On the size of the ideal boundary
of a finite Riemann surface

Abstract. The ideal boundary of a non-compact Riemann surface $R_0$ becomes visible if $R_0$ is embedded into some compact surface $R$ which naturally should have the same genus $g$ as $R_0$. All these compactifications of $R_0$ can be compared in a certain quotient space of $\mathbb{C}^g$. With respect to the canonical metric in this space the diameters of all models of the ideal boundary of $R_0$ are known to be bounded (cf. [4]) by a number depending only on $R_0$.

In this paper we prove that the diameter of each component has either a positive lower bound, depending only on $R_0$, or this component appears to be a single point in any compactification $R$.

Introduction. There are several definitions of the ideal boundary of Riemann surfaces (cf. [2]). In this article we consider a finitely connected, non-compact Riemann surface $R_0$ of finite genus $g$. If $\iota : R_0 \to R$ is a conformal embedding of $R_0$ into some compact surface $R$ of genus $g$, then we call the boundary $\partial \iota(R_0) \subset R$ the ideal boundary of $R_0$ with respect to the compactification $(R, \iota)$ of $R_0$. We will ask for properties of this ideal boundary which are independent of $(R, \iota)$ and such characteristics of $R_0$. As in [4] we use a suitable Jacobian manifold, a quotient space of $\mathbb{C}^g$, in

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which each embedding $\nu(R_0) \subset R$ can again be embedded. On the Jacobian manifold we have a natural metric, induced by the euclidean metric on $\mathbb{C}^g$. With respect to this metric we may compare the diameter of the ideal boundaries which we obtain for all the different embeddings in any surfaces $R$ as described above. In [4] is proved that there is some uniform bound for all these diameters.

The ideal boundary, realized as a portion of a compact surface $R$, consists of components. Because $R_0$ is provided as a finitely connected surface we have only finitely many components of the ideal boundary. It is easy to verify that there is a one-to-one correspondence of these components if we consider two or more different embeddings $\nu_1 : R_0 \to R_1, \nu_2 : R_0 \to R_2$. In this sense we understand the components of the ideal boundary of $R_0$. The purpose of this article is to show that for each such component we have (besides the supremum obtained in [4]) also a non trivial infimum for the diameter of the corresponding subset of the Jacobian manifold, which is valid for all such compactifications $R$ of $R_0$. If the infimum is 0, then the component in view is always (i.e. on each such $R$) a singleton.

1. Notations and Definitions. Let, as before, $R_0$ denote some finitely connected non-compact Riemann surface of finite genus $g > 0$. Then we can fix $g$ pairs of piecewise smooth curves $a_j^0, b_j^0$ such $\chi_0 = \{a_j^0, b_j^0\}_{j=1}^g$ represents a canonical homology basis modulo dividing cycles on $R_0$ (cf. [1]). Now we consider some compact Riemann surface $R$ of genus $g$ together with some conformal embedding $\nu : R_0 \to R$ and define

$$\nu(a_j^0) = : a_j \quad \text{and} \quad \nu(b_j^0) = : b_j \quad (1 \leq j \leq g)$$

It can be easily seen that the $g$ pairs of curves $\chi = \{a_j, b_j\}_{j=1}^g$ represent a canonical homology basis for $R$.

We say that the triple $\mathcal{R} = (R, \chi, \nu)$ gives a conformal compactification of the (marked) Riemann surface $(R_0, \chi_0)$.

Remark: For each $j, 1 \leq j \leq g$ there is one and only one closed holomorphic differential $\phi^{(j)}$ on $R$ with

$$(1) \quad \int_{a_k} \phi^{(j)} = \delta_{jk}, \quad \int_{b_k} \phi^{(j)} = : \tau_{jk} \quad (j,k = 1,2,\ldots,g),$$

where $\delta_{jk}$ denotes the Kronecker symbol(cf. [3] III.2.8).

We write $\tau_k(R, \chi)$ resp. $\epsilon_k$ for the $k$th column of the matrix $(\tau_{jk})$ resp. $(\delta_{jk})$.

Let $\Pi$ stand for the linear span with integer coefficients of the $2g$ vectors $\tau_1, \tau_2, \ldots, \tau_g, \epsilon_1, \epsilon_2, \ldots, \epsilon_g$
and we call 
\[
\text{Jac } (R, \chi) := \mathbb{C}^g / \Pi
\]
the Jacobian manifold of the marked Riemann surface \((R, \chi)\). We have the canonical projection \(\pi : \mathbb{C}^g \to \text{Jac } (R, \chi)\).

Now we fix some point \(p^0\) on \(R\) and take for each \(p \in R\) a piecewise smooth curve \(\gamma_p \) on \(R\) with initial point \(p^0\) and endpoint \(p\). This defines a map \(\tilde{\Phi}_R : R \to \mathbb{C}^g\) via
\[
\tilde{\Phi}_R(p) = \left( \int_{\gamma_p} \phi^{(1)}, \int_{\gamma_p} \phi^{(2)}, \ldots, \int_{\gamma_p} \phi^{(g)} \right).
\]

Note that the image \(\tilde{\Phi}_R(p)\) depends on \(p\) and on the contour \(\gamma_p\). However, the composition map \(\Phi_R := \pi \circ \tilde{\Phi}_R : R \to \text{Jac } (R, \chi)\) turns out to be independent of the special choice of \(\gamma_p\).

Relating to the conformal compactification \(R = (R, \chi, \iota)\) of \((R_0, \chi_0)\) we define the ideal boundary of \(R_0\) as the topological boundary of the set \(\iota(R_0) \subset R\), i.e.
\[
\partial_R R_0 := \iota(R_0) \setminus \iota(R_0).
\]
The set \(R \setminus \iota(R_0)\) consists, by the assumption on \(R_0\) and the compactness of \(R\), of finitely many components \(B^1_R, \ldots, B^n_R\). Now we consider another conformal compactification \(S\) instead of \(R\), which gives the components \(B^1_S, \ldots, B^n_S\). Then, by means of pairwise disjoint, simple closed curves on \(R_0\) whose images under \(\iota_R\) resp. \(\iota_S\) separate the components \(B^j_R\) on \(R\) as well as \(B^j_S\) on \(S\), we get a one-to-one correspondence of the sets \(B^j_R\) and \(B^j_S\) for \(j = 1, \ldots, n\). In this sense we can speak of the \(n\) components \(B^1, \ldots, B^n\) (with respect to some fixed denumeration) of the ideal boundary \(\partial_R R_0\) independently of \(R\). Moreover, let
\[
\Delta_R R_0 := \Phi_R(\partial_R R_0) \text{ as well as } \Delta^j_R R_0 := \Phi_R(\partial B^j_R) \quad (j = 1, \ldots, n).
\]

We denote by \(d_R(M)\) the diameter of a subset \(M\) of \(\text{Jac}(R, \chi)\) with respect to the canonically induced metric of \(\mathbb{C}^g\).

2. Universal bounds.

**Theorem 1.** Let \((R_0, \chi_0)\) denote a non compact, finitely connected, marked Riemann surface of finite genus \(g > 0\) with the ideal boundary components \(B^1, \ldots, B^n\) (defined as above). Then there exist numbers \(c_j, C_j\) \((j = 1, \ldots, n)\) such that
\[
c_j \leq d_R(\Delta^j_R R_0) \leq C_j \quad (j = 1, \ldots, n)
\]
for all conformal compactifications \(R = (R, \chi, \iota)\) of \((R_0, \chi_0)\). Each lower bound \(c_j\) can be taken strictly positive except for the case where \(B^j_R \subset R\) is a singleton for some (and thus for all) conformal compactification of \((R_0, \chi_0)\).

In the proof we will need the following
Lemma. Let $\Omega$ denote a doubly connected domain in the complex plane, bounded by the piecewise smooth Jordan curves $\Gamma_1, \Gamma_2$. For each $m \in \mathbb{N}$ let some complex-valued function $f_m$, continuous on $\overline{\Omega}$ and holomorphic on $\Omega$ be given. We assume that the sequence $f_m$ is uniformly bounded on $\Omega$ and tends to some constant $c$ uniformly on $\Gamma_2$. Let $f$ denote the limit function of some locally convergent subsequence of $f_m$ on $\Omega$. Then $f \equiv c$ on $\Omega$ or $\Gamma_2$ consists of a single point.

Proof. We assume that the cycle $\Gamma := \Gamma_1 - \Gamma_2$ represents a positively oriented parametrization of $\partial \Omega$, where the boundary of the unbounded component $C_1$ of $\mathbb{C} \setminus \Omega = C_1 \cup C_2$ is given by $\Gamma_1$. By Cauchy’s formula we have for $m \in \mathbb{N}$, $z \in \Omega$

$$f_m(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f_m(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f_m(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f_m(\zeta)}{\zeta - z} d\zeta =: g_1^m(z) - g_2^m(z).$$

Each function $g_1^m$ admits an analytic continuation on $I(\Gamma_1) := \Omega \cup C_2$. Because $\Gamma_1$ has winding number 1 with respect to the points on $\Gamma_2$ and $f_m \to c$ uniformly on $\Gamma_2$ we have $g_1^m \to c$ as $m \to \infty$ on this contour.

The functions $g_1^m$ are uniformly bounded on $I(\Gamma_1)$. By Montel’s theorem we may assume that the sequence $g_1^m$ is locally uniformly convergent on $I(\Gamma_1)$. The limit function $g$ is obviously an analytic continuation of $f = \lim f_m$ on $I(\Gamma_1)$. But we have just proved $g \equiv c$ on $\Gamma_2$. So, if $\Gamma_2$ is a continuum, we conclude $g \equiv c$ on $I(\Gamma_1)$, and thus $f \equiv c$ on $\Omega$. □

Now we are ready to give the proof of Theorem 1. According to [4, Satz 2] there exists some $C$ with $d_R(\Delta_R R_0) \leq C$ simultaneously for all conformal compactifications $\mathcal{R} = (R, \chi, \iota)$ of $(R_0, \chi_0)$. Since $\Delta_j R_0 \subset \Delta_R R_0 (j = 1, \ldots, n)$, we get the existence of the upper bounds $\tilde{C}_j$ already by the mentioned result in [4].

Now we fix some $j \in \{1, \ldots, n\}$ and assume that there is no strictly positive lower bound $c_j$. This means, there exists some sequence of conformal compactifications $\mathcal{R}_m = (R_m, \chi_m, \iota_m)$ of $(R_0, \chi_0)$ in the described sense with the property

$$d_{\mathcal{R}_m}(\Delta^j_{\mathcal{R}_m} R_0)) \to 0 \text{ as } m \to \infty. \tag{2}$$

On the Riemann surface $R^j_m := R_m \setminus B^j_{R_m}$ we can find some domain $\Lambda^0_m$ with the following properties:

(i) $\Lambda^0_m$ has genus $g$,

(ii) $B^\mu_{R_m} \subset \Lambda^0_m$ for $\mu = 1, \ldots, j - 1, j + 1, \ldots, n$,

(iii) $\partial \Lambda^0_m$ can be parametrized as a Jordan curve $\omega^0_m$ on $R^j_m$. 


In \( R^j \backslash \Lambda_0 \) we fix another Jordan curve \( \omega^1 \), homotopic to \( \omega^0 \) on \( R^j \). By \( A_m \) we denote the domain bounded by these curves and let \( \Lambda^1_m := \Lambda^0_m \cup A_m \).

As proved (with slight modifications) in [4], p.42, the following estimate is valid:

\[
(3) \quad d_{R_m}(\Phi_{R_m}(R_m \backslash \Lambda_m^1)) \leq B,
\]

where \( B \) depends only on \( A_m \) and the periods \( \tau_{\nu \nu} \). Note that we can give the conformal annulus \( A_m \) via \( \iota_m \) by the curves \( C^0 := \iota^{-1}(\omega^0_0) \) and \( C^1 := \iota^{-1}(\omega^1_0) \) on \( R_0 \) as well as on \( R_m \). Thus \( B \) is determined by considerations purely on the Riemann surface \( R_0 \) and we may assume that the boundary curves \( C^0, C^1 \) are the same for all \( m \in \mathbb{N} \).

Note that (3) can also be expressed as:

\[
(4) \quad \text{The variation of } \Phi_{R_m} \circ \iota_m \text{ on } M_m := R_m \backslash \Lambda_m^1 \text{ is uniformly bounded.}
\]

The set \( M_m \) is, for each \( m \in \mathbb{N} \), a simply connected domain. We may assume that for all \( m \) the starting point \( p_0^m \) of the contours in the definition of \( \Phi_{R_m} \) belongs to \( M_m \) and also that for each \( p \in M_m \) the contour \( \gamma_p \) is a curve in \( M_m \). Moreover, we take \( p_0^m = \iota_m(p_0) \) where \( p_0 \) is some fixed point on \( R_0 \). By the monodromy theorem the value \( \Phi_{R_m}(p) \) for \( p \in M_m \) comes out to be independent of the special choice of the contours \( \gamma_p \).

The set \( H := \iota_m^{-1}(M_m \cap \iota_m(R_0)) \) is a planar domain on \( R_0 \) and does not depend on \( m \).

Let \( G \subset \mathbb{C} \) be a domain bounded by Jordan curves which admits a conformal map \( \theta \) of \( G \) onto \( H \). It follows from our construction that the boundary of \( G \) consists of two components. One of them, which we denote by \( \Gamma_1 \), corresponds under \( \theta \) to the Jordan curve \( C_1 \) on \( R_0 \), the other one, \( \Gamma_2 \), to the ideal boundary component \( B^j \) of \( R_0 \).

The functions \( f_m := \Phi_{R_m} \circ \iota_m \circ \theta \) map \( G \) holomorphically in \( \mathbb{C}^g \) and have a continuous extension on \( \Gamma_1 \) and \( \Gamma_2 \). From (2) we know that the sequence \( f_m \) tends on \( \Gamma_2 \) uniformly to some constant. The functions \( f_m \) are uniformly bounded on \( G \), as follows from (4) and the normalization

\[
f_m(\theta^{-1}(p_0)) = \Phi_{R_m}(\iota_m(p_0)) = \tilde{\Phi}_{R_m}(p_0^m) = 0.
\]

We apply Montel’s theorem to the coordinate functions of \( f_m \) and may assume that the sequence \( f_m \) itself is locally convergent on \( G \). By our Lemma we see that the limit function \( f \) is constant, or \( \Gamma_2 \) consists of a single point.

But the first case cannot happen: the canonical lifting of the function \( f_m \) on \( H \subset R_0 \) is given by \( F_m := \tilde{\Phi}_{R_m} \circ \iota_m \) and has an unrestricted analytic
continuation on \( R \) along every curve on \( R \) starting in \( H \). This defines an analytic element \( F_m \) on \( R \). On the universal covering surface \( \Sigma_0 \) of \( R \) this element \( \tilde{F}_m \) appears as a holomorphic function \( F_m^* : \Sigma_0 \to \mathbb{C} \). Let this be done for all \( m \in \mathbb{N} \). By (4) and the definition of the functions \( \tilde{\Phi}_{R,m} \) we see that the functions \( F_m^* \) are uniformly bounded on every compact subset of \( \Sigma_0 \). This shows that the sequence \( F_m^* \) tends, locally uniformly on \( \Sigma_0 \), to a constant as \( m \to \infty \) if the sequence \( f_m \) does the same on \( G \). But this contradicts (cf.(1))

\[
\int_{a_k} \phi^{(k)} = 1 \quad (k = 1, \ldots, g).
\]

Thus \( \Gamma_2 \) is a constant curve. By elementary considerations we see that in this case \( B_{R} \subset R \) must be a singleton for all conformal compactifications of \( R_0 \) in the described sense. \( \square \)

**References**


