe. In this article we wished to present our version of an introduction to the subject which is by no means complete nor the only one possible. We discussed merely a few among the best known and, on the other hand, very general, problems related to symmetries. We have no space to develop for instance more advanced symmetry problems of single-particle, collective and many-body Hamiltonians and the related ‘interface’ domain. Also, the so called dynamical symmetries are important in all fields of physics but their presentation was totally left out.

Another category of symmetry problems that we could not treat for the lack of space are related to non-linear systems, to the General Theory of Relativity, field theory, fluid dynamics and other domains. The problem of intrinsic symmetries and their relations to space-time symmetries (no-go theorem) is still an open problem of physics. Another type of symmetries which we had to
omitted in our story are the so called the gauge symmetries playing an important role in our imagination of interactions in the universe.

We hope that our reader who begins his or her interest in physics will be encouraged to invest in their own deeper and more complete studies of the fascinating issue of symmetries in the Universe as well as the symmetry breaking phenomena.

REFERENCES