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On upper semicontinuity of geometric difference
of multifunctions

Abstract. The short proof of upper semicontinuity of geometric difference
of multifunctions is given.

Let $X$ and $Y$ be two topological spaces. A multifunction (or a set-valued
map) $F : X \to Y$ is a mapping from $X$ to the nonempty subsets of $Y$; thus,
for each $x \in X$, $F(x)$ is a nonempty set in $Y$.

We say that $F$ is upper semicontinuous (usc) at $x \in X$ if for any open set
$V$ containing $F(x)$ there exists a neighborhood $U$ of $x$ such that $F(y) \subset V$
for any $y \in U$. $F$ is usc on $X$ if it is usc at each $x \in X$.

We say that $F$ is lower semicontinuous (lsc) at $x \in X$ if for any open
set $V$ which meets $F(x)$ there exists a neighborhood $U$ of $x$ such that
$F(y) \cap V \neq \emptyset$ for every $y \in U$. $F$ is lsc on $X$ if it is lsc at any $x \in X$.

If a multifunction $F : X \to Y$ is compact-valued, i.e. if for every $x \in X$,
the set $F(x)$ is a compact set in $Y$, and if $X$ and $Y$ satisfy the "first axiom
of countability", then we have the following useful conditions, which are
equivalent to usc and lsc, respectively.

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**Proposition 1.** ([4, Proposition 4.1, p. 48]). A multifunction $F : X \to Y$ is usc at $x \in X$ if and only if for any sequence $\{x_n\}$ in $X$ converging to $x$ and for any sequence $\{y_n\}$ of elements of $F(x_n)$ there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to $y \in F(x)$.

**Proposition 2.** ([3, Proposition II-2-1, p. 15]). A multifunction $F : X \to Y$ is lsc at $x \in X$ if and only if for any $y \in F(x)$ and for any sequence $\{x_n\}$ in $X$ converging to $x$ there exists a sequence $\{y_n\}$ of elements of $F(x_n)$ converging to $y$.

Now let $Y$ be a linear topological space. For $A \subset Y, B \subset Y$ and $\lambda \in \mathbb{R}$ we put

$$A + B = \{a + b : a \in A, b \in B\},$$

$$\lambda A = \{\lambda a : a \in A\},$$

$$A - B = A + (-1)B.$$

The geometric difference (or Minkowski subtraction [1], [2], [5]) of the set $A$ and $B$ is denoted by $A \Delta B$ and defined by setting

$$A \Delta B = \{y \in Y : y + B \subset A\}.$$

**Remark.** It is worth noting here that the set $A \Delta B$ is different from $A - B$.

In [1] the following theorem is proved

**Theorem 1.** ([1, Theorem 2.1, p. 165]). Let $X$ be a complete metric space, $Y$ a separable Banach space and let $F, G : X \to Y$ be weakly compact-valued multifunction. If $F : X \to Y$ is weakly usc, $G$ weakly lsc and a multifunction $H : X \to Y$ is defined by $H(x) = F(x) \Delta G(x) \neq \emptyset$ for any $x \in X$, then the multifunction $H$ is weakly usc, provided $H(X)$ is contained in some weakly compact set in $Y$.

We will give a certain generalisation of this result. Moreover, our proof seems to be shorter and simpler.

**Theorem 2.** Let $X$ be a topological space with "the first axiom of countability", $Y$ a metrisable linear topological space and let $F, G : X \to Y$ be compact-valued multifunctions. If $F$ is usc, $G$ is lsc then the multifunction $H : X \to Y$ defined by $H(x) = F(x) \Delta G(x) \neq \emptyset$ for any $x \in X$ is usc.

**Proof.** Obviously the multifunction $H = F \Delta G$ is compact-valued. Therefore, by Proposition 1, it suffices to show that for every $x \in X$ and for any sequence $\{x_n\} \subset X$ converging to $x$ and for any sequence $\{y_n\} \subset Y$ such
that $y_n \in H(x_n)$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which converges to $y \in H(x)$.

So, let $x \in X$ and suppose that $\{x_n\} \subset X$ converges to $x$. Let $\{y_n\} \subset Y$ be such that $y_n \in H(x_n)$. We have $y_n + G(x_n) \subset F(x_n)$. From lower semicontinuity of $G$ at $x$ it follows (by Proposition 2) that for each $z \in G(x)$ there exists a sequence $\{z_n\} \subset Y$ with $z_n \in G(x_n)$ which converges to $z$. Thus we have $u_n = y_n + z_n \in F(x_n)$. Since $F$ is usc at $x$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging to $u \in F(x)$.

Hence the subsequence $\{y_{n_k}\}$ of $\{y_n\}$, where $y_{n_k} = u_{n_k} - z_{n_k}$, converges to $y = u - z$ and $y + z = u \in F(x)$.

Since $z \in G(x)$ was chosen arbitrarily, $y + G(x) \subset F(x)$, which gives $y \in F(x) - G(x) = H(x)$.

By Proposition 2, the multifunction $H = F - G : X \rightarrow Y$ is usc at $x$ and the proof of Theorem 2 is complete. □

References


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