ANNALES
UNIVERSITATIS MARIAE CURIE – SKŁODOWSKA
LUBLIN – POLONIA

VOL. LIV, 6 SECTIO A 2000

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Regularity theorems for linearly invariant families of holomorphic mappings in $\mathbb{C}^n$

Dedicated to Professor Z. Lewandowski on his 70th birthday

Abstract. The authors give a theorem concerning results which state that the mapping having the highest rate of growth of the Jacobian, in a linearly invariant family of locally biholomorphic mappings, have this growth regular.

1. Introduction. Regularity theorems are well known in different families of holomorphic functions of one variable; see e.g. [BIE], [BAZ], [CAM], [HAY], [KRZ], [LEB], [MIL], [ST1], [ST2]. For example, in the class $\mathcal{S}$ of normalized univalent functions in the open unit disc Δ a regularity theorem is as follows:

Theorem 1 ([HAY], [KRZ]). *For every continuous function $g : \Delta \longrightarrow \mathbb{C}$ and $r \in [0,1)$, put $M(r,g) = \max_{|\zeta|=r} |g(\zeta)|$. If $f \in \mathcal{S}$, then there exist the limits*

$$\lim_{r\to 1^-} \frac{(1-r)^2}{r} M(r,f), \quad \lim_{r\to 1^-} \frac{(1-r)^3}{1+r} M(r,f');$$

1991 *Mathematics Subject Classification.* Primary 32H02; Secondary 30C55.
Key words and phrases. locally biholomorphic mappings, linearly invariant families, regularity theorems.

they both equal the same number $\delta_f = \delta \in [0,1]$ *and* $\delta = 1$ *only for the Koebe function* $K_\eta(\zeta) = \zeta(1-\zeta e^{-i\eta})^{-2}$. *Moreover, if* $f \in \mathcal{S}$ *and* $\delta \neq 1$, *then functions* $\frac{(1-r)^2}{r}M(r,f)$, $\frac{(1-r)^3}{1+r}M(r,f')$ *decrease on the interval* $[0,1)$, *but if* $f \in \mathcal{S}$ *and* $\delta \neq 0$, *then for every* $\theta \in [0,2\pi)$ *functions* $\frac{(1-r)^2}{r}\left|f(re^{i\theta})\right|$, $\frac{(1-r)^3}{1+r}\left|f'(re^{i\theta})\right|$ *do not increase and there exists a unique number* $\theta_f \in [0,2\pi)$ *such that*

$$\lim_{r\to 1^-}\frac{(1-r)^2}{r}\left|f(re^{i\theta})\right| = \lim_{r\to 1^-}\frac{(1-r)^3}{1+r}\left|f'(re^{i\theta})\right| = \begin{cases} \delta \text{ for } \theta = \theta_f \\ 0 \text{ for } \theta \neq \theta_f \end{cases}.$$

Similar regularity theorems for any linearly invariant families of finite order (of locally univalent functions in the unit disc Δ) have been given in papers [CAM] and [ST1], [ST2].

In this paper we will consider the case of holomorphic mappings in $\mathbb{C}^n$.

2. Preliminaries. Let us denote by B^n the unit ball $\{z = (z_1, ..., z_n) \in \mathbb{C}^n : \langle z,z\rangle^{\frac{1}{2}} < 1\}$, where $\langle\cdot,\cdot\rangle$ is the euclidean inner product; for $r > 0$ let $B^n_r := rB^n$. Let $\mathcal{A}$ be the set of all biholomorphic automorphisms of the ball B^n. If $D^k f(z)$ is the k-th Fréchet differential of the mapping f at the point z, then $J_f(z) := \det Df(z)$, but $D^2 f(z)(w,\cdot)$ is a linear bounded operator from $\mathbb{C}^n$ into itself, which is obtained by the restriction of the symmetrical bilinear operator $D^2 f(z)$ to $w \times \mathbb{C}^n$. Let $\mathcal{LS}^n$ stand for the family of all holomorphic mappings $f : B^n \longrightarrow \mathbb{C}^n$ normalized by the conditions

$$J_f(z) \neq 0,\ Df(0) = I,\ f(0) = 0.$$

For every $\varphi \in \mathcal{A}$ we will consider an operator Λ_φ defined on the set $\mathcal{LS}^n$ as follows:

$$\Lambda_\varphi(f)(z) = (D\varphi(0))^{-1}(Df(\varphi(0)))^{-1}(f(\varphi(z)) - f(\varphi(0))), \quad z \in B^n.$$

A family $\mathfrak{M} \subset \mathcal{LS}^n$ is called linearly invariant family if for every $f \in \mathfrak{M}$ and every $\varphi \in \mathcal{A}$ the mapping $\Lambda_\varphi(f)$ also belongs to $\mathfrak{M}$; (usually, we will write $\mathfrak{M} \in \mathcal{LIF}$). The quantity

$$\operatorname{ord}\mathfrak{M} = \frac{1}{2}\sup_{f\in\mathfrak{M}}\max_{\|w\|=1}\left|\operatorname{tr} D^2 f(0)(w,\cdot)\right|$$

is called the order of a family $\mathfrak{M} \in \mathcal{LIF}$. This definition of the order of a family $\mathfrak{M} \in \mathcal{LIF}$ comes from J.A. Pfaltzgraff (see [PFA]), but a similar idea has been presented in [BFG] by R.W. Barnard, C.H. FitzGerald and S. Gong.

In this paper we will only consider the case when ord $\mathfrak{M} < \infty$.

In [PFA] it is shown that if ord $\mathfrak{M} = \alpha$ for a family $\mathfrak{M} \in \mathcal{LIF}$, then $\alpha \geq \frac{n+1}{2}$ and the following inequality holds for $f \in \mathfrak{M}$

$$(2.1) \qquad \frac{(1-\|z\|)^{\alpha-\frac{n+1}{2}}}{(1+\|z\|)^{\alpha+\frac{n+1}{2}}} \leq |J_f(z)| \leq \frac{(1+\|z\|)^{\alpha-\frac{n+1}{2}}}{(1-\|z\|)^{\alpha+\frac{n+1}{2}}}\,, \quad z \in B^n.$$

A complete proof of the sharpness of estimates (2.1) is given in our paper [LST].

For $n = 2$ the above result was obtained by R.W. Barnard, C.H. FitzGerald and S. Gong in [BFG], but under the additional assumption that all mappings $f \in \mathfrak{M}$ are biholomorphic.

Let $f \in \mathcal{LS}^n$; the order of the family $\mathfrak{M}_f := \{\Lambda_\varphi(f) : \varphi \in \mathcal{A}\}$ belonging to $\mathcal{LIF}$ will be called the order of the mapping f. In [GLS] it was shown that the number ord f determines the rate of growth of $\left|J_{\Lambda_\varphi(f)}(z)\right|$. To be more precise, ord f is the infimum of all numbers α such that for every $\varphi \in \mathcal{A}$ and $z \in B^n$ holds the following estimate

$$(2.2) \qquad \left|J_{\Lambda_\varphi(f)}(z)\right| \leq \frac{(1+\|z\|)^{\alpha-\frac{n+1}{2}}}{(1-\|z\|)^{\alpha+\frac{n+1}{2}}}\,.$$

We will use the following universal linearly invariant family

$$\mathfrak{U}_\alpha := \bigcup \{\mathfrak{M} \in \mathcal{LIF} \ : \text{ord } \mathfrak{M} \leq \alpha\}\,.$$

3. Regularity theorem. For every continuous function $g : B^n \longrightarrow \mathbb{C}$ and $r \in [0,1)$ put, similarly as above,

$$M(r,g) = \max_{\|z\|=r} |g(z)|\,.$$

Theorem 2. *If $f \in \mathfrak{U}_\alpha$, then:*

(i) $M(r,J_f)\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}}$ *is a non-increasing function on the interval* $[0,1)$ *and for every $v \in \partial B^n$* $\left|J_f(rv)\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}}\right|$ *is also non-increasing on* $[0,1)$.

(ii) *There exists a vector $v_0 = v_0(f) \in \partial B^n$ and a number $\delta_0 = \delta_0(f) \in [0,1]$ such that*

$$(3.1) \qquad \lim_{r\to 1^-} M(r,J_f)\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}} = \delta_0 = \lim_{r\to 1^-} |J_f(rv_0)|\,\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}}\,,$$

$$\begin{aligned} &\lim_{r\to 1^-} M(r, \frac{d}{dr}J_f(rv))\frac{(1-r)^{\alpha+\frac{n+3}{2}}}{((n+1)r+2\alpha)(1+r)^{\alpha-\frac{n+3}{2}}} = \delta_0 \\ &\limsup_{r\to 1^-} \left|\frac{d}{dr}J_f\,(rv_0)\right| \frac{(1-r)^{\alpha+\frac{n+3}{2}}}{((n+1)r+2\alpha)(1+r)^{\alpha-\frac{n+3}{2}}} = \delta_0 \end{aligned}, \tag{3.2}$$

$$\begin{aligned} &\lim_{r\to 1^-} \int_0^r M(\rho,\ \frac{d}{d\rho}J_f(\rho v))d\rho\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}} = \delta_0 \\ &\lim_{r\to 1^-} \int_0^r \left|\frac{d}{d\rho}J_f(\rho v_o))\right| d\rho\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}} = \delta_0 \end{aligned}. \tag{3.3}$$

The vector $v_0 = v_0(f) \in \partial B^n$ will be called the direction of the maximal growth of the mapping $f \in \mathfrak{U}_\alpha$.

(iii) *If in part* (ii) $v_0 = (1, 0, ..., 0)$, *then* $\delta_0 = \delta_0(f) = 1$ *if and only if*

$$J_f\,(z_1v_0) = \frac{(1+z_1)^{\alpha-\frac{n+1}{2}}}{(1-z_1)^{\alpha+\frac{n+1}{2}}} := F(z_1), \quad z_1 \in \Delta. \tag{3.4}$$

However, if $n > 1$, then there exist infinitely many mappings $f \in \mathfrak{U}_\alpha$, for which relation (3.4) *is fulfilled.*

Proof. For an arbitrarily fixed point $a \in B^n$, let $s = \sqrt{1-\|a\|^2}$ and for $z \in B^n$

$$P_a(z) = \begin{cases} a\frac{\langle z,a\rangle}{\|a\|^2} & \text{for } a \neq 0 \\ 0 & \text{for } a = 0 \end{cases},\ \varphi_a(z) = \frac{a - sz + (s-1)P_a(z)}{1-\langle z, a\rangle}.$$

Then, (see [RUD]):

$$\varphi_a \in \mathcal{A},\ \varphi_a(0) = 0,\ D\varphi_a(0) = -s(I + (s-1)P_a),\ D\varphi_a(0)(a) = -s^2a,$$

$$|J_{\varphi_a}(z)| = \left(\frac{s^2}{|1-\langle z,a\rangle|^2}\right)^{\frac{n+1}{2}}, \quad |J_{\varphi_a}(0)| = s^{n+1}, \quad |J_{\varphi_a}(a)| = s^{-(n+1)}.$$

Let us fix $f \in \mathfrak{U}_\alpha$ and $v \in \partial B^n$. Then, using the above properties of the mappings φ_a, for every $t \in [0, 2\pi)$ and every $a \in B^n - \{0\}$ such that $\frac{a}{\|a\|} = v$, we obtain the following relations

$$\frac{d}{d\rho}J_f(\varphi_a(\rho e^{it}v))_{|\rho=0} =$$

$$DJ_f(a)D\varphi_a(0)(e^{it}v) = DJ_f(a)D\varphi_a(0)(a)(\frac{e^{it}}{\|a\|}) = DJ_f(a)(a)(-s^2\frac{e^{it}}{\|a\|}).$$

Thus,

$$\frac{d}{d\rho} J_f(\varphi_a(\rho e^{it} v))_{|\rho=0} = -s^2 e^{it} DJ_f(a)(v). \tag{3.5}$$

On the other hand, ord $\mathfrak{U}_\alpha = \alpha$ and for $\varphi \in \mathcal{A}$

$$J_{\Lambda_\varphi(f)}(z) \;=\; \frac{J_f(\varphi(z))J_\varphi(z)}{J_f(\varphi(0))J_\varphi(0)},$$

so putting in (2.2) $z = \rho e^{it} v = \rho e^{it} \frac{a}{\|a\|}$ and $\varphi = \varphi_a$ we have

$$\log \left| \frac{J_f(\varphi_a(\rho e^{it} v)) J_{\varphi_a}(\rho e^{it} v)}{J_f(a) J_{\varphi_a}(0)} \right| \le \log \frac{(1+\rho)^{\alpha - \frac{n+1}{2}}}{(1-\rho)^{\alpha + \frac{n+1}{2}}}.$$

This inequality remains true also after differentiation with respect to ρ at the point $\rho = 0$. Therefore, using elementary calculations and the properties of the mapping φ_a, we obtain for every $t \in [0, 2\pi)$

$$\Re \left[e^{it} \left(\frac{-s^2 DJ_f(a)(v)}{J_f(a)} + \|a\| \, (n+1) \right) \right] \le 2\alpha. \tag{3.6}$$

We will prove now the claim (i) of our theorem.

Let $t = \pi$ and let $r := \|a\|$ vary in the interval $[0, 1)$. Then the above inequality can be rewritten in the following equivalent form

$$\Re \frac{\frac{d}{dr} J_f(rv)}{J_f(rv)} - \frac{(n+1)r + 2\alpha}{1 - r^2} \le 0.$$

Since the left side of this inequality is the derivative of the function

$$\log |J_f(rv)| - \int_0^r \frac{(n+1)\rho + 2\alpha}{1 - \rho^2} d\rho = \log \left(|J_f(rv)| \, \frac{(1-r)^{\alpha + \frac{n+1}{2}}}{(1+r)^{\alpha - \frac{n+1}{2}}} \right),$$

with respect to r, $\log \left(|J_f(rv)| \, \frac{(1-r)^{\alpha + \frac{n+1}{2}}}{(1+r)^{\alpha - \frac{n+1}{2}}} \right)$ is a non-increasing function of the variable $r \in [0, 1)$. This gives the second part of claim (i).

Now let $r_1, r_2 \in [0, 1)$ be fixed but arbitrary numbers such that $r_1 < r_2$. Since $\partial(r_2 B^n)$ is a compact set, there exists a point $v_2 \in \partial B^n$ such that $M(r_2, J_f) = |J_f(r_2 v_2)|$. Using the second part of (i) (proved above), we have

$$\begin{aligned} M(r_1, J_f) \frac{(1-r_1)^{\alpha + \frac{n+1}{2}}}{(1+r_1)^{\alpha - \frac{n+1}{2}}} &\ge |J_f(r_1 v_2)| \, \frac{(1-r_1)^{\alpha + \frac{n+1}{2}}}{(1+r_1)^{\alpha - \frac{n+1}{2}}} \\ \ge |J_f(r_2 v_2)| \, \frac{(1-r_2)^{\alpha + \frac{n+1}{2}}}{(1+r_2)^{\alpha - \frac{n+1}{2}}} &= M(r_2, J_f) \frac{(1-r_2)^{\alpha + \frac{n+1}{2}}}{(1+r_2)^{\alpha - \frac{n+1}{2}}}. \end{aligned}$$

Hence

$$M(r_1, J_f)\frac{(1-r_1)^{\alpha+\frac{n+1}{2}}}{(1+r_1)^{\alpha-\frac{n+1}{2}}} \geq M(r_2, J_f)\frac{(1-r_2)^{\alpha+\frac{n+1}{2}}}{(1+r_2)^{\alpha-\frac{n+1}{2}}}.$$

This proves the first part of claim (i).

Now we will prove claim (ii) of our theorem.

We start with the proof of equality (3.1).

Part (i), (proved above), implies that there exist both limits in (3.1). If we denote the first limit by δ_0 and the second limit by δ_1, then $\delta_0, \delta_1 \in [0,1]$, because $M(0, J_f) = |J_f(0)| = 1$. It is sufficient to prove that $\delta_0 = \delta_1$ for some $v_0 \in \partial B^n$. For every $r \in [0,1)$ the function $|J_f(z)|$ is continuous on the compact set $\partial(rB^n)$, so there exists a point $v(r) \in \partial(B^n)$ such that $M(r, J_f) = |J_f(rv(r))|$. Let (r_ν) be an increasing sequence of numbers $r_\nu \in [0,1)$, convergent to 1 and such that the corresponding sequence (v_ν) of points $v(r_\nu) \in \partial(B^n)$ tends to a point $v_0 \in \partial B^n$ if ν tends to infinity. Let $r \in [0,1)$ be fixed but arbitrary. Then $r \in [0, r_\nu)$ for sufficiently large ν, so by the definition of $v(r)$ and by part (i)

$$\begin{aligned} M(r, J_f)\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}} &\geq |J_f(rv_\nu)|\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}} \\ \geq |J_f(r_\nu v_\nu)|\frac{(1-r_\nu)^{\alpha+\frac{n+1}{2}}}{(1+r_\nu)^{\alpha-\frac{n+1}{2}}} &= M(r_\nu, J_f)\frac{(1-r_\nu)^{\alpha+\frac{n+1}{2}}}{(1+r_\nu)^{\alpha-\frac{n+1}{2}}}. \end{aligned}$$

If $\nu \to \infty$, then from the above, in view of continuity of $|J_f|$ and in view of the definition of δ_0, we have

$$M(r, J_f)\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}} \geq |J_f(rv_0)|\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}} \geq \delta_0.$$

If $r \to 1^-$, then using the definition of numbers δ_0, δ_1, we obtain $\delta_0 \geq \delta_1 \geq \delta_0$. This proves the announced equality $\delta_0 = \delta_1$.

Now we will prove equalities (3.2).

Since t is arbitrary, (3.6) implies

$$s^2\left|\frac{DJ_f(a)(v)}{J_f(a)}\right| - \|a\|(n+1) \leq 2\alpha.$$

Thus, after introducing the variable $r := \|a\|$, ranging over the interval $[0,1)$, we have

$$\left|\frac{\frac{d}{dr}J_f(rv)}{J_f(rv)}\right| \leq \frac{(n+1)r + 2\alpha}{1-r^2}.$$

Therefore, by (2.1) we obtain

$$\tag{3.7} \begin{aligned} \left|\frac{d}{dr}J_f(rv)\right| &\le ((n+1)r+2\alpha)(1-r^2)\,|J_f(rv)| \\ &\le ((n+1)r+2\alpha)\frac{(1+r)^{\alpha-\frac{n+3}{2}}}{(1-r)^{\alpha+\frac{n+3}{2}}}. \end{aligned}$$

This implies the existence of a finite upper limit in (3.2) which is denoted by δ_2. We will show that $\delta_2=\delta_0$. From the definition of the number δ_2 it follows that for every $\varepsilon>0$ there exists a number $r_0\in[0,1)$ such that

$$\left|\frac{d}{dr}J_f(rv_0)\right| \le (\delta_2+\varepsilon)((n+1)r+2\alpha)(1-r^2)\frac{(1+r)^{\alpha-\frac{n+3}{2}}}{(1-r)^{\alpha+\frac{n+3}{2}}}$$

for every $r\in[r_0,1)$ and $v_0\in\partial B^n$. From this we obtain

$$\begin{aligned} |J_f(rv_0)|-|J_f(r_0v_0)| &= [\exp(\Re\log J_f(\rho v_0))]_{\rho=r_0}^{\rho=r} \\ &= \int_{r_0}^{r}|J_f(\rho v_0)|\,\Re\frac{\frac{d}{d\rho}J_f(\rho v)}{J_f(\rho v)}d\rho \le \int_{r_0}^{r}\left|\frac{d}{d\rho}J_f(\rho v_0)\right|d\rho \\ &\le (\delta_2+\varepsilon)\int_{r_0}^{r}((n+1)\rho+2\alpha)\frac{(1+\rho)^{\alpha-\frac{n+3}{2}}}{(1-\rho)^{\alpha+\frac{n+3}{2}}}d\rho \\ &= (\delta_2+\varepsilon)\left[\frac{(1+r)^{\alpha-\frac{n+1}{2}}}{(1-r)^{\alpha+\frac{n+1}{2}}}-\frac{(1+r_0)^{\alpha-\frac{n+1}{2}}}{(1-r_0)^{\alpha+\frac{n+1}{2}}}\right]. \end{aligned}$$

Thus

$$\tag{3.8} |J_f(rv_0)|-|J_f(r_0v_0)| \le (\delta_2+\varepsilon)\left[\frac{(1+r)^{\alpha-\frac{n+1}{2}}}{(1-r)^{\alpha+\frac{n+1}{2}}}-\frac{(1+r_0)^{\alpha-\frac{n+1}{2}}}{(1-r_0)^{\alpha+\frac{n+1}{2}}}\right].$$

Multiplying both sides of this inequality by $\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}}$, we obtain as $r\to 1^-$,

$$\lim_{r\to1^-}|J_f\,(rv_0)|\,\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}} \le \delta_2+\varepsilon,$$

which, in view of the definition of δ_0, gives $\delta_0\le\delta_2$. From inequality (3.7) it also follows that for $v_0\in\partial B^n$ and $r\in[0,1)$

$$\tag{3.9} \left|\frac{d}{dr}J_f(rv_0)\right|\frac{(1-r)^{\alpha+\frac{n+3}{2}}}{((n+1)r+2\alpha)(1+r)^{\alpha-\frac{n+3}{2}}} \le |J_f\,(rv_0)|\,\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}}.$$

From this by the definition of δ_0, δ_2, we deduce that $\delta_2 \le \delta_0$. Hence $\delta_2 = \delta_0$.

Similarly, we show that $\delta_3 = \delta_0$, where

$$\delta_3 = \lim_{r\to 1^-} M(r, \frac{d}{dr} J_f(rv)) \frac{(1-r)^{\alpha+\frac{n+3}{2}}}{((n+1)r+2\alpha)(1+r)^{\alpha-\frac{n+3}{2}}}.$$

It remains to show that the first limit appearing in (3.2) does exist, but we will do it latter.

Now, we will prove equalities (3.3).

First, observe that we can replace the integrals

$$\int_0^r M(\rho,\ \frac{d}{d\rho} J_f(\rho v)) d\rho \frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}}\ , \quad \int_0^r \left|\frac{d}{d\rho} J_f(\rho v_o)\right| d\rho \frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}}$$

by the integrals

$$\int_{r_0}^r M(\rho,\ \frac{d}{d\rho} J_f(\rho v)) d\rho \frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}}\ , \quad \int_{r_0}^r \left|\frac{d}{d\rho} J_f(\rho v_o)\right| d\rho \frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}},$$

with an $r_0 \in [0, 1)$. This follows directly from the additivity of the integral and the fact that $\lim_{r\to 1^-} (1-r)^{\alpha+\frac{n+1}{2}} = 0$.

We now start with the proof of the first equality in (3.3). From (3.8) it follows that for every $\varepsilon > 0$ there exists a number $r_0 \in [0, 1)$ such that for $r \in [r_0, 1)$

$$|J_f(rv_0)| - |J_f(r_0 v_0)| \le p(r) \le (\delta_2 + \varepsilon) \left[\frac{(1+r)^{\alpha-\frac{n+1}{2}}}{(1-r)^{\alpha+\frac{n+1}{2}}} - \frac{(1+r_0)^{\alpha-\frac{n+1}{2}}}{(1-r_0)^{\alpha+\frac{n+1}{2}}} \right],$$

with

$$p(r) = \int_{r_0}^r M(\rho,\ \frac{d}{d\rho} J_f(\rho v)) d\rho. \tag{3.10}$$

Multiplying both sides of the last inequality by $\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}}$, we obtain, as $r \to 1^-$,

$$\delta_0 \le \liminf_{r\to 1^-} p(r) \frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}} \le \limsup_{r\to 1^-} p(r) \frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}} \le \delta_0 + \varepsilon.$$

Thus

$$\lim_{r\to 1^-} p(r) \frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}} = \delta_0. \tag{3.11}$$

Now, we will prove the second equality in (3.3). From (3.8) it follows that

$$|J_f(rv_0)| - |J_f(r_0v_0)| \leq \int_{r_0}^{r} \left| \frac{d}{d\rho} J_f(\rho v_o)) \right| d\rho \leq p(r).$$

Multiplying both sides of the last inequality by $\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}}$, we have, as $r \to 1^-$,

$$\delta_0 \leq \lim_{r\to 1^-} \int_{r_0}^{r} \left| \frac{d}{d\rho} J_f(\rho v_o)) \right| d\rho \frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}} \;\leq\; \delta_0.$$

Thus

$$\lim_{r\to 1^-} \int_{r_0}^{r} \left| \frac{d}{d\rho} J_f(\rho v_o)) \right| d\rho \frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}} \;=\; \delta_0$$

Now we will show the existence of the first limit appearing in (3.2).

To this end we use the following two results:

Lemma 1 ([HAR, Thm. 112]). *Let p be a differentiable function of the variable $r \in [0,1)$ such that $p'(r)$ does not decrease. If for a positive real number $\beta > 0$, $\lim_{r\to 1^-} p(r)(1-r)^{\beta} = \gamma > 0$, then $\lim_{r\to 1^-} p'(r)(1-r)^{\beta+1} = \beta\gamma$.*

Lemma 2 ([CHA]). *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and:*

(i) *$h = (h_1, ..., h_n) : \Omega \to \mathbb{C}^n$ is a holomorphic mapping in Ω and continuous on $\overline{\Omega}$, having no zeros on $\partial\Omega$, whereas in Ω it has only isolated zeros of order k in the following sense: $h(a) = 0$, the functions h_j, $j = 1, ..., n$ expand in some neighborhood $\|z - a\| < r$ in a series of homogeneous polynomials $\sum_{l=k}^{\infty} Q_{jl}(z-a)$ and the system of equations $Q_{jk}(w) = 0$, $j = 1, ..., n$ has only the trivial solution,*

(ii) *$g : \Omega \to \mathbb{C}$ is a holomorphic function in Ω, continuous on $\overline{\Omega}$, such that if a is an isolated zero of order k of the mapping h, then the function g has a zero of order no less than k at a.*

Then the function

$$p(z) = \limsup_{\overline{\Omega}\ni w\to z} \frac{|g(w)|}{\|h(w)\|}, \quad z \in \overline{\Omega},$$

satisfies the maximum principle in Ω in the following sense

$$\sup_{z\in\overline{\Omega}} p(z) = \sup_{z\in\partial\Omega} p(z).$$

Now, observe that Lemma 2, (Chądzyński's maximum principle), gives the equality

$$\max_{\|z\|\le r}\frac{|DJ_f(z)(z)|}{\|z\|}=\max_{\|z\|=r}\frac{|DJ_f(z)(z)|}{\|z\|}.$$

We conclude from this that $M(r,\frac{d}{dr}J_f(rv))$ is a non-decreasing function of the variable $r\in[0,1)$, because

$$M(r,\frac{d}{dr}J_f(rv))=\max_{\|z\|=r}\frac{|DJ_f(z)(z)|}{\|z\|}.$$

This property of $M(r,\frac{d}{dr}J_f(rv))$ shows that the function p, defined in (3.10), is differentiable, $p'(r)=M(r,\frac{d}{dr}J_f(rv))$ for $r\in[0,1)$ and p' does not decrease. We can now apply Lemma 1. Then, from (3.11) we obtain

$$\lim_{r\to1^-}M(r,\frac{d}{dr}J_f(rv))\frac{(1-r)^{\alpha+\frac{n+3}{2}}}{((n+1)r+2\alpha)(1+r)^{\alpha-\frac{n+3}{2}}}$$
$$=\lim_{r\to1^-}p'(r)(1-r)^{\alpha+\frac{n+3}{2}}\frac{1}{((n+1)r+2\alpha)(1+r)^{\alpha-\frac{n+3}{2}}}=\delta_0.$$

This completes the proof of part (ii) of our theorem.

We will now prove part (iii).

If f belongs to $\mathfrak{U}_\alpha$ and satisfies condition (3.6), then $\delta_0=1$, because

$$\lim_{r\to1^-}|J_f\,(rv_0)|\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}}=1.$$

Let us now assume that for a mapping $f^*\in\mathfrak{U}_\alpha$ we have $\delta_0(f^*)=1$ and $v_0=(1,0,...,0)$, that is

$$\lim_{r\to1^-}|J_{f^*}\,(rv_0)|\frac{(1-r)^{\alpha+\frac{n+1}{2}}}{(1+r)^{\alpha-\frac{n+1}{2}}}=1.$$

Then, from part (i) of claim it follows that

$$|J_{f^*}\,(rv_0)|=\frac{(1+r)^{\alpha+\frac{n+1}{2}}}{(1-r)^{\alpha-\frac{n+1}{2}}}=F(r), \tag{3.12}$$

because $J_{f^*}\,(0)=1$. Let us denote $J_{f^*}\,(z_1v_0)=F(z_1)e^{i\psi(z_1)}$, where $\psi(z_1)$ is a function holomorphic in the unit disc Δ. From (3.12) it follows that the values of ψ are real for $z_1 = r \in [0,1)$. Let $a_1 \in \Delta$, $a = a_1 v_0$ and $g^*(z) = \Lambda_{\varphi_a}(f^*)(z)$. Then, $\varphi_a(z) = \frac{a_1 - z_1}{1-\overline{a_1}z_1} v_0$ and

$$|J_{g^*}(z_1 v_0)| = \frac{\left|J_{f^*}\left(\frac{a_1-z_1}{1-\overline{a_1}z_1} v_0\right)\right|}{|J_{f^*}(a_1 v_0)|\,|1-\langle z_1 v_0, a\rangle|^{n+1}}$$

$$= \frac{\left|F\left(\frac{a_1-z_1}{1-\overline{a_1}z_1}\right)\exp[i\psi\left(\frac{a_1-z_1}{1-\overline{a_1}z_1}\right)\right|}{|F(a_1)\exp[i\psi(a_1)|\,|1-\overline{a_1}z_1|^{n+1}}.$$

Consequently,

$$\begin{aligned} &\Re \log J_{g^*}(z_1 v_0) = \\ &\Re\left\{\log F\left(\frac{a_1-z_1}{1-\overline{a_1}z_1}\right) - \log F(a_1) - (n+1)\log(1-\overline{a_1}z_1)\right. \\ &\left. +i\left[\psi\left(\frac{a_1-z_1}{1-\overline{a_1}z_1}\right) - \psi(a_1)\right]\right\}. \end{aligned} \tag{3.13}$$

Let us put $z_1 = \rho e^{is}$, $s \in R$, $\rho \in [0,1)$ in the above equality. If we denote $w_s = e^{is} v_0 \in \partial B^n$, then after the differentiation of (3.13) with respect to ρ, we obtain at $\rho = 0$,

$$\Re\left\{\frac{d}{d\rho}\log J_{g^*}(\rho w_s)\right\}_{|\rho=0} = \Re\left\{\frac{\frac{d}{d\rho}J_{g^*}(\rho w_s)\,|_{\rho=0}}{J_{g^*}(0)}\right\}$$

$$= \Re e^{is}\left\{\frac{F'(a_1)}{F(a_1)}(|a_1|^2-1) + (n+1)\overline{a_1} + i\psi'(a_1)(|a_1|^2-1)\right\}.$$

Since $g^* \in \mathfrak{U}_\alpha$, we get from (3.7)

$$\left|\frac{d}{d\rho}J_{g^*}(\rho w_s)\,|_{\rho=0}\right| \le 2\alpha.$$

Thus

$$\left|\frac{F'(a_1)}{F(a_1)}(|a_1|^2-1) + (n+1)\overline{a_1} + i\psi'(a_1)(|a_1|^2-1)\right| \le 2\alpha.$$

Choosing $a_1 = r \in [0,1)$ we obtain

$$\left|2\alpha + i(1-r^2)\psi'(r)\right| \le 2\alpha, \tag{3.14}$$

because

$$\frac{F'(r)}{F(r)} = \frac{(n+1)r + 2\alpha}{1 - r^2}.$$

However, $\psi(r)$ is real, and so is $\psi'(r)$. Thus, inequality (3.14) holds only if $\psi'(r) = 0$. This equality with arbitrary $r \in [0, 1)$ and the uniqueness theorem imply $\psi'(z_1) = 0$ for $z_1 \in \Delta$. Therefore, by the normalization $\psi(0) = 0$ we obtain $\psi(z_1) = 0$. Consequently, $J_{f^*}(z_1 v_0) = F(z_1)$.

In [GLS] it was shown that the mapping

$$f(z) = (\int_0^{z_1} h_1(\zeta)d\zeta,\ z_2 h_2(z_1), ..., z_n h_n(z_1)),\ z = (z_1, ..., z_n) \in B^n,$$

belongs to $\mathfrak{U}_\alpha$ for all nonvanishing functions $h_j(z_1)$, $j = 1, ..., n$, holomorphic in Δ and fulfilling the condition

$$\prod_{j=1}^{n} h_j(z_1) = \frac{(1+z_1)^{\alpha - \frac{n+1}{2}}}{(1-z_1)^{\alpha + \frac{n+1}{2}}}\ , \quad z_1 \in \Delta.$$

Therefore $\delta_0 = \delta_0(f) = 1$ for this mapping f and every nonvanishing function h_1 holomorphic in Δ which generates such an f with $\delta_0(f) = 1$. □

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