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Parallelograms inscribed in a curve having a circle as $\frac{\pi}{2}$-isoptic

Abstract. Jean-Marc Richard observed in [7] that maximal perimeter of a parallelogram inscribed in a given ellipse can be realized by a parallelogram with one vertex at any prescribed point of ellipse. Alain Connes and Don Zagier gave in [4] probably the most elementary proof of this property of ellipse. Another proof can be found in [1]. In this note we prove that closed, convex curves having circles as $\frac{\pi}{2}$-isoptics have the similar property.

1. Introduction. Let $C$ be a closed and strictly convex curve. We fix an interior point of $C$ as an origin of a coordinate system. Denote $e^{it} = (\cos t, \sin t)$, $ie^{it} = (-\sin t, \cos t)$. The function $p: \mathbb{R} \to \mathbb{R}$

$$p(t) = \sup_{z \in C} \langle z, e^{it} \rangle$$

is called the support function of $C$. For a strictly convex curve $p$ is differentiable. We assume that the function $p$ is of class $C^2$ and the curvature of $C$ is positive. We have the following equation of $C$ in terms of its support function

$$(1.1) \quad z(t) = p(t)e^{it} + \dot{p}(t)ie^{it}.$$ 

Then $||z|| = \sqrt{p^2(t) + \dot{p}^2(t)}$ and $R(t) = p(t) + \dot{p}(t)$ is a radius of curvature of $C$ at $t$.

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The \( \alpha \)-isoptic of \( C \) consists of those points in the plane from which the curve is seen under the fixed angle \( \alpha \) (for the geometric properties of isoptics see [2], [3], [5], [6], [8]). Suppose that the \( \frac{\pi}{2} \)-isoptic of \( C \) is a circle of radius \( r \) with the center in the origin of a coordinate system. Then

\[
p^2(t) + p^2 \left( t + \frac{\pi}{2} \right) = r^2,
\]

and

\[
p^2(t + \pi) + p^2 \left( t + \frac{\pi}{2} \right) = r^2,
\]

so \( p(t) = p(t + \pi) \) and the center of the circle is a center of symmetry of \( C \). The curve (1.1) has a circle with the center in the origin of a coordinate system as an \( \frac{\pi}{2} \)-isoptic if and only if (1.2) holds good.

**Example 1.1.** Let \( C \) be an ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). Then

\[
p(t) = \sqrt{a^2 \cos^2 t + b^2 \sin^2 t},
\]

\[
z(t) = (x(t), y(t)) = \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} e^{it} + \frac{\sin t \cos t (b^2 - a^2)}{\sqrt{a^2 \cos^2 t + b^2 \sin^2 t}} ie^{it}
\]

is its equation in terms of a support function and \( p^2(t) + p^2 \left( t + \frac{\pi}{2} \right) = a^2 + b^2 \).

2. **Extremal property of the perimeter of inscribed parallelograms.**

Assume that a curve \( C \) given by (1.1) has a circle with a center in an origin of a coordinate system as an \( \frac{\pi}{2} \)-isoptic. Then we have (1.2) and

\[
p(t)p'(t) + p \left( t + \frac{\pi}{2} \right) \dot{p} \left( t + \frac{\pi}{2} \right) = 0.
\]

Fix \( t \) and consider inscribed parallelogram with \( z(t) \) as one of the vertices. There exists \( \alpha \) such that \( z(t + \alpha), -z(t), -z(t + \alpha) \) are its remaining vertices and

\[
d_t(\alpha) = |z(t + \alpha) - z(t)| + |z(t + \alpha) + z(t)|
\]

is a half of a perimeter of parallelogram.

**Theorem 2.1.** Let \( C \) be a strictly convex curve having a circle with a center in an origin of a coordinate system as an \( \frac{\pi}{2} \)-isoptic and let \( d_t(\alpha) \) be the function given by (2.2). Then

- (i) \( d'_t \left( \frac{\pi}{2} \right) = 0 \), where prime denotes the derivative with respect to \( \alpha \),
- (ii) \( d \left( \frac{\pi}{2} \right) = d_t \left( \frac{\pi}{2} \right) \) does not depend on \( t \).

**Proof.** We have

\[
e^{i(t+\alpha)} = \cos \alpha e^{it} + \sin \alpha ie^{it},
\]

\[
ie^{i(t+\alpha)} = -\sin \alpha e^{it} + \cos \alpha ie^{it},
\]

\[
z(t + \alpha) = (p(t + \alpha) \cos \alpha - \dot{p}(t + \alpha) \sin \alpha) e^{it} + (p(t + \alpha) \sin \alpha + \dot{p}(t + \alpha) \cos \alpha) ie^{it}.
\]
Let
\[ A = p(t + \alpha) \cos \alpha - \hat{p}(t + \alpha) \sin \alpha - p(t), \]
\[ B = p(t + \alpha) \sin \alpha + \hat{p}(t + \alpha) \cos \alpha - \hat{p}(t), \]
\[ C = p(t + \alpha) \cos \alpha - \hat{p}(t + \alpha) \sin \alpha + p(t), \]
\[ D = p(t + \alpha) \sin \alpha + \hat{p}(t + \alpha) \cos \alpha + \hat{p}(t). \]

Then
\[ d_t(\alpha) = \sqrt{A^2 + B^2} + \sqrt{C^2 + D^2} \]
and
\[ d'_t(\alpha) = (p(t + \alpha) + \hat{p}(t + \alpha)) \]
\[ \times \left( \frac{\hat{p}(t + \alpha) + p(t) \sin \alpha - \hat{p}(t) \cos \alpha}{\sqrt{A^2 + B^2}} + \frac{\hat{p}(t + \alpha) - p(t) \sin \alpha + \hat{p}(t) \cos \alpha}{\sqrt{C^2 + D^2}} \right). \]

Putting \( \alpha = \frac{\pi}{2} \), we get
\[ d'_t\left(\frac{\pi}{2}\right) = R \left( t + \frac{\pi}{2} \right) \left( \frac{\hat{p}(t + \frac{\pi}{2}) + p(t)}{\sqrt{(p(t) + \hat{p}(t + \frac{\pi}{2}))^2 + (p(t + \frac{\pi}{2}) - \hat{p}(t))^2}} \right) + \frac{\hat{p}(t + \frac{\pi}{2}) - p(t)}{\sqrt{(p(t) - \hat{p}(t + \frac{\pi}{2}))^2 + (p(t + \frac{\pi}{2}) + \hat{p}(t))^2}} \right). \]

From (2.1) we have
\[ \hat{p}\left( t + \frac{\pi}{2} \right) = -\frac{p(t)\hat{p}(t)}{p(t + \frac{\pi}{2})}, \]
and since
\[ \left( p\left( t + \frac{\pi}{2} \right) - \hat{p}(t) \right) \left( p\left( t + \frac{\pi}{2} \right) + \hat{p}(t) \right) \]
\[ = p^2 \left( t + \frac{\pi}{2} \right) - \hat{p}^2(t) = r^2 - (p^2(t) + \hat{p}^2(t)) \]
\[ = r^2 - \|z(t)\|^2 > 0, \]
we obtain
\[ \text{sgn} \left( p\left( t + \frac{\pi}{2} \right) - \hat{p}(t) \right) = \text{sgn} \left( p\left( t + \frac{\pi}{2} \right) + \hat{p}(t) \right). \]
Hence
\[ \frac{\hat{p}(t + \frac{\pi}{2}) + p(t)}{\sqrt{(p(t) + \hat{p}(t + \frac{\pi}{2}))^2 + (p(t + \frac{\pi}{2}) - \hat{p}(t))^2}} \]
\[ + \frac{\hat{p}(t + \frac{\pi}{2}) - p(t)}{\sqrt{(p(t) - \hat{p}(t + \frac{\pi}{2}))^2 + (p(t + \frac{\pi}{2}) + \hat{p}(t))^2}} \]
2.1

which proves the first part of Theorem 2.1.

Let

\[ h(t) = d_t \left( \frac{\pi}{2} \right) = \sqrt{\left( p(t) + \dot{p} \left( t + \frac{\pi}{2} \right) \right)^2 + \left( \dot{p}(t) - p \left( t + \frac{\pi}{2} \right) \right)^2} \]

Then

\[ \dot{h}(t) = \frac{R(t)(\dot{p}(t) - p(t + \frac{\pi}{2})) + R(t + \frac{\pi}{2})(p(t) + \dot{p}(t + \frac{\pi}{2}))}{\sqrt{(p(t) + \dot{p}(t + \frac{\pi}{2}))^2 + (\dot{p}(t) - p(t + \frac{\pi}{2}))^2}} \]

\[ + \frac{R(t + \frac{\pi}{2})(\dot{p}(t + \frac{\pi}{2}) - p(t)) + R(t)(\dot{p}(t) + p(t + \frac{\pi}{2}))}{\sqrt{(p(t) - \dot{p}(t + \frac{\pi}{2}))^2 + (\dot{p}(t) + p(t + \frac{\pi}{2}))^2}} \]

\[ = R \left( t + \frac{\pi}{2} \right) \left( \frac{p(t) + \dot{p}(t + \frac{\pi}{2})}{\sqrt{(p(t) + \dot{p}(t + \frac{\pi}{2}))^2 + (\dot{p}(t) - p(t + \frac{\pi}{2}))^2}} \right) \]

\[ - \frac{p(t) - \dot{p}(t + \frac{\pi}{2})}{\sqrt{(p(t) - \dot{p}(t + \frac{\pi}{2}))^2 + (\dot{p}(t) + p(t + \frac{\pi}{2}))^2}} \]

\[ + R(t) \left( \frac{\dot{p}(t) - p(t + \frac{\pi}{2})}{\sqrt{(\dot{p}(t) + p(t + \frac{\pi}{2}))^2 + (\dot{p}(t) - p(t + \frac{\pi}{2}))^2}} \right) \]

\[ + \frac{\dot{p}(t) + p(t + \frac{\pi}{2})}{\sqrt{(\dot{p}(t) - p(t + \frac{\pi}{2}))^2 + (\dot{p}(t) + p(t + \frac{\pi}{2}))^2}} \right). \]
Since the first summand is equal to zero for each \( t \) and the second summand is equal to the first at \( t + \frac{\pi}{2} \), they are equal to zero.

3. The converse theorem. In this section we shall prove the converse of Theorem 2.1. For this purpose we define the function \( d(t) = d(t, \frac{\pi}{2}) \).

**Theorem 3.1.** Let \( C \) be a closed and strictly convex curve of class \( C^2 \) with positive curvature having a center of symmetry. Suppose that an origin of a coordinate system is in the center of \( C \) and \( d(t, \frac{\pi}{2}) = 0 \). Then \( d(t) = 0 \) and \( \frac{\pi}{2} \)-isoptic of \( C \) is a circle.

**Proof.** The equality \( d(t, \frac{\pi}{2}) = 0 \) is equivalent to

\[
\frac{\dot{p}(t + \frac{\pi}{2}) + p(t)}{\sqrt{(p(t) + \dot{p}(t + \frac{\pi}{2}))^2 + (p(t + \frac{\pi}{2}) - \dot{p}(t))^2}} = \frac{p(t) - \dot{p}(t + \frac{\pi}{2})}{\sqrt{(p(t) - \dot{p}(t + \frac{\pi}{2}))^2 + (p(t + \frac{\pi}{2}) + \dot{p}(t))^2}}.
\]

The equality (3.1) for \( t + \frac{\pi}{2} \) gives

\[
\frac{\dot{p}(t) + p(t + \frac{\pi}{2})}{\sqrt{(p(t + \frac{\pi}{2}) + \dot{p}(t))^2 + (p(t) - \dot{p}(t + \frac{\pi}{2}))^2}} = \frac{p(t + \frac{\pi}{2}) - \dot{p}(t)}{\sqrt{(p(t + \frac{\pi}{2}) - \dot{p}(t))^2 + (p(t) + \dot{p}(t + \frac{\pi}{2}))^2}}.
\]

From (3.1) and (3.2) we get

\[
\frac{\dot{p}(t + \frac{\pi}{2}) + p(t)}{p(t + \frac{\pi}{2}) - \dot{p}(t)} = \frac{p(t) - \dot{p}(t + \frac{\pi}{2})}{\dot{p}(t) + p(t + \frac{\pi}{2})},
\]

or equivalently

\[
p(t)\dot{p}(t) + p\left(t + \frac{\pi}{2}\right) + p\left(t + \frac{\pi}{2}\right)\dot{p}\left(t + \frac{\pi}{2}\right) = 0,
\]

which gives

\[
p^2(t) + p^2\left(t + \frac{\pi}{2}\right) = \text{const}.
\]

**Example 3.1** ([5]). Let \( p(t) = \cos\left(\frac{\pi}{4} + k\sin(2t)\right) \). For \( k \) sufficiently small \( p(t) \) is a support function of a closed and strictly convex curve having a circle as \( \frac{\pi}{2} \)-isoptic and different from an ellipse.

**Example 3.2.** Let \( p(t) = \sqrt{a\sin^2 t + b\cos^2 t + c} \), for positive \( a, b, c \). For \( c \) sufficiently big \( p(t) + \dot{p}(t) > 0 \) for each \( t \) and \( p^2(t) + p^2\left(t + \frac{\pi}{2}\right) = a + b + 2c \) so \( p(t) \) is a support function of a closed and strictly convex curve having a circle as \( \frac{\pi}{2} \)-isoptic. This curve cannot be an ellipse because an origin of a coordinate system is its center of symmetry and \( p(t) \) is a periodic function.
with a period $\frac{\pi}{3}$. Hence its curvature function is also periodic with the same period and this curve has more then four vertices. More generally we can take $p(t) = \sqrt{a \sin^2 mt + b \cos^2 mnt} + c$, where $m$ and $m$ are odd integers and $a$, $b$, $c$ are positive.

**Remark 3.1.** Let $C$ be an ellipse. We fix a diameter $PP'$ and consider an ellipse $C'$ with focuses at $P$ and $P'$ which is tangent to $C$. Then points $Q$ and $Q'$ of tangency give a diameter such that a perimeter of parallelogram $PQP'Q'$ is maximal. The common tangent of $C$ and $C'$ at $Q$ (resp. $Q'$) makes equal angles with the sides $PQ$ and $P'Q$ (resp. $PQ'$ and $P'Q'$). This means that for parallelogram of maximal perimeter a tangent at any vertex makes equal angles with adjoining sides. This is a part of a more general fact.

Let $C$ be any closed and convex curve given in an arbitrary parametrization $z = z(t)$ of class $C^1$. Fix the points $z(t_1)$ and $z(t_2)$. Let $z(t_0)$ be such a point that the perimeter of the triangle $z(t_1)z(t_2)z(t_0)$ is maximal. Then the tangent at $t_0$ makes equal angles with the sides $z(t_0)z(t_1)$ and $z(t_0)z(t_2)$.

Indeed,

\[
\frac{d}{dt}(|z(t) - z(t_1)| + |z(t) - z(t_2)|) = \frac{\langle z(t) - z(t_1), \dot{z}(t) \rangle}{|z(t) - z(t_1)|} + \frac{\langle z(t) - z(t_2), \dot{z}(t) \rangle}{|z(t) - z(t_2)|} = \frac{|z(t) - z(t_1)||\dot{z}(t)|\cos \angle(z(t) - z(t_1), \dot{z}(t))}{|z(t) - z(t_1)|} + \frac{|z(t) - z(t_2)||\dot{z}(t)|\cos \angle(z(t) - z(t_2), \dot{z}(t))}{|z(t) - z(t_2)|} = |\dot{z}(t)| |\cos \angle(z(t) - z(t_1), \dot{z}(t)) + \cos \angle(z(t) - z(t_2), \dot{z}(t))|.
\]

For $t = t_0$ we obtain

\[
\cos \angle(z(t_0) - z(t_1), \dot{z}(t_0)) + \cos \angle(z(t_0) - z(t_2), \dot{z}(t_0)) = 0.
\]

**References**


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