WOJCIECH ZYGMUNT

On a functional equation

Abstract. The existence of continuous solutions of the functional equation
\[ \phi(\phi(x)) = 2\phi(x) - x + p \]

is studied.

The object of the paper is to investigate the functional equation of the form
\[ (1) \quad \phi(\phi(x)) = 2\phi(x) - x + p, \]
where \( \phi \) is the unknown function and \( p \) a real constant. The equation (1) is a particular case of the equation \( \phi(\phi(x)) = g(x, \phi(x)) \), which has been studied in [K1, p. 282], [K2], [K3] and [F]. Here we look for continuous solutions of (1) defined on the whole real line \( \mathbb{R} \). We show that this equation has a solution only when \( p = 0 \). In this case the only solutions are \( \phi(x) = x + \alpha \), \( \alpha \in \mathbb{R} \).

To prove our main result we will need the following lemmas.

2000 Mathematics Subject Classification. 39B22.
Key words and phrases. Functional equation.
Lemma 1. If $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous solution of (1), then $\phi$ is strictly increasing and $\phi(\mathbb{R}) = \mathbb{R}$.

Proof. Observe first that any solution of (1) (even not continuous) must be a one-to-one map. Indeed, if $\phi$ satisfies (1) and $\phi(x) = \phi(y)$, then

$$2\phi(x) - x + p = \phi(\phi(x)) = \phi(\phi(y)) = 2\phi(y) - y + p = 2\phi(x) - y + p.$$ 

Thus $x = y$. Consequently, each continuous solution $\phi$ of (1) is either strictly decreasing or strictly increasing. In the former case, one of the following conditions holds.

1) $\lim_{x \to -\infty} \phi(x) = +\infty$ and $\lim_{x \to -\infty} \phi(x) = -\infty$,
2) $\lim_{x \to -\infty} \phi(x) = +\infty$ and $\lim_{x \to -\infty} \phi(x) = b > -\infty$,
3) $\lim_{x \to -\infty} \phi(x) = a > -\infty$ and $\lim_{x \to -\infty} \phi(x) = -\infty$,
4) $\lim_{x \to -\infty} \phi(x) = a > -\infty$ and $\lim_{x \to -\infty} \phi(x) = b > -\infty$, $a > b$.

We claim that any of 1) - 4) is not possible. Indeed, if 1) holds, then we would get

$$\lim_{x \to -\infty} \phi(\phi(x)) = \lim_{x \to -\infty} (2\phi(x) - x + p) = +\infty.$$ 

On the other hand, by the continuity of $\phi$,

$$\lim_{x \to -\infty} \phi(\phi(x)) = \phi(\lim_{x \to -\infty} \phi(x)) = -\infty,$$ 

a contradiction. In case 2) we have

$$-\infty \neq \phi(b) = \phi(\lim_{x \to +\infty} \phi(x)) = \lim_{x \to +\infty} \phi(\phi(x)) = \lim_{x \to +\infty} (2\phi(x) - x + p) = -\infty,$$ 

a contradiction. Similar analysis can be applied to show that neither 3) nor 4) is possible. This proves that $\phi$ cannot be strictly decreasing. So it is strictly increasing. Our next claim is that $\phi(\mathbb{R}) = \mathbb{R}$. The following four cases are possible:

5) $\lim_{x \to -\infty} \phi(x) = -\infty$ and $\lim_{x \to -\infty} \phi(x) = \infty$,
6) $\lim_{x \to -\infty} \phi(x) = -\infty$ and $\lim_{x \to -\infty} \phi(x) = b < \infty$,
7) $\lim_{x \to -\infty} \phi(x) = a > -\infty$ and $\lim_{x \to -\infty} \phi(x) = +\infty$,
8) $\lim_{x \to -\infty} \phi(x) = a > -\infty$ and $\lim_{x \to -\infty} \phi(x) = b < \infty$, $a < b$.

As above one can show that the last three cases give a contradiction. To see that 5) is possible, we rewrite (1) in the form

$$\phi(\phi(x)) + x = 2\phi(x) + p.$$ 

□
Lemma 2. Equation (1) has a solution if and only if $p = 0$.

Proof. If $p = 0$, then the identity function is a solution of (1). Suppose now that (1) has a solution $\phi$ and let $x_0 \in \mathbb{R}$ be arbitrarily chosen. If $\phi(x_0) = x_0$, then

$$x_0 = \phi(x_0) = \phi(\phi(x_0)) = 2\phi(x_0) - x_0 + p = x_0 + p.$$ 

Thus $p = 0$. If $\phi(x_0) \neq x_0$, then we construct a sequence $\{x_n\}_{-\infty}^\infty$ by setting $x_n = \phi^n(x_0)$, where $\phi^n$ denotes the nth iterate of $\phi$, that is,

$$\phi^0(x) = x, \quad \phi^{n+1}(x) = \phi(\phi^n(x)), \quad \phi^{n-1}(x) = \phi^{-1}(\phi^n(x)),$$

where $\phi^{-1}$ is the inverse of $\phi$. Since $x_1 \neq x_0$, we have $x_1 = x_0 + r$ with some nonzero $r$. Then the sequence $\{x_n\}_{-\infty}^\infty$ is, by Lemma 1, strictly increasing if $r > 0$, and strictly decreasing if $r < 0$. Moreover, for each integer $n$,

$$x_{n+2} = 2x_{n+1} - x_n + p,$$

or, equivalently,

$$x_{n+2} - x_{n+1} = x_{n+1} - x_n + p.$$

By induction, we obtain

$$x_n = x_0 + n\left(r + \frac{(n-1)}{2}p\right), \quad n = 0, \pm 1, \ldots.$$ 

So, if $p \neq 0$, then for sufficiently large $|n|$ the terms $x_n$ are either greater than $x_0$ or less then $x_0$, which contradicts strict monotonicity of $\{x_n\}_{-\infty}^\infty$. Thus $p = 0$. □

Now we are ready to prove our main result.

Theorem. The only functions continuous on $\mathbb{R}$ and satisfying the equation

$$\phi(\phi(x)) = 2\phi(x) - x$$

are

$$\phi(x) = x + \alpha, \quad \alpha \in \mathbb{R}.$$ 

Proof. It is clear that the identity function $\phi(x) = x$ is a solution of (3). Now, suppose that $\phi$, different from identity, is a solution of (3). Then there exists an $x_0$ such that $\phi(x_0) \neq x_0$. As in Lemma 2 we define the sequence $\{x_n\}_{-\infty}^\infty$ with $x_n = \phi^n(x_0)$. By (2),

$$x_{n+1} = x_n + r \quad \text{and} \quad x_n = x_0 + nr,$$
where \( r = x_1 - x_0 = \phi(x_0) - x_0 \). Choose \( y_0 \) from the open interval with end points \( x_0 \) and \( x_1 \) and consider the sequence \( \{y_n\}_{-\infty}^{\infty} \) with \( y_n = \phi^n(y_0) \). If we set \( \rho = y_1 - y_0 \), then by (2),

\[
y_{n+1} = y_n + \rho \quad \text{and} \quad y_n = y_0 + n\rho.
\]

Since \( \phi \), as a solution of (3), is strictly increasing, we see that each \( y_n \) is between \( x_n \) and \( x_{n+1} \). Note that the sequences \( \{x_n\}_{-\infty}^{\infty} \) and \( \{y_n\}_{-\infty}^{\infty} \) are both either strictly decreasing or strictly increasing. Suppose, for example, that the latter case holds. If \( \rho \neq r \), then for some \( n \), \( x_n \) would not be in the interval (\( x_n, x_{n+1} \)), a contradiction. An analogous reasoning can be applied in the other case. Thus we see that \( \rho = r \). Consequently, if \( \phi \) is a continuous solution of (3), then \( \phi(x) - x = \alpha \) with some real constant \( \alpha \). □

References


Faculty of Mathematics and Natural Sciences KUL
Al. Raclawickie 14
20-950 Lublin, Poland
e-mail: wzgymunt@kul.lublin.pl

Received November 28, 2001