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Another proof of boundedness of the Cesàro operator on H^p

Dedicated to Professor Zdzisław Lewandowski on his 70th birthday

Abstract. We give a new short proof of the boundedness of the Cesáro operator on H^p, $0 < p < \infty$.

1. Introduction. Let H^p, $0 < p < \infty$, be the standard Hardy space on the unit disc $\mathbb{D}$. For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in H^p, the Cesàro operator is given by the formula

$$\mathcal{C}(f)(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n.$$

It follows from [H], [S1], [S2] and [M] that the Cesàro operator is a bounded operator on H^p for $0 < p < \infty$. In [S2] the author proved the case $p = 1$ and he remarked that the proof cannot be adapted for the other values of p. Here we present a modification of his proof which works for all positive p.

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H^∞, the space of bounded analytic functions on $\mathbb{D}$, is not mapped into itself by the Cesàro operator. However, $\mathcal{C}$ is a bounded operator from H^∞ into the space BMOA ([DS], see also [EX, p. 191]). Recently J. Shi and G. Ren [SR] have proved that $\mathcal{C}$ is a bounded operator on a mixed norm space $H_{p,q}(\varphi)$, and as a special case, it is bounded on the weighted Bergman space.

For an analytic function f on $\mathbb{D}$ and for $0 < p, q, \gamma < \infty$, we define

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}, \quad \|f\|_p = \sup_{0 \le r < 1} M_p(r, f)$$

and

$$M_{p,q,\gamma}(f) = \left(\int_0^1 (1-r)^{q\gamma-1} M_p^q(r, f) dr \right)^{1/q}$$

Our proof is based on the Hardy-Littlewood inequality and its dual inequality due to T. M. Flett. We state these results as the following lemmas.

Lemma HL [HL, p. 411]. *Let f be an analytic function on $\mathbb{D}$ and let*

$$0 < p < q, \quad \alpha = \frac{1}{p} - \frac{1}{q}, \quad l \ge p.$$

Then

$$M_{q,l,\alpha}^l(f) = \int_0^1 M_q^l(r, f)(1-r)^{l\alpha-1} dr \le C M_p^l(r, f).$$

The next two lemmas are special cases of Theorem 2 in [F, p.750].

Lemma F1. *Let $0 < p \le 2$, and let f be an analytic function on $\mathbb{D}$ such that $f(0) = 0$. If $M_{p,p,1}(f') < \infty$, then $f \in H^p$ and*

$$\|f\|_p \le C M_{p,p,1}(f').$$

Lemma F2. *Let $0 < s < p < \infty$, $\gamma = 1 - \left(\frac{1}{s} - \frac{1}{p}\right) > 0$ and let f be an analytic function on $\mathbb{D}$ such that $f(0) = 0$. If $M_{s,p,\gamma}(f') < \infty$, then $f \in H^p$ and*

$$\|f\|_p \le C M_{s,p,\gamma}(f').$$

2. The Proof. Put $F(z) = z\mathcal{C}f(z)$. A computation shows that $F'(z) = \frac{1}{1-z}f(z),\ z \in \mathbb{D}$. Assume first that $0 < p \leq 2$. If $\beta > 1,\ \ \frac{1}{\beta} + \frac{1}{\beta'} = 1$ and $p\beta' > 1$, then the Hölder inequality and the lemma in [D, p.65] give

$$\begin{aligned}
M^p_{p,p,1}(F') &= \int_0^1 (1-r)^{p-1} M_p^p(r, F')dr \\
&= \int_0^1 (1-r)^{p-1} \int_0^{2\pi} \frac{|f(re^{i\theta})|^p}{|1-re^{i\theta}|^p} d\theta dr \\
&\leq \int_0^1 (1-r)^{p-1} M^p_{\beta p}(r, f) \left(\int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}|^{p\beta'}} \right)^{1/\beta'} dr \\
&\leq \int_0^1 (1-r)^{\frac{1}{\beta'}-1} M^p_{p\beta}(r, f)dr = \int_0^1 (1-r)^{-\frac{1}{\beta}} M^p_{p\beta}(r, f)dr \\
&\leq C\|f\|_p^p,
\end{aligned}$$

where the last inequality follows from Lemma HL. Thus, by Lemma F1, $\|F\|_p^p \leq C\|f\|_p^p$.

Now assume that $2 < p < \infty$. If $\frac{1}{p+1} < \alpha < 1$, then $\gamma = 1 - \left(\frac{1}{\alpha p} - \frac{1}{p} \right) > 0$ and

$$M^p_{p\alpha,p,\gamma}(F') = \int_0^1 (1-r)^{p-\frac{1}{\alpha}} \left(\int_0^{2\pi} \frac{|f(re^{i\theta})|^{\alpha p}}{|1-re^{i\theta}|^{\alpha p}} d\theta \right)^{1/\alpha} dr.$$

Take $\beta > 1$, α as above and such that $\alpha\beta > 1$ and $p\alpha\beta' > 1$ (e.g. $\alpha = \frac{2}{p+1},\ \beta = p+1$). Then, in much the same way as in the first case, one can get

$$M^p_{p\alpha,p,\gamma}(F') \leq \int_0^1 (1-r)^{-\frac{1}{\alpha\beta}} M^p_{p\alpha\beta}(r, f)dr \leq C\|f\|_p^p.$$

Thus Lemma F2 implies the desired result.

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