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On the valency of a polynomial in $H\overline{H}$

To Professor Z. Lewandowski on his 70^{th} birthday

Abstract. In this note we discuss the valency of a function f which is the product of an analytic polynomial and the conjugate of another analytic polynomial.

1. Introduction. At the second international workshop on planar harmonic mappings at the Technion , Haifa, January 7-13, 2000, the following question was posed. Let f be the product of an analytic polynomial p_n of order n and the conjugate of an analytic polynomial q_m of order m. Determine the maximal valency of f. Such a function is termed a logharmonic polynomial. Under the mild assumption that $p_n = const\ q_m$, the cardinality of the zeros of $f - w$ is finite for all $w \in \mathbb{C}$ [1]. Observe that harmonic polynomials do not inherit this property as the following example shows.

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The quadratic harmonic polynomial $f(z) = z + \overline{z} + z^2 - \overline{z}^2$ maps the whole imaginary axis to the origin.

A function f that is the product of an entire analytic function and the conjugate of another entire analytic function is called a logharmonic mapping and it shall be denoted by $f \in H\overline{H}$. A function $f \in H\overline{H}$ that vanishes at a point z_0 is of the form

$$f(z) = (z - z_0)^n (\overline{z - z_0})^m h(z)\overline{g(z)} \tag{1}$$

where h and g are entire analytic functions non-vanishing in a neighbourhood of z_0.We shall then say that f has a zero of order (n_k, m_k) at z_0. The valency of f at z_0 is defined by $VZ(f, z_0) = |n - m|$ provided that $n \neq m$. It is obvious that this definition becomes senseless if $n = m > 0$. If f does not vanish at z_0 then f is said to have a zero of order (0,0). Before we can define the valency of f at an arbitrary point of $\mathbb{C}$, some further investigation on the behaviour of $f - w$ is necessary.

2. The valency of a polynomial in $H\overline{H}$. Let $f = p_n\overline{q}_m$ be a logharmonic polynomial in $H\overline{H}$. Then f is a solution of the non-linear system of elliptic partial differential equations

$$\overline{f_{\overline{z}}} = a\frac{\overline{f}}{f} f_z \tag{2}$$

where $a(z) = \frac{q_n'(z)p_n(z)}{q_n(z)p_n'(z)}$ is either a rational function or $a \equiv \infty$. In the latter case, p_n is a constant. Let D be a subdomain of $\mathbb{C}$. We define

$$\begin{aligned} S_L(D) &= \{z : |a(z)| < 1\} \\ S_E(D) &= \{z : |a(z)| = 1\} \\ S_G(D) &= \{z : |a(z)| > 1\}. \end{aligned}$$

If f is not a constant then f is an open mapping on $\mathbb{C} \setminus S_E(D)$.Hence, the similarity principle holds on $S_L(D)$ (respectively on $S_G(D)$). This is to say that on B, a subdomain of $S_L(D)$ (respectively of $S_G(D)$), the function f can be represented as

$$f(z) = A(\chi(z)) \tag{3}$$

where χ is a homeomorphism defined on G and A is an analytic function defined on $\chi(G)$. Thus, f behaves like an analytic function on $S(D) = S_L(D) \cup S_G(D)$. In particular, the zeros of $f - w$ form an isolated set in $S(D)$, and can be counted by the argument principle.This will be explored in the next section.

Definition 1. Let $f = p_n\overline{q}_m \in H\overline{H}$ and let D be a subdomain of $S_L(\mathbb{C})$ (respectively of $S_G(\mathbb{C})$). For $z_0 \in S(D) = S_L(\mathbb{C}) \cup S_G(\mathbb{C})$ we define:

(1) $NZ(f-w, D)$ is the cardinality of the zeros of $f-w$ in D;
(2) $VZ(f-w, z) = VZ(A-w, \chi(z))$ is the valency of $f-w$ at z if $f-w$ vanishes at z. In other words, this valency is the order of the zero of $A-w$ at $\chi(z)$;
(3) $VZ(f-w, D) = \sum_{z\in D} VZ(f-w, z)$ is the number of zeros of f multiplicity counted;
(4) $V(f, z_0) = VZ(f-f(z_0), z_0)$ is the valency of f at an arbitrary point of $S_L(\mathbb{C}) \cup S_G(\mathbb{C})$;
(5) $V(f, D) = \max_{w\in\mathbb{C}} VZ(f-w, D) = \max_{w\in\mathbb{C}} NZ(f-w, D)$;
(6) $V(f, \mathbb{C}) = \max_{w\in\mathbb{C}} NZ(f-w, \mathbb{C})$.

Remarks.

1. Item 3 of the above definition is compatible with the earlier definition of $VZ(f, z_0) = |n_0 - m_0|$ by using the representation (1).

2. If $VZ(f-w_0, z_0) = k$, then there is a $\delta-$ neighborhood of z_0 and an ϵ -neighbourhood of w_0 such that $NZ(f-w, |z-z_0| < \delta) = k$ for all $0 < |w-w_0| < \epsilon$.

The following argument principle for polynomials in $H\overline{H}$ is shown in [1]

Theorem A. *Let $f = p_n\overline{q}_m$ be a polynomial in $H\overline{H}$ and $w \in \mathbb{C}$ be fixed. Assume that $f-w$ does not vanish on the set $S_E(\mathbb{C})$. For $n > m$, we have*

$$VZ(f-w, S_L(\mathbb{C})) - VZ(f-w, S_G(\mathbb{C})) = n-m. \tag{4}$$

As we have mentioned in the introduction, the cardinality of the zeros of $f-w$ in $\mathbb{C}$ is finite for all $w \in \mathbb{C}$, unless $p_n = const.q_n$. Applying Bezout's Theorem on the common zeros of two real-valued polynomials of two real variables, we conclude that $(n+m)^2$ is an upper bound for the valency $V(f, \mathbb{C})$. We think that this upper bound is too large. Indeed, the maximal valency of $A(z-c)\overline{(z-d)}, c \neq d$, is 2 (see the next section). The examples in section 3 give the impression that the best upper bound for $VZ(f, \mathbb{C})$ is $(n+m)$. However, we show in section 4 that this value has to be much larger.

3. The cases of small and large values of *w*. We first start with polynomials that vanish at a single point. In this case

$$f(z) = const.t(z-p)^n(\overline{z-p})^m.$$

If $n \neq m$, then we have $VZ(f-w, \mathbb{C}) = |n-m|$ for all $w \in \mathbb{C}$. Hence $V(f, \mathbb{C}) = |n-m| \leq n+m$.

Consider now the case $n = 1, m = 1$ and $p_1 \neq const.q_1$. Then, without loss of generality, we may assume that $f(z) = \overline{z}(z-b), b > 0$. The equation $x^2 + y^2 - bx + iby = u + iv$ implies $y = v/b$ and the equation $x^2 + bx = u - (v/b)^2$ has at most two solutions for x if u and v are given . Hence, $V(f, \mathbb{C}) \leq 2 = n + m$.

Next, we consider the case of small values of w. Suppose that f vanishes at the points $z_k, 1 \leq k \leq N$, with order $(n_k, m_k), n_k \neq m_k$. Then $n_k > m_k$ implies that $a(z_k) = 0$ and $n_k < m_k$ implies that $a(z_k) = \infty$. Consider the disks $\Delta_k = \{z : |z - z_k| \leq \delta\}$, such that $\Delta_k \cap \Delta_j = \phi$ if $k \neq j$ and $\Delta_k \cap S_E(\mathbb{C}) = \phi$ for all $k, 1 \leq k \leq N$. Furthermore, choose δ so small such that $|f_z| + |f_{\overline{z}}|$ does not vanish on $\Delta_k \setminus \{z_k\}$. It then follows that f is locally univalent and $|n_k - m_k|-$valent on $\Delta_k \setminus \{z_k\}$. Define $M = \inf\{t = |f(z)| : z \in \mathbb{C} \setminus \cup_{k=1}^{N} \Delta_k\}$. Then $M > 0$ and we conclude that $VZ(f - w, \mathbb{C}) = \sum_{k=1}^{N} |n_k - m_k| \leq n + m$ for all $w, |w| < M$.

Now, we consider the case of large values of w. Let $f = p_n\overline{q}_m$ be a polynomial in $H\overline{H}$ and assume that $n \neq m$. With no loss of generality, we may assume that $n > m$. Then we have $a(\infty) = m/n < 1$ and there is an $R > 0$ such that $\Delta_R = \{z : |z| \geq R\}$ is contained in $S_L(\mathbb{C})$. Define $M = \sup\{t = |f(z)| : z \in \mathbb{C} \setminus \Delta_R\}$. Fix w with $|w| > M$. Next, choose $R_1 > R$ such that $M_1 = \inf\{t = |f(z)| : |z| = R_1\} > |w|$. We now apply the classical argument principle and get

$$\begin{aligned} NZ(f - w, \mathbb{C}) &= NZ(f - w, R < |z| < R_1) \leq VZ(f - w, R < |z| < R_1) \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\ \arg[f(R_1 e^{it}) - w] - \frac{1}{2\pi} \int_0^{2\pi} d\ \arg[f(Re^{it}) - w] \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\ \arg[f(R_1 e^{it}) - w] = n - m \leq n + m. \end{aligned}$$

4. The case $m = 1$. We first start with the example $f(z) = \overline{z}(z-2)(z-4)$. Then $a(z) = \frac{(z-2)(z-4)}{z(2z-6)}$ and $a'(z) = \frac{(z^2+8)(6)-32z}{z^2(2z-6)^2}$ which is positive on the negative real axes. We have $f(3) = -3, a(3) = \infty, f(-1) = -15$ and $a(-1) = \frac{15}{8}$. Hence, there is a z_1 on the interval $(-1, 0)$ such that $f(z_1) = -3$ and $a(z_1) > 1$. In other words, we have $VZ(f + 3, S_G(|z| \leq 10)) \geq 2$. We now apply the generalized argument principle stated in Theorem A and we get for sufficiently large R

$$VZ(f + 3, S_L(|z| \leq R)) - VZ(f + 3, S_G(|z| \leq R)) = n - m = 1$$

which implies $VZ(f + 3, S_L(|z| \leq R)) \geq 3$ and hence,

$$V(f, \mathbb{C}) \geq VZ(f + 3, S_L(|z| \leq R)) + VZ(f + 3, S_G(|z| \leq R)) = 5.$$

This eliminates the conjecture that the maximum valency is at most $n+m=3$.

We conjecture that in the case of $m=1$ the maximal valency is $3n-1$. In the following example we show that it is at least $3n-3$. Consider the polynomial $f(z)=|z|^2(\frac{z^{n-1}}{n}-1)=\overline{z}p_n(z)$. Then we have $p'_n(z)=z^{n-1}-1$ and $a(z)=\frac{p_n(z)}{zp'_n(z)}$. At the points $z_k=e^{2\pi ik/(n-1)}, 1\leq k\leq n-1$, we have $a(z_k)=\infty$ and $f(z_k)=-\frac{n-1}{n}$. Hence $VZ(f+\frac{n-1}{n},S_G(|z|<2))\geq n-1$. We now apply the generalized argument principle and get for sufficiently large R

$$VZ(f+\frac{n-1}{n},S_L(|z|\leq R))-VZ(f+\frac{n-1}{n},S_G(|z|\leq R))=n-1.$$

This implies $VZ(f+\frac{n-1}{n},S_L(|z|\leq R))\geq 2n-2$ and hence,

$$\begin{aligned}V(f,\mathbb{C})&\geq VZ(f+\frac{n-1}{n},S_L(|z|\leq R))+VZ(f+\frac{n-1}{n},S_G(|z|\leq R))\\&\geq 3n-3.\end{aligned}$$

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