On estimating the coefficient product $A_1A_2A_3$ for real bounded non-vanishing univalent functions

Abstract. The class of the title is sufficiently limited for allowing certain estimations for combinations of the three first coefficients $A_1$, $A_2$ and $A_3$. The negative sign of $A_2$ implies complications which, however, in the present treatment will be governed, when estimating the product $A_1A_2A_3$.

1. Introduction. In [2] the observations of J. Śładkowska [1] were utilized in determining the first coefficient bodies for functions $F$ which are univalent and bounded with the condition of non-vanishedness. Denote the class of these functions by $S'(B)$. Another condition will be a restriction to real coefficients $A_ν$. The subclass thus introduced is denoted by $S'_R(B)$:

$$
S'_R(B) = \{ F \mid F(z) = B + A_1z + \ldots, \quad z \in U \supset F(U) \not\ni O, \quad 0 < B < 1, A_1 > 0 \},
$$

Here $U$ is the unit disc centered at the origin and $B$ is the leading coefficient, characterizing the function through the image of the origin: $B = F(O)$. The class notation repeats those of the normalized bounded

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univalent functions $f$:

$$\begin{align*}
S(b) &= \{ f \mid f(z) = b(z + a_2z^2 + \cdots), \, z \in U, \, |f(z)| < 1, \, 0 < b < 1 \}, \\
S_R(b) &= S(b).
\end{align*}$$

Again, $S_R(b)$ means the real subclass of $S(b)$.

The observation on Śladkowska combined the above real classes together through the function $L$:

$$\begin{align*}
L &= L(z) = K^{-1} \left[ \frac{4B}{(1-B)^2} \left( K(z) + \frac{1}{4} \right) \right], \\
K &= K(z) = \frac{z}{(1-z)^2}.
\end{align*}$$

Here $K$ is the left Koebe-function and hence $L(U)$ is a unit disc with a left radial slit from the point $-1$ to the origin. The one-to-one correspondence

$$L \circ f \in S'_R(B), \quad L^{-1} \circ F \in S_R(b)$$

will be governed by aid of the development of $L$:

$$\begin{align*}
y &= L(z) = B + B_1z + B_2z^2 + B_3z^3 + \ldots, \\
B_1 &= \frac{4B(1-B)}{1+B}, \\
B_2 &= \frac{8B(1-B)}{(1+B)^3} (1-2B-B^2), \\
B_3 &= \frac{4B(1-B)}{(1+B)^5} (3 - 20B + 18B^2 + 12B^3 + 3B^4),
\end{align*}$$

yielding

$$\begin{align*}
b &= \frac{A_1}{B_1}, \\
a_2 &= \frac{A_2}{A_1} - \frac{B_2}{B_1^2} A_1, \\
a_3 &= \frac{A_3}{A_1} - 2 \frac{B_2}{B_1^2} A_2 + \left( \frac{B_2^2}{B_1^2} - \frac{B_3}{B_1^3} \right) A_1^2.
\end{align*}$$

The knowledge concerning the coefficient bodies of $S_R(b)$ allows determining the corresponding bodies of $S'_R(B)$ [2]. They are denoted by $(A_2, A_1)$ and $(A_3, A_2, A_1)$. For $(A_2, A_1)$ we have

$$(A_2, A_1) = \left\{ (A_1, A_2) \mid -2A_1 + \frac{A_2^2}{B(1-B^2)} \leq A_2 \leq 2A_1 - \frac{2+B}{1-B^2} A_1^2, \quad 0 < A_1 < B_1 \right\}.$$
The body \((A_3, A_2, A_1)\) is defined on \((A_2, A_1)\) so that
\[
E \leq A_3 \leq F,
\]
where in the whole \((A_2, A_1)\),
\[
E = A_3 = \frac{A_2^2}{A_1} - A_1 + \frac{A_1^3}{(1 - B^2)^2}.
\]
The extremal domains connected to \(E\) are of left-right radial-slit types \([2]\).

For \(F\) the area of definition is divided in three parts I, II and III visualized in Figure 1. The dividing lines \(I \cap II\) and \(II \cap III\) are determined by the limits
\[
R^2 \left[ B_2 - 2B_1 |\ln R| \right] \leq A_2 \leq R^2 \left[ B_2 + 2B_1 |\ln R| \right],
\]
where \(R = A_1/B_1\).

The slit-type boundary functions extremizing \(F\) are similarly visualized in Figure 1.

![Figure 1](image_url)

Observe that according to the extremal types the region II is split in two parts, \(II_1\) and \(II_2\) by the dividing line
\[
A_2 = R^2 \left[ B_2 + 2B_1 \frac{1 - 6B + B^2}{(1 + B)^2} \ln R \right].
\]
In the following denote

\[ D_1 = B_3/B_1 - 2B_2^2/B_1^2. \]

By using this notation we have for \( F \) in the regions I and III (cf. [2]):

\[
\begin{cases}
A_3 = \left[ a_3 + 2 \frac{B_2}{B_1^2} A_2 + D_1 R^2 \right] A_1 = F, \\
A_2 = A_1 a_2 + B_2 R^2, \\
a_2 = 2 \delta (R - \sigma + \sigma \ln \sigma); \quad \sigma \in [R, 1], \\
a_3 = 1 - R^2 + a_2^2 + 2 \delta \sigma a_2 + 2(\sigma - R)^2.
\end{cases}
\]

Here \( \delta = 1 \) for I and \( \delta = -1 \) for III.

In II, \( F \) is defined by (cf. [2])

\[
\begin{cases}
A_3 = \left[ a_3 + 2 \frac{B_2}{B_1^2} A_2 + D_1 R^2 \right] A_1 = F, \\
a_2 = A_2 A_1 - \frac{B_2}{B_1} R, \\
a_3 = 1 - R^2 + \left( 1 + \frac{1}{\ln R} \right) a_2^2.
\end{cases}
\]

2. Maximizing \( A_1 A_2 A_3 \). In some former papers, e.g. [3], a few simple functionals of the coefficients \( A_\nu \) were considered. They were chosen to be independent of the sign of \( A_2 \). The present functional is free of that restriction. Thus

\[
A_2 \geq 0: \quad A_1 A_2 E \leq A_1 A_2 A_3 \leq A_1 A_2 F, \\
A_2 \leq 0: \quad A_1 A_2 F \leq A_1 A_2 A_3 \leq A_1 A_2 E.
\]

Consider first the local extremal point connected with \( A_1 A_2 E \):

\[
Q = A_1 A_2 A_3 \leq A_1 A_2 E = A_2^3 + \left( \frac{A_1^4}{(1 - B^2)^2} - A_1^2 \right) A_2.
\]

Differentiating this we obtain for the local extremal:

\[
Q = \frac{\sqrt{3}}{36} (1 - B^2)^3; \quad A_1 = \frac{1 - B^2}{\sqrt{2}}, \quad A_2 = -\frac{1 - B^2}{2\sqrt{3}}, \quad A_3 = -\frac{\sqrt{2}}{6} (1 - B^2).
\]

The extremal point lies above the lower boundary arc \( \partial I \) of \( (A_2, A_1) \) if

\[
-\frac{1 - B^2}{2\sqrt{3}} - \left[ -2 A_1 + \frac{A_1^2}{B(1 - B^2)} \right] A_1 = \frac{1 - B^2}{\sqrt{2}} \geq 0
\]

\[
\Downarrow
\]

\[
B \geq \tilde{c} = \frac{6\sqrt{2} + \sqrt{3}}{23} = 0.444231834.
\]
For the upper boundary arc \( \partial III \) of \((A_2, A_1)\) we require
\[
\left[ 2A_1 - \frac{2 + B}{1 - B^2} A_1^2 \right]_{A_1 = \frac{1 - B^2}{\sqrt{2}}} \geq -\frac{1 - B^2}{2\sqrt{3}},
\]
which holds for the whole interval \( 0 < B < 1 \).

For an interval below \( \tilde{c} \) the extremal point will be located on the lower boundary arc \( \partial I \),
\[
\partial I : A_2 = -2A_1 + \frac{A_1^2}{B(1 - B^2)},
\]
where according to (3),
\[
Q = -6A_1^3 + \frac{11A_1^4}{B(1 - B^2)} - \frac{6 + 2B^2}{B^2(1 - B^2)^2} A_1^5 + \frac{1 + B^2}{B^3(1 - B^2)^3} A_1^6.
\]
For the local extremal point on \( \partial I \) we thus have
\[
-9[B(1 - B^2)]^3 + 22[B(1 - B^2)]^2 A_1
\]
\[
-5[B(1 - B^2)](3 + B^2)A_1^3 + 3(1 + B^2)A_1^3 = 0.
\]
This condition is satisfied at the point (4) for \( B = \tilde{c} \).

Next, determine the local extremal point of \( Q = A_1A_2F \) in the regions I and III. From (1) deduce
\[
\begin{cases}
\frac{1}{2A_1^2} \cdot \frac{\partial Q}{\partial A_1} = h_0 + h_1A_1 + h_2A_1^2 = 0; \\
h_0 = \delta \ln \sigma(1 + 12s^2 + 12\sigma s + 2\sigma^2), \\
h_1 = 4 \ln \sigma(3s + \sigma)S, \\
h_2 = \delta \ln \sigma(13/B_1^2 + 12\delta B_2/B_1^2 + 2B_2^2/B_1^3 + B_3/B_1^3).
\end{cases}
\]
Further
\[
\begin{cases}
\frac{1}{A_1^2} \cdot \frac{\partial Q}{\partial A_1} = k_0 + k_1A_1 + k_2A_1^2 + k_3A_1^3 = 0; \\
k_0 = 6\delta s(1 + 4s^2 + 4\sigma s + 2\sigma^2), \\
k_1 = 4(1 + 12s^2 + 4\sigma s + 2\sigma^2)S, \\
k_2 = 10\delta s(2S^2 + 5/B_1^2 + 4\delta B_2/B_1^3 + B_3/B_1^3), \\
k_3 = 6(5/B_1^2 + 4\delta B_2/B_1^3 + B_3/B_1^3)S.
\end{cases}
\]
Here
\[
s = \sigma \ln \sigma - \sigma, \ S = 2\delta/B_1 + B_2/B_1^2
\]
and \( \delta = 1 \) for I and \( \delta = -1 \) for III.

From (7)
\[
A_1 = \frac{-h_1 + \delta \cdot \sqrt{h_1^2 - 4h_0h_2}}{2h_2},
\]
which, when substituted in (8), yields in the local extremal case $\sigma$ and hence $A_1$, too.

There remains the maximizing of $Q = A_1A_2F$ in II. By aid of the abbreviations

$$A_1/B_1 = R, \ H = 1 + 1/\ln R;$$
$$D_2 = B_3/B_1 - B_2^2/B_1^2 - 1, \ D_3 = B_3/B_1 + 2B_2^2/B_1^2 - 1,$$
we obtain from (2)

$$\begin{align*}
-\ln^2 R \cdot \frac{\partial Q}{\partial A_1} & = a_2^2 + 4\frac{B_2}{B_1} R \ln R \cdot a_2 - 2\ln^2 R (1 + 2R^2D_2), \\
\frac{1}{A_1^2} \cdot \frac{\partial Q}{\partial A_2} & = 3Ha_2^2 + 2\frac{B_2}{B_1} (H + 2)Ra_2 + 1 + D_3R^2.
\end{align*}$$

This yields the necessary extremal conditions for determining $A_1$ and $A_2$:

$$\begin{align*}
3Ha_2^2 + G_1a_2 + G_2 &= 0, \\
3Ha_2^2 + G_3a_2 + G_4 &= 0,
\end{align*}$$

$$a_2 = \frac{G_4 - G_2}{G_1 - G_3} \Rightarrow A_2 = A_1a_2 + B_2R^2,$$
$$3Ha_2^2 + G_3a_2 + G_4 = 0;$$

$$\begin{align*}
G_1 &= 12H \frac{B_2}{B_1} R \ln R, \\
G_2 &= -6H \ln^2 R (1 + 2R^2D_2), \\
G_3 &= 2\frac{B_2}{B_1} (H + 2)R, \\
G_4 &= 1 + D_3R^2.
\end{align*}$$

(9)

3. Maximalization results. In Table 1 there is a list of maximal points and values for increasing values of $B$. Observe, that the sign $-$ in the region-notation implies maximizing with negative $A_2$, i.e. the maximum is obtained from $A_1A_2E$ which means explicit expression (4) for max $Q$. Similarly, + indicates maximalization with positive $A_2$, from $A_1A_2F$, yielding results in implicit form.

There exist the following max max-cases:

$$\text{max max } Q = 0.037487883; \ B = b_1 = 0.105067336 \in \text{P},$$
$$\text{max max } Q = 0.026754453; \ B = b_2 = 0.397998215 \in \partial I.$$

The maximizing point varies with increasing values of $B$. Crossing the boundaries between different regions of the body $(A_3, A_2, A_1)$ occurs at the
On estimating the coefficient product $A_1 A_2 A_3 \ldots$

points $c_2$ and $c_3$:

$$B = c_2 = 0.185727645 \in II_+ \cap III_+,$$
$$B = c_3 = 0.453697122 \in I_- \cap II_-.$$

At

$$B = d = 0.312534879 \in III_+, \partial I$$

the maximalization occurs simultaneously on the upper surface $III_+$ and on the lower boundary $\partial I$, determining at the same time

$$\min \max Q = 0.021714369; \ B = d \in III+ \partial I.$$ 

Such double maximal points may be called Twin Peaks on the surface of the coefficient body $(A_3, A_2, A_1)$.

Table 1.

<table>
<thead>
<tr>
<th>$B$</th>
<th>Region</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>max $Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>$P$</td>
<td>0.039208</td>
<td>0.075326</td>
<td>0.105567</td>
<td>0.000312</td>
</tr>
<tr>
<td>0.1</td>
<td>$P$</td>
<td>0.327273</td>
<td>0.427348</td>
<td>0.266517</td>
<td>0.037275</td>
</tr>
<tr>
<td>0.105067 = $b_1$</td>
<td>$P$</td>
<td>0.340353</td>
<td>0.434133</td>
<td>0.253711</td>
<td>0.037487</td>
</tr>
<tr>
<td>0.1051</td>
<td>$P$</td>
<td>0.340436</td>
<td>0.434173</td>
<td>0.253625</td>
<td>0.037487</td>
</tr>
<tr>
<td>0.105369 = $c_1$</td>
<td>$II_+ \cap P$</td>
<td>0.341122</td>
<td>0.434504</td>
<td>0.252918</td>
<td>0.037487</td>
</tr>
<tr>
<td>0.14</td>
<td>$II_+$</td>
<td>0.356935</td>
<td>0.412379</td>
<td>0.244326</td>
<td>0.035963</td>
</tr>
<tr>
<td>0.185728 = $c_2$</td>
<td>$II_+ \cap III_+$</td>
<td>0.355339</td>
<td>0.388176</td>
<td>0.233366</td>
<td>0.032189</td>
</tr>
<tr>
<td>0.2</td>
<td>$II_+$</td>
<td>0.350186</td>
<td>0.383550</td>
<td>0.230412</td>
<td>0.030947</td>
</tr>
<tr>
<td>0.3</td>
<td>$II_+$</td>
<td>0.312908</td>
<td>0.348136</td>
<td>0.208226</td>
<td>0.022683</td>
</tr>
<tr>
<td>0.312535 = $d$</td>
<td>$III_+$</td>
<td>0.308088</td>
<td>0.343354</td>
<td>0.205272</td>
<td>0.021714</td>
</tr>
<tr>
<td>0.312535 = $d$</td>
<td>$\partial I$</td>
<td>0.455939</td>
<td>-0.174732</td>
<td>-0.272563</td>
<td>0.021714</td>
</tr>
<tr>
<td>0.35</td>
<td>$\partial I$</td>
<td>0.495114</td>
<td>-0.192058</td>
<td>-0.262990</td>
<td>0.025008</td>
</tr>
<tr>
<td>0.38</td>
<td>$\partial I$</td>
<td>0.522565</td>
<td>-0.205232</td>
<td>-0.247032</td>
<td>0.026493</td>
</tr>
<tr>
<td>0.39</td>
<td>$\partial I$</td>
<td>0.530866</td>
<td>-0.209495</td>
<td>-0.240097</td>
<td>0.026702</td>
</tr>
<tr>
<td>0.397998 = $b_2$</td>
<td>$\partial I$</td>
<td>0.537182</td>
<td>-0.212860</td>
<td>-0.233981</td>
<td>0.026754</td>
</tr>
<tr>
<td>0.4</td>
<td>$\partial I$</td>
<td>0.538716</td>
<td>-0.213697</td>
<td>-0.232372</td>
<td>0.026751</td>
</tr>
<tr>
<td>0.444232 = $\bar{c}$</td>
<td>$\partial I \cap I_-$</td>
<td>0.567565</td>
<td>-0.231707</td>
<td>-0.189188</td>
<td>0.024880</td>
</tr>
<tr>
<td>0.45</td>
<td>$I_-$</td>
<td>0.563918</td>
<td>-0.230218</td>
<td>-0.187973</td>
<td>0.024403</td>
</tr>
<tr>
<td>0.453697 = $c_3$</td>
<td>$I_- \cap II_-$</td>
<td>0.561555</td>
<td>-0.229254</td>
<td>-0.187185</td>
<td>0.024098</td>
</tr>
<tr>
<td>0.46</td>
<td>$II_-$</td>
<td>0.557483</td>
<td>-0.227591</td>
<td>-0.185828</td>
<td>0.023578</td>
</tr>
<tr>
<td>0.5</td>
<td>$II_-$</td>
<td>0.530330</td>
<td>-0.216506</td>
<td>-0.176777</td>
<td>0.020297</td>
</tr>
<tr>
<td>0.6</td>
<td>$II_-$</td>
<td>0.452548</td>
<td>-0.184752</td>
<td>-0.150849</td>
<td>0.012612</td>
</tr>
<tr>
<td>0.7</td>
<td>$II_-$</td>
<td>0.366242</td>
<td>-0.147222</td>
<td>-0.120208</td>
<td>0.006382</td>
</tr>
<tr>
<td>0.8</td>
<td>$II_-$</td>
<td>0.254558</td>
<td>-0.103923</td>
<td>-0.084853</td>
<td>0.002425</td>
</tr>
<tr>
<td>0.9</td>
<td>$II_-$</td>
<td>0.134350</td>
<td>-0.054848</td>
<td>-0.044783</td>
<td>0.000330</td>
</tr>
<tr>
<td>0.99</td>
<td>$II_-$</td>
<td>0.014071</td>
<td>-0.005745</td>
<td>-0.004690</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
The point $\tilde{c}$ from (5) defines an interval $d \leq B \leq \tilde{c}$ in which the maximizing point lies on $\partial I$. From this onwards, in the interval $\tilde{c} < B < 1$, the regions $I_-$ or $II_-$ take care of the maximalization.

If $B$ is sufficiently close to 0 the point $P$ assumes the role of the maximizing point. In order to find the shifting point $c_1 = II_+ \cap P$ let $A_1$ tend to $B_1$ so that

$$A_1 = B_1(1 - h), \quad h \to +0.$$ 

From (9) we see that

$$a_2 = -\frac{B_1 \ln R}{2B_2}(1 + D_3) + O(h), \quad O(h) \to 0 \text{ for } h \to 0;$$

$$\frac{1}{A_1 A_2} \cdot \frac{\partial Q}{\partial A_1} = K(B) + O(h),$$

where

$$K(B) = \frac{B_1^2}{4B_2^2}(1 + D_3)^2 - 4D_2 - 2D_3 - 4.$$ 

Hence $\frac{\partial Q}{\partial A_1} = 0$ yields for $B = c_1$ the condition $K(B) = 0$, i.e.

$$8B_1^2B_2^2 - 20B_1B_2^2B_3 + B_1^2B_3^2 + 4B_2^4 = 0$$

$$\Downarrow$$

$$B = c_1 = 0.105369060 \in II_+ \cap P.$$ 

The explicit part of the above estimation is collected as follows.

**Result.** In $S'_R(B)$ the maximum of $A_1 A_2 A_3$ for the interval

$$0.444031833 = \frac{6\sqrt{2} + \sqrt{3}}{23} = \tilde{c} \leq B < 1$$

occurs on the lower surface of the body $(A_3, A_2, A_1)$:

$$\max A_1 A_2 A_3 = \frac{\sqrt{3}}{36}(1 - B^2)^3,$$

at the point

$$A_1 = \frac{1 - B^2}{\sqrt{2}}, \quad A_2 = -\frac{1 - B^2}{2\sqrt{3}}, \quad A_3 = -\frac{\sqrt{2}}{6}(1 - B^2).$$

In Figure 2 there is the graph connected with the values of the Table 1.
4. Minimalization results. According to the Section 2 the minimum of 
\( Q = A_1 A_2 A_3 \) is obtained from the expressions

\[
\begin{align*}
A_1 A_2 E & \text{ for } A_2 \geq 0, \\
A_1 A_2 F & \text{ for } A_2 \leq 0.
\end{align*}
\]

Actually, only the last alternative will be realized. Therefore, the sign –, characterizing the region-notation, can be omitted.

<table>
<thead>
<tr>
<th>B</th>
<th>Region</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( A_3 )</th>
<th>( \min Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>( \partial I )</td>
<td>0.034231</td>
<td>-0.044968</td>
<td>0.024882</td>
<td>-0.000038</td>
</tr>
<tr>
<td>0.1</td>
<td>( \partial I )</td>
<td>0.262374</td>
<td>0.170606</td>
<td>-0.133010</td>
<td>-0.005954</td>
</tr>
<tr>
<td>0.2</td>
<td>( \partial I )</td>
<td>0.489747</td>
<td>0.269737</td>
<td>-0.213725</td>
<td>-0.028234</td>
</tr>
<tr>
<td>0.27</td>
<td>( \partial I )</td>
<td>0.612783</td>
<td>0.274543</td>
<td>-0.222069</td>
<td>-0.037360</td>
</tr>
<tr>
<td>0.274376 = ( \beta_1 )</td>
<td>( \partial I )</td>
<td>0.619290</td>
<td>0.273003</td>
<td>-0.221185</td>
<td>-0.037395</td>
</tr>
<tr>
<td>0.28</td>
<td>( \partial I )</td>
<td>0.627436</td>
<td>0.270719</td>
<td>-0.219810</td>
<td>-0.037337</td>
</tr>
<tr>
<td>0.284717 = ( \gamma_1 )</td>
<td>( \partial I \cap P )</td>
<td>0.634079</td>
<td>0.268541</td>
<td>-0.218451</td>
<td>-0.037197</td>
</tr>
<tr>
<td>0.285</td>
<td>P</td>
<td>0.634319</td>
<td>0.267964</td>
<td>-0.218773</td>
<td>-0.037186</td>
</tr>
<tr>
<td>0.289393 = ( \gamma_2 )</td>
<td>P ( \cap \partial III )</td>
<td>0.637958</td>
<td>0.258988</td>
<td>-0.223541</td>
<td>-0.036934</td>
</tr>
<tr>
<td>0.29</td>
<td>( \partial III )</td>
<td>0.637558</td>
<td>0.258804</td>
<td>-0.223569</td>
<td>-0.036890</td>
</tr>
<tr>
<td>0.3</td>
<td>( \partial III )</td>
<td>0.630918</td>
<td>0.255757</td>
<td>-0.223967</td>
<td>-0.036140</td>
</tr>
<tr>
<td>0.4</td>
<td>( \partial III )</td>
<td>0.559821</td>
<td>0.224215</td>
<td>-0.221370</td>
<td>-0.027786</td>
</tr>
<tr>
<td>0.489950 = ( \delta )</td>
<td>( \partial III )</td>
<td>0.489238</td>
<td>0.194240</td>
<td>-0.209355</td>
<td>-0.019895</td>
</tr>
<tr>
<td>0.489958 = ( \delta )</td>
<td>I</td>
<td>0.308716</td>
<td>-0.325655</td>
<td>0.197891</td>
<td>-0.019895</td>
</tr>
<tr>
<td>0.5</td>
<td>I</td>
<td>0.314515</td>
<td>-0.327111</td>
<td>0.199710</td>
<td>-0.020547</td>
</tr>
<tr>
<td>0.554728 = ( \gamma_3 )</td>
<td>I ( \cap \Pi )</td>
<td>0.371011</td>
<td>-0.307904</td>
<td>0.207974</td>
<td>-0.023758</td>
</tr>
<tr>
<td>0.6</td>
<td>( \Pi )</td>
<td>0.414995</td>
<td>-0.290090</td>
<td>0.218305</td>
<td>-0.026281</td>
</tr>
<tr>
<td>0.66</td>
<td>( \Pi )</td>
<td>0.428346</td>
<td>-0.292403</td>
<td>0.223806</td>
<td>-0.028032</td>
</tr>
<tr>
<td>0.667947 = ( \beta_2 )</td>
<td>( \Pi )</td>
<td>0.428169</td>
<td>-0.292795</td>
<td>0.223822</td>
<td>-0.028060</td>
</tr>
<tr>
<td>0.67</td>
<td>( \Pi )</td>
<td>0.428053</td>
<td>-0.292886</td>
<td>0.223798</td>
<td>-0.028058</td>
</tr>
<tr>
<td>0.7</td>
<td>( \Pi )</td>
<td>0.423061</td>
<td>-0.293516</td>
<td>0.222059</td>
<td>-0.027574</td>
</tr>
<tr>
<td>0.790542 = ( \gamma_4 )</td>
<td>( \Pi \cap P )</td>
<td>0.369911</td>
<td>-0.278305</td>
<td>0.199329</td>
<td>-0.020521</td>
</tr>
<tr>
<td>0.8</td>
<td>P</td>
<td>0.355556</td>
<td>-0.272154</td>
<td>0.199590</td>
<td>-0.019313</td>
</tr>
<tr>
<td>0.9</td>
<td>P</td>
<td>0.189474</td>
<td>-0.169004</td>
<td>0.149698</td>
<td>-0.004794</td>
</tr>
<tr>
<td>0.99</td>
<td>P</td>
<td>0.019899</td>
<td>-0.019699</td>
<td>0.019500</td>
<td>-0.000008</td>
</tr>
</tbody>
</table>

There appears that the minimum may occur also on the upper boundary \( \partial III \) of \( (A_2, A_1) \);
\[ \partial \text{III} : \quad A_2 = 2A_1 - \frac{2 + B}{1 - B^2}A_1^2 \]

\[ \Downarrow \]

\[ Q = A_1A_2E \]

\[ = 6A_1^3 - 11 \frac{2+B}{1-B^2} A_1^4 + 2 \frac{1+3(2+B)^2}{(1-B^2)^2} A_1^5 - \frac{2+B}{(1-B^2)^3} \left[ 1 + (2 + B)^2 \right] A_1^6. \]

Thus, for the local extremal point on \( \partial \text{III} \) there holds

\[ 9(1 - B^2)^2 - 22(2 + B)(1 - B^2)A_1 \]

\[ \begin{align*}
&+ 5 \left[ 1 + 3(2 + B)^2 \right] A_1^2 - 3 \frac{(2 + B)(1 + (2 + B)^2)}{1 - B^2} A_1^3 = 0.
\end{align*} \]
In Table 2 there is a collection of minimal points. Some of them deserve to be mentioned separately.

\[
\min \min Q = -0.037395325; \quad B = \beta_1 = 0.274376470 \in \partial I,
\]
\[
\min \min Q = -0.028059590; \quad B = \beta_2 = 0.667947135 \in \Pi.
\]

The tip P assumes the role of minimizing point three times. Shifting from \(\partial I\) to P occurs at \(B = \gamma_1\). This point is found from (6) by aid of the limit process \(A_1 \to B_1\), i.e. at (6) we have to take \(A_1 = B_1\). Similarly, (11) with \(A_1 = B_1\) yields the shifting point \(B = \gamma_2\) from \(\partial \Pi\) to P. At \(B = \gamma_4\) we move from \(\Pi\) to P by aid of (10). Between \(\gamma_2\) and \(\gamma_4\) there exists still another shifting point \(\gamma_3\) of the type \(I \cap \Pi\). The results are:

\[
\begin{align*}
\gamma_1 &= 0.284716560 \in \partial I \cap P, \\
\gamma_2 &= 0.289392233 \in P \cap \partial \Pi, \\
\gamma_3 &= 0.554728151 \in I \cap \Pi, \\
\gamma_4 &= 0.790541920 \in \Pi \cap P.
\end{align*}
\]

Finally, at

\[
B = \delta = 0.489949658 \in \partial \Pi, \quad I
\]

there occur two simultaneous minima. We may speak about Twin Pits which, at the same time, happen to yield

\[
\max \min Q = -0.019894996; \quad B = \delta \in \partial \Pi, \quad I.
\]

The results of the Table 2 are visualized in Figure 2. In it the points of twin peaks and twin pits are pointed out by dotted circles.

**References**


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