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CHRISTIAN POMMERENKE and ALEXANDER VASIL'EV

On bounded univalent functions and the angular derivative

Dedicated to Professor Zdzisław Lewandowski on his 70-th birthday

Abstract. In this paper we study bounded univalent functions $f(z)$ that map the unit disk into itself such that $f(0) = 0$. In particular we are concerned with the functions for which the angular limit and the angular derivative exist at certain points of the unit circle. For such functions we obtain several explicit estimates many of which are sharp. We apply two different methods to derive them. One is based on the the Schiffer-Tammi analogue of the Grunsky inequality, the other one uses the method of modules of curve families and the extremal partition of domains.

1. Introduction. Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and let $\mathbb{T} = \partial\,\mathbb{D}$. We consider conformal maps f of $\mathbb{D}$ into $\mathbb{D}$. The angular limit

$$f(\zeta) = \lim_{z\to\zeta,\, z\in\Delta} f(z), \quad \Delta \text{ is any Stolz angle at } \zeta,$$

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exists for almost all $\zeta \in \mathbb{T}$, the exceptional set has even zero capacity. In general, very little can be said about the existence of the angular derivative

$$f'(\zeta) = \lim_{z \to \zeta,\, z \in \Delta} f'(z), \tag{1.1}$$

see e.g. [10, Chapter 6] for a discussion.

The situation becomes much better when we restrict ourselves to the set

$$A = \{\zeta \in \mathbb{T} : \, f(\zeta) \text{ exists and } |f(\zeta)| = 1\}, \tag{1.2}$$

because the angular derivative exists for every $\zeta \in A$ by the Julia-Wolff Lemma [10, Proposition 4.13], even without the assumption that f is injective in $\mathbb{D}$. It may, however, be infinite. In our case of univalent functions, it follows from the McMillan Twist Theorem [5], [10, Theorem 6.18] that $f'(\zeta) \neq \infty$ for almost all $\zeta \in A$.

Moreover, it follows from [1, Corollary 6.4] that f is continuous and injective in $A \setminus E_0$ where $\operatorname{cap} E_0 = 0$, $f(E_0)$ is countable, and f is bilipschitz in $A \setminus E_\varepsilon$ where E_ε has measure less than ε by McMillan's theorem and by Corollary 3.4 below.

We will be concerned with explicit estimates many of which will be sharp. We use the normalization $f(z) = \alpha z + \dots$ with $0 < \alpha \leq 1$. Our results will be based on two methods.

In Section 2 we use the Schiffer-Tammi [11] analogue of the Grunsky inequality to derive two positive semi-definite quadratic forms involving z_ν, $f(z_\nu)$, and $f'(z_\nu)$ for $\nu = 1, \dots, n$. In Section 3 we employ these quadratic forms (for $n = 2$) to derive various estimates, in particular for the case when the angular derivative is finite at a given point.

In Section 4 we turn to the method of modules and quadratic differentials that goes back to Teichmüller; see e.g. [3], [10, Chapter 8]. We use a theorem by G. Kuz'mina [4] and E. Emel'yanov [2] about the reduced module of extremal partitions. In order to apply this theorem we have to calculate certain reduced modules for domains bounded by critical trajectories of the quadratic differentials

$$-A\frac{z-c}{z^2(z-a)}dz^2 \quad \text{and} \quad -A\frac{(z-b)^2}{z^2(z-a_1)(z-c_1)}dz^2, \quad A > 0;$$

see Theorems 4.1 and 4.2. We use this to give a complete description of the domain of values of $(|f(r)|, \alpha)$ for functions $f(z) = \alpha z + \dots$ with $f(1) = 1$ and $|f'(1)| = \beta$ where $r \in (0,1)$ and β are given (Theorem 5.1). We denote this class by $\mathcal{M}^1(\beta)$.

In many cases, the extremal function is the classical conformal map

$$p_\alpha(z) = \frac{4\alpha z}{\left(1 - z + \sqrt{(1-z)^2 + 4\alpha z}\right)^2} = \alpha z + \dots \tag{1.3}$$

of $\mathbb{D}$ onto $\mathbb{D} \setminus [-1, -\alpha/(1+\sqrt{1-\alpha})^2]$. It satisfies the identities

$$\frac{p_\alpha(z)}{(1-p_\alpha(z))^2} = \frac{\alpha z}{(1-z)^2}, \quad \frac{p_\alpha(z)}{(1+p_\alpha(z))^2} = \frac{\alpha z}{(1-z)^2+4\alpha z}, \tag{1.4}$$

$$z\frac{p_\alpha'(z)}{p_\alpha(z)} = \frac{1+z}{\sqrt{(1-z)^2+4\alpha z}}. \tag{1.5}$$

In Theorem 5.1, however, the extremal function is more complicated; the extremal domain is $\mathbb{D}$ minus slits with two endpoints in $\mathbb{D}$.

2. The Schiffer-Tammi inequality. Let the function $f(z) = \alpha z + \dots$ $(0 < \alpha \le 1)$ be univalent in $\mathbb{D}$ and let $f(\mathbb{D}) \subset \mathbb{D}$. We define $a_{jk} = a_{kj}$ $(j, k = 0, 1, \dots)$ and $a^*_{jk} = \overline{a^*_{kj}}$ $(j, k = 1, 2, \dots)$ by

$$\log \frac{f(z)-f(\zeta)}{z-\zeta} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk} z^j \zeta^k, \quad (z, \zeta \in \mathbb{D}), \tag{2.1}$$

$$-\log\left[1 - f(z)\overline{f(\zeta)}\right] = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a^*_{jk} z^j \bar{\zeta}^k, \quad (z, \zeta \in \mathbb{D}). \tag{2.2}$$

Schiffer and Tammi [11] have shown that

$$\operatorname{Re}\Big[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}\lambda_j\lambda_k\Big] + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a^*_{jk}\lambda_j\bar{\lambda}_k \le \sum_{k=1}^{\infty}\frac{|\lambda_k|^2}{k} \tag{2.3}$$

for $\lambda_0 \in \mathbb{R}$, $\lambda_k \in \mathbb{C}$ $(k = 1, 2, \dots)$; the case $\lambda_0 = 0$ is due to Nehari [7]. See also [12, p.174] and [9, Theorem 4.2].

We derive two positive semi-definite quadratic forms from the Schiffer-Tammi inequality (2.3); compare [9, Corollary 4.3].

Theorem 2.1. *Let $f(z) = \alpha z + \dots$ be univalent in $\mathbb{D}$ and $f(\mathbb{D}) \subset \mathbb{D}$. If $z_\nu \in \mathbb{D}$ and $w_\nu = f(z_\nu)$ for $\nu = 1, \dots, n$ and if $x_\nu \in \mathbb{R}$, $(\nu = 0, \dots, n)$, then*

$$x_0^2 \log\frac{1}{\alpha} + 2\sum_{\nu=1}^{n} x_0 x_\nu \arg\frac{w_\nu}{z_\nu} + \sum_{\mu=1}^{n}\sum_{\nu=1}^{n} x_\mu x_\nu \log\left|\frac{\alpha z_\mu z_\nu}{w_\mu w_\nu}\cdot\frac{w_\mu - w_\nu}{z_\mu - z_\nu}\cdot\frac{1-w_\mu\bar{w}_\nu}{1-z_\mu\bar{z}_\nu}\right| \ge 0, \tag{2.4}$$

$$(2.5)\quad x_0^2\log\frac{1}{\alpha}+2\sum_{\nu=1}^{n}x_0x_\nu\log\left|\frac{w_\nu}{z_\nu}\right| + \sum_{\mu=1}^{n}\sum_{\nu=1}^{n}x_\mu x_\nu\log\left|\frac{z_\mu-z_\nu}{w_\mu-w_\nu}\cdot\frac{1-w_\mu\bar{w}_\nu}{1-z_\mu\bar{z}_\nu}\right|\geq 0.$$

Proof. (a) First we set

$$\lambda_0=-x_0,\ \lambda_k=i\sum_{\nu=1}^{n}x_\nu z_\nu^k,\ \ (k=1,2,\dots)$$

and conclude from the definitions (2.1) and (2.2) that

$$a_{00}=\log\,\alpha,\ \sum_{k=1}^{\infty}a_{k0}\lambda_k=i\sum_{\nu=1}^{n}x_\nu\log\,\frac{w_\nu}{\alpha z_\nu},$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}\lambda_j\lambda_k=-\sum_{\mu=1}^{n}\sum_{\nu=1}^{n}x_\mu x_\nu\log\left(\frac{\alpha z_\mu z_\nu}{w_\mu w_\nu}\cdot\frac{w_\mu-w_\nu}{z_\mu-z_\nu}\right),$$

$$(2.6)\qquad \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a^*_{jk}\lambda_j\bar{\lambda}_k=-\sum_{\mu=1}^{n}\sum_{\nu=1}^{n}x_\mu x_\nu\log\left(1-w_\mu\bar{w}_\nu\right),$$

$$(2.7)\qquad \sum_{k=1}^{\infty}\frac{1}{k}|\lambda_k|^2=-\sum_{\mu=1}^{n}\sum_{\nu=1}^{n}x_\mu x_\nu\log\left(1-z_\mu\bar{z}_\nu\right).$$

Hence (2.4) follows from the Schiffer-Tammi inequality (2.3)

(b) Now we set

$$\lambda_0=-x_0+\sum_{\nu=1}^{n}x_\nu,\ \ \lambda_k=\sum_{\nu=1}^{n}x_\nu z_\nu^k,\ \ (k=1,2,\dots).$$

We deduce from (2.1) and (2.3) that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}\lambda_j\lambda_k=x_0^2\log\,\alpha-2x_0\sum_{\nu=1}^{n}x_\nu\log\,\frac{w_\nu}{z_\nu}+\sum_{\mu=1}^{n}\sum_{\nu=1}^{n}x_\mu x_\nu\log\,\frac{w_\mu-w_\nu}{z_\mu-z_\nu}$$

while (2.6) and (2.7) continue to hold without change. Hence (2.5) follows from the Schiffer-Tammi inequality (2.3). □

The inequality (2.4) also holds if $z_\nu \in A \subset \mathbb{T}$ and $f'(z_\nu) \neq \infty$; the inequality (2.5) becomes trivial. To see this we apply (2.4) to rz_ν $(0 < r < 1)$ and then let $r \to 1$. The Julia-Wolff Lemma [10, Proposition 4.13] shows that

$$f'(rz_\nu) \to f'(z_\nu), \quad \frac{1 - |f(rz_\nu)|^2}{1 - r^2} \to |f'(z_\nu)|. \tag{2.8}$$

In the limit many terms of (2.4) become simpler.

The quadratic form

$$\sum_{\mu=0}^{n} \sum_{\nu=0}^{n} \delta_{\mu\nu} x_\mu x_\nu$$

is positive semidefinite if and only if its principal determinants satisfy

$$\det \begin{vmatrix} \delta_{jj} & \delta_{j,j+1} & \dots & \delta_{jk} \\ \vdots & \vdots & & \vdots \\ \delta_{kj} & \delta_{k,j+1} & \dots & \delta_{kk} \end{vmatrix} \geq 0 \ \text{ for } \ 0 \leq j \leq k \leq n.$$

For $j = 0$, $k = 1$ and $j = 1$, $k = 2$ respectively, we obtain from (2.4) and (2.9)

Corollary 2.2. *Let $f(z) = \alpha z + \dots$ be univalent in $\mathbb{D}$ and $f(\mathbb{D}) \subset \mathbb{D}$. If $z \in \mathbb{D}$ and $w = f(z)$, then*

$$\left(\arg \frac{w}{z}\right)^2 \leq \log \frac{1}{\alpha} \cdot \log \left| \frac{\alpha z^2}{w^2} f'(z) \frac{1 - |w|^2}{1 - |z|^2} \right|, \tag{2.10}$$

and if $z_\nu \in \mathbb{D}$, $w_\nu = f(z_\nu)$ $(\nu = 1, 2)$, then

(2.11)

$$\left(\log \left| \frac{\alpha z_1 z_2}{w_1 w_2} \cdot \frac{w_1 - w_2}{z_1 - z_2} \cdot \frac{1 - w_1 \bar{w}_2}{1 - z_1 \bar{z}_2} \right| \right)^2 \leq \prod_{\nu=1}^{2} \log \left| \frac{\alpha z_\nu^2}{w_\nu^2} f'(z_\nu) \frac{1 - |w_\nu|^2}{1 - |z_\nu|^2} \right|.$$

These inequalities are sharp. If we choose for f the function p_α defined in (1.3), then $p_\alpha(x)$ is real for $-1 < x < 1$ and we obtain from (1.4) and (1.5)

$$\frac{\alpha x^2}{p_\alpha(x)^2} p'_\alpha(x) \frac{1 - p_\alpha(x)^2}{1 - x^2} = 1. \tag{2.12}$$

Hence, the right-hand sides in (2.10) and (2.11) are zero so that equality holds.

3. Some estimates involving the angular derivative. Now we derive some consequences of the Schiffer-Tammi inequality in the case where our bounded univalent function has a finite angular derivative at a point $\zeta \in A$; see (1.1) and (1.2).

Theorem 3.1. *Let $f(z) = \alpha z + \dots$ be univalent in $\mathbb{D}$ and $f(\mathbb{D}) \subset \mathbb{D}$. We assume that*

$$\zeta \in \mathbb{T}, \quad f(\zeta) \in \mathbb{T}, \quad |f'(\zeta)| = \beta < \infty. \tag{3.1}$$

Then $\alpha \cdot \beta^2 \geq 1$, and if $z \in \mathbb{D}$, then

$$\frac{|f(\zeta) - f(z)|^2}{1 - |f(z)|^2} \frac{1 - |z|^2}{|\zeta - z|^2} \leq \frac{\sqrt{\alpha}\beta(1 + |z|)}{\sqrt{(1 - |z|)^2 + 4\alpha|z|}}, \tag{3.2}$$

$$\frac{1 - |f(z)|}{1 - |z|} \leq \beta \Big| \frac{f(z)}{z} \Big|^{1/2} \left(\frac{|\zeta - z|}{1 - |z|} \right)^2, \tag{3.3}$$

$$\Big| z \frac{f'(z)}{f(z)} \Big| \leq \sqrt{\alpha}\beta \frac{1 + |z|}{\sqrt{(1 - |z|)^2 + 4\alpha|z|}} \left(\frac{|\zeta - z|}{1 - |z|} \right)^2. \tag{3.4}$$

The Julia-Wolff Lemma [10, Proposition 4.13] shows that the left-hand side of (3.2) is $\leq \beta$ for all analytic functions $f : \mathbb{D} \to \mathbb{D}$. Furthermore, it follows from (3.7) and (3.8) below that, for univalent functions,

$$\frac{1 - |z|^2}{1 - |f(z)|^2} |f'(z)| \leq \frac{\alpha(1 + |z|)^2}{(1 - |z|)^2 + 4\alpha|z|} \quad (z \in \mathbb{D}); \tag{3.5}$$

it is well-known that the left-hand side is ≤ 1 for all analytic $f : \mathbb{D} \to \mathbb{D}$.

All inequalities (3.2)–(3.5) are sharp in the restricted sense that we have equality if

$$f = p_\alpha, \quad \zeta = 1, \quad 0 \leq z < 1$$

where p_α is defined by (1.3). This is easy to check using (1.4) and (1.5); note that now $\alpha\beta^2 = 1$. But it is by no means clear whether the factor $|\zeta - z|^2/(1 - |z|)^2$ in (3.3) and (3.4) is the right one.

For the proof we need two inequalities for univalent functions $f(z) = \alpha z + \dots$ with $f(\mathbb{D}) \subset \mathbb{D}$, namely that, for $|z| = r < 1$,

$$-p_\alpha(-r) \leq |f(z)| \leq p_\alpha(r), \tag{3.6}$$

$$\frac{1 + |f(z)|}{1 - |f(z)|} \cdot \frac{1 - r}{1 + r} \leq \Big| z \frac{f'(z)}{f(z)} \Big| \leq \frac{1 - |f(z)|}{1 + |f(z)|} \cdot \frac{1 + r}{1 - r}. \tag{3.7}$$

The first inequality is due to Pick [8]. See e.g. [6] for the second one.

We also need a consequence of (3.6). Since $x(1+x)^{-2}$ is increasing in $0 \le x \le 1$, it follows from (3.6) and (1.4) that, for $|z| = r < 1$,

$$\frac{|f(z)|}{(1+|f(z)|)^2} \le \frac{p_\alpha(r)}{(1+p_\alpha(r))^2} = \frac{\alpha r}{(1-r)^2 + 4\alpha r}. \tag{3.8}$$

Proof of Theorem 3.1. We put $w = f(z)$ and $\omega = f(\zeta)$. We apply (2.4) in Theorem 2.1 with $n = 2$ and

$$x_0 = 0;\ \ x_1 = 1,\ z_1 = z;\ \ x_2 = -1,\ z_2 = r\zeta\ \ (0 < r < 1)$$

and let $r \to 1$. Using (2.8) we obtain

$$\log\left|\frac{\alpha z^2}{w^2}\, f'(z)\, \frac{1-|w|^2}{1-|z|^2}\right| - 2\log\left|\frac{\alpha z}{w}\left(\frac{w-\omega}{z-\zeta}\right)^2\right| + \log(\alpha\beta^2) \ge 0$$

and, therefore,

$$\left|\frac{w-\omega}{z-\zeta}\right|^4 \le \beta^2 |f'(z)| \frac{1-|w|^2}{1-|z|^2}. \tag{3.9}$$

We use (3.7) to estimate $|f'(z)|$ from above and obtain

$$\left|\frac{w-\omega}{z-\zeta}\right|^4 \le \beta^2 \left|\frac{w}{z}\right| \left(\frac{1-|w|^2}{1-|z|^2}\right)^2 \left(\frac{1+|z|}{1+|w|}\right)^2,$$

and the assertion (3.2) follows from (3.8).

Furthermore, we deduce from (3.9) that

$$\left(\frac{1-|w|}{1-|z|}\right)^4 \le \beta^2 |f'(z)| \frac{1-|w|^2}{1-|z|^2} \left(\frac{|\zeta - z|}{1-|z|}\right)^4$$

and therefore, by (3.7),

$$\left(\frac{1-|w|}{1-|z|}\right)^3 \le \beta^2 |f'(z)| \frac{1+|w|}{1+|z|} \left(\frac{|\zeta-z|}{1-|z|}\right)^4 \le \beta^2 \left|\frac{w}{z}\right| \frac{1-|w|}{1-|z|} \left(\frac{|\zeta-z|}{1-|z|}\right)^4 \tag{3.10}$$

which implies the assertion (3.3).

Finally we see from (3.7) and the first inequality (3.10) that

$$\begin{aligned} |f'(z)|^3 &\le \left|\frac{w}{z}\right|^3 \left(\frac{1+|z|}{1+|w|}\right)^3 \left(\frac{1-|w|}{1-|z|}\right)^3 \\ &\le \beta^2 \left|\frac{w}{z}\right|^3 |f'(z)| \left(\frac{1+|z|}{1+|w|}\right)^2 \left(\frac{|\zeta-z|}{1-|z|}\right)^4. \end{aligned}$$

We now divide by $|f'(z)|$ and apply (3.8) to obtain (3.4). □

Theorem 3.2. *Let $f(z) = \alpha z + \dots$ be univalent in $\mathbb{D}$ and $f(\mathbb{D}) \subset \mathbb{D}$. If $z_\nu \in \mathbb{D}$, $w_\nu = f(z_\nu)$ $(\nu = 1, 2)$, then*

$$(3.11)\qquad \frac{1}{\alpha}\prod_{\nu=1}^{2}\Big|\frac{w_\nu}{z_\nu}\Big|\left(\frac{1-|z_\nu|^2}{1-|w_\nu|^2}\right)^{1/2} \le \Big|\frac{w_1-w_2}{z_1-z_2}\Big| \le \prod_{\nu=1}^{2}\left(\frac{1-|w_\nu|^2}{1-|z_\nu|^2}\right)^{1/2}.$$

If the function f is odd and $z_2 = -z_1$, then the first inequality (3.11) reduces to $|w_1|/(1-|w_1|^2) \le \alpha|z_1|/(1-|z_1|^2)$ and we have equality for the functions

$$f(z) = [e^{i\theta} p_{\alpha^2}(e^{-i\theta}z^2)]^{1/2} = \alpha z + \dots,\ \ \theta = 2i \arg z_1,$$

(see (1.3) and (1.4)). Hence, the lower estimate (3.11) is sharp for $z_2 = -z_1$.

It is possible to obtain a slightly better but more complicated upper estimate in (3.11) but this still does not appear to be sharp.

Corollary 3.3. *If $z_\nu \in \mathbb{T}$, $f(z_\nu) \in \mathbb{T}$, and $|f'(z_\nu)| \ne \infty$ $(\nu = 1, 2)$, then*

$$(3.12)\qquad \frac{1}{\alpha|f'(z_1)f'(z_2)|^{1/2}} \le \Big|\frac{f(z_1)-f(z_2)}{z_1-z_2}\Big| \le |f'(z_1)f'(z_2)|^{1/2}.$$

This is an immediate consequence of (3.11) applied to rz_ν with $r \to 1$; see (2.8). There is another case of equality in the lower estimate (3.12), namely when

$$f = p_\alpha,\ \ z_1 = e^{-it},\ \ z_2 = e^{it},\ \ 0 \le \sin\frac{t}{2} \le \sqrt{\alpha}.$$

It follows from (1.3) and (1.5) that

$$(3.13)\qquad \operatorname{Im} p_\alpha(e^{\pm it}) = \pm\frac{2s}{\alpha}\sqrt{\alpha - s^2},\ \ |p'_\alpha(e^{it})| = \sqrt{\frac{1-s^2}{\alpha - s^2}},$$

where $s = \sin(t/2)$, so that we have equality in the lower estimate. We have equality in the upper estimate (3.12) in the trivial case $z_1 = z_2$.

Proof of Theorem 3.2. We conclude from (2.4) in Theorem 2.1 with $x_0 = 0$, $x_1 = x_2 = 1$ that

$$\alpha^4\Big|\frac{z_1z_2}{w_1w_2}\Big|^4 |f'(z_1)f'(z_2)|\frac{1-|w_1|^2}{1-|z_1|^2}\cdot\frac{1-|w_2|^2}{1-|z_2|^2}\cdot\Big|\frac{w_1-w_2}{z_1-z_2}\Big|^2\cdot\Big|\frac{1-w_1\bar{w}_2}{1-z_1\bar{z}_2}\Big|^2 \ge 1\,,$$

and it follows from (2.5) with $x_0 = 0$, $x_1 = 1$, $x_2 = -1$ that

$$\frac{1-|w_1|^2}{(1-|z_1|^2)|f'(z_1)|}\cdot\frac{1-|w_2|^2}{(1-|z_2|^2)|f'(z_2)|}\cdot\Big|\frac{w_1-w_2}{z_1-z_2}\Big|^2\cdot\Big|\frac{1-z_1\bar{z}_2}{1-w_1\bar{w}_2}\Big|^2 \ge 1,$$

and the lower estimate (3.11) follows after multiplying these two inequalities.

We deduce from (2.4) with $x_0 = 0$, $x_1 = 1$, $x_2 = -1$ that

$$|f'(z_1)f'(z_2)| \cdot \frac{1-|w_1|^2}{1-|z_1|^2}\frac{1-|w_2|^2}{1-|z_2|^2} \cdot \Big|\frac{z_1-z_2}{w_1-w_2}\Big|^2 \cdot \Big|\frac{1-z_1\bar{z}_2}{1-w_1\bar{w}_2}\Big|^2 \geq 1$$

and the upper estimate (3.11) now follows from the well-known estimates

$$\Big|\frac{w_1-w_2}{1-w_1\bar{w}_2}\Big| \leq \Big|\frac{z_1-z_2}{1-z_1\bar{z}_2}\Big|, \quad |f'(z_\nu)| \leq \frac{1-|w_\nu|^2}{1-|z_\nu|^2}$$

valid for all analytic functions from $\mathbb{D}$ to $\mathbb{D}$. $\square$

We defined A in (1.2). For $1 \leq \beta < \infty$ and $1 < \lambda < \infty$, we furthermore define

$$A(\beta) = \{\zeta \in A : \ |f'(\zeta)| \leq \beta\}, \tag{3.14}$$

$$G(\beta,\lambda) = \{z \in \mathbb{D} : \ \frac{|\zeta - z|}{1-|z|} < \lambda \ \text{for some} \ \zeta \in A(\beta)\}. \tag{3.15}$$

It follows from (3.3) and (2.8) that $A(\beta)$ is closed, and if $A(\beta) \neq \emptyset$, then $G(\beta,\lambda)$ is a subdomain of $\mathbb{D}$ that contains 0 and a Stolz angle of fixed size at each $\zeta \in A(\beta)$.

Corollary 3.4. *Let $f(z) = \alpha z + \dots$ be univalent in $\mathbb{D}$ and $f(\mathbb{D}) \subset \mathbb{D}$. Then f is bilipschitz in $A(\beta) \cup G(\beta,\lambda)$ for every β and λ; more precisely,*

$$\frac{1}{\alpha\beta} \leq \Big|\frac{f(z_1)-f(z_2)}{z_1-z_2}\Big| \leq \beta \ , \ \textit{for} \ z_1, z_2 \in A(\beta), \tag{3.16}$$

$$\frac{\sqrt{\alpha}}{8\beta\lambda^2} \leq \Big|\frac{f(z_1)-f(z_2)}{z_1-z_2}\Big| \leq \beta\lambda^2 \ \ \textit{for} \ z_1, z_2 \in G(\beta,\lambda). \tag{3.17}$$

The fact that f is bilipschitz in $G(\beta,\lambda)$ is an unpublished result of Steffen Rohde. The estimate (3.16) is an immediate consequence of Corollary 3.3 and is, therefore, in some sense sharp. The estimate (3.17), however, is certainly not sharp and it is not even clear whether λ^2 is the right power.

Proof. We see from the inequality (3.3) in Theorem 3.1 that

$$\frac{1-|w|^2}{1-|z|^2} \leq \beta\frac{1+|w|}{1+|z|}\Big|\frac{w}{z}\Big|^{1/2}\left(\frac{|\zeta - z|}{1-|z|}\right)^2 \leq \beta\lambda^2\Big|\frac{w}{z}\Big|^{1/2} \leq \beta\lambda^2 \tag{3.18}$$

for $z \in \mathbb{D}$, $w = f(z)$. Hence, the upper estimate (3.17) follows from Theorem 3.2.

Furthermore, we obtain from Theorem 3.2 and (3.18)

$$\left|\frac{w_1 - w_2}{z_1 - z_2}\right| \geq \left|\frac{w_1 w_2}{\alpha z_1 z_2}\right| \left|\frac{z_1 z_2}{w_1 w_2}\right|^{1/4} \frac{1}{\beta \lambda^2} = \frac{1}{\alpha \beta \lambda^2} \left|\frac{w_1 w_2}{z_1 z_2}\right|^{3/4},$$

which implies the lower estimate (3.17) because $|w_\nu| \geq \alpha |z_\nu|/4$ by the Koebe distortion theorem. □

4. Modules and quadratic differentials. Let S_0 be a multiply connected domain in $\mathbb{C}$ with n punctures and with possibly l hyperbolic boundary components, $2n+3l > 6$. We define on S_0 an admissible system of curves $(\gamma_1, \ldots, \gamma_m)$ of two types. The curves from this system are not freely homotopic to each other in pairs and not homotopic to a point of S_0. The first type (I) consists of simple loops, each of which is homotopic to a puncture of S_0. The second one (II) consists of arcs with fixed endpoints on a boundary of S_0 (possibly punctures) that are not homotopic to the boundary point. All curves from the admissible system do not intersect.

A doubly connected parabolic domain D_j on S_0 bounded by a puncture of S_0 and a non-degenerate continuum is said to be of first homotopic type γ_j if any simple loop on S_0 separating the boundary components of D_j is freely homotopic to the curve γ_j of the first type from the admissible system given. A simply connected domain D_k on S_0 with at least two boundary points on ∂S_0 is said to be of second homotopy type γ_k if γ_k is an arc with endpoints on ∂S_0 and if any arc in D_k connecting these points is homotopic to γ_k.

A system of non-overlapping doubly connected parabolic domains and simply connected domains $(D_1, \ldots, D_m)$ on S_0 is said to have homotopy type $(\gamma_1, \ldots, \gamma_m)$ if $(\gamma_1, \ldots, \gamma_m)$ is an admissible curve system on S_0 and for any $j \in \{1, \ldots, m\}$ the domain D_j has the homotopy type γ_j of the first or the second type.

Let $D \subset \overline{\mathbb{C}}$ be a simply connected hyperbolic domain, $a \in D$, $|a| < \infty$. We construct a doubly connected domain $D_\varepsilon = D \setminus \{|z - a| \leq \varepsilon\}$ for a sufficiently small ε. The quantity

$$m(D, a) := \lim_{\varepsilon \to 0} \left(M(D_\varepsilon) + \frac{1}{2\pi} \log \varepsilon \right)$$

is said to be the *reduced module* of the domain D where $M(D_\varepsilon)$ is the module of the doubly connected domain D_ε with respect to the family of curves separating its boundary components (see [3]).

By the Riemann mapping theorem there is a unique conformal mapping $w = f(z)$ from D onto the disk $\{|w| < R\}$, $R < \infty$ such that

$f(a) = 0, f'(a) = 1$. The number R is said to be the conformal radius of D with respect to the point a. We denote it by $R(D, a)$. Then, $m(D, a) = \frac{1}{2\pi} \log R(D, a)$. If $f(z)$ is a conformal mapping from D such that $|f(a)| < \infty$, then $m(f(D), f(a)) = m(D, a) + \frac{1}{2\pi} \log |f'(a)|$

Now we define the reduced module $m(D, \infty)$ of a simply connected domain D, $\infty \in D$ with respect to infinity as the reduced module of the image of D under the mapping $1/z$ with respect to the origin

$$m(D, \infty) = -\frac{1}{2\pi} \log R(D, \infty).$$

So, if D is a simply connected hyperbolic domain, $a \in D$, $|a| < \infty$ and $f(z) = A_{-1}/(z - a) + A_0 + A_1(z - a) + \dots$ is a conformal mapping from D, then $m(f(D), \infty) = m(D, a) - \frac{1}{2\pi} \log |A_{-1}|$.

Now we define the reduced module of a "bigon". For details we refer to the papers by E. Emel'yanov [2], G. Kuz'mina [4], and A. Solynin [14]. The term "bigon" which appears in a paper of A. Solynin is an incorrectly coined neologism composed of Latin "bi-" and Greek "-gon" for "gonia"= angle. Greek part should be rather replaced by a suffix "-angle" derived from Latin "angulus". Therefore in what follows we prefer to use "biangle" instead of "bigon".

Let D be a hyperbolic simply connected domain from $\mathbb{C}$ with two finite fixed boundary points a, b (maybe with the same support) on its boundary. It is called a *biangle*. Let $S(a, \varepsilon)$ be a connected component of $D \cap \{|z - a| < \varepsilon\}$ such that $a \in \partial S(a, \varepsilon)$. Denote by D_ε the set $D \setminus \{S(a, \varepsilon_1) \cup S(b, \varepsilon_2)\}$ for sufficiently small $\varepsilon_{1,2}$ and by $M(D_\varepsilon)$ the module of the family of arcs in D_ε joining the boundary arcs of $S(a, \varepsilon_1)$ and $S(b, \varepsilon_2)$ situated on the circles $|z - a| = \varepsilon_1$ and $|z - b| = \varepsilon_2$. If the limit

$$m(D, a, b) = \lim_{\varepsilon_{1,2} \to 0} \left(\frac{1}{M(D_\varepsilon)} + \frac{1}{\varphi_a} \log \varepsilon_1 + \frac{1}{\varphi_b} \log \varepsilon_2 \right),$$

exists, where $\varphi_a = \sup \Delta_a$ and $\varphi_b = \sup \Delta_b$ are the inner angles and $\Delta_{a,b}$ is the Stolz angle inscribed in D at a or b respectively, then it is called the *reduced module of the biangle* D. Following [10] the points a, b will be called corners of D. Various conditions guarantee the existence of this module (see [14]). The existence of the limit is the local characteristic of the domain D (see [2,4,14]). If the domain D is conformal at the points a and b (see [10, p. 80]) then this condition is necessary and sufficient for the limit to exist. More general [14], suppose that there exists a conformal map $f(z)$ from the domain $S(a, \varepsilon_1) \subset D$ onto a circular sector so that there exists the angular limit $f(a)$ which is the vertex of this sector with the angle φ_a. If the function f has the angular finite non-zero derivative $f'(a)$ we say that the domain D is also *conformal at the point* a (compare [10, p. 80]). The biangle D is conformal at the points a, b if and only if the limit in the definition of $m(D, a, b)$ exists.

Suppose that there exists a conformal map $f(z)$ from the biangle D (which is conformal at a, b) onto a biangle D' so that there exist the angular limits $f(a)$, $f(b)$ with the inner angles ψ_a and ψ_b at the corners $f(a)$ and $f(b)$ which we also understand as the supremum over all Stolz angles inscribed in D' with corners at $f(a)$ or $f(b)$ respectively. If the function f has angular finite non-zero derivatives $f'(a)$ and $f'(b)$ then $\varphi_a = \psi_{f(a)}$, $\varphi_b = \psi_{f(b)}$, and the reduced module exists and is changed [2,4,14] according to the rule

$$m(f(D), f(a), f(b)) = m(D, a, b) + \frac{1}{\psi_a} \log |f'(a)| + \frac{1}{\psi_b} \log |f'(b)|$$

If we suppose, moreover, that f has the expansion

$$f(z) = w_1 + (z - a)^{\psi_a/\varphi_a}(c_1 + c_2(z - a) + \dots)$$

in a neighbourhood of a and the expansion

$$f(z) = w_2 + (z - b)^{\psi_b/\varphi_b}(d_1 + d_2(z - a) + \dots)$$

in a neighbourhood of b, then the reduced module of D is changed according to the rule

$$m(f(D), f(a), f(b)) = m(D, a, b) + \frac{1}{\psi_a} \log |c_1| + \frac{1}{\psi_b} \log |d_1|.$$

Obviously, one can extend this definition to the case of corners with the infinite support.

Now we give another definition of the same quantity of reduced module of a biangle that will be more convenient while applying the symmetrization. Denote by D'_ε the domain obtained from the biangle D by fixing two connected arcs δ_a and δ_b starting from a, b that lie on one and the same side of its boundary within disks $\{|z - b| < \varepsilon_1\}\}$ and $\{|z - b| < \varepsilon_2\}\}$ for sufficiently small $\varepsilon_{1,2}$. Denote by $M(D'_\varepsilon)$ the module of the family of arcs in D'_ε joining δ_a and δ_b.

Lemma 4.1. *The following equality*

$$\lim_{\varepsilon_{1,2} \to 0} \left(\frac{1}{M(D'_\varepsilon)} + \frac{1}{\varphi_a} \log \varepsilon_1 + \frac{1}{\varphi_b} \log \varepsilon_2 \right) = m(D, a, b) + \frac{2}{\pi} \log 4,$$

holds where φ_a and φ_b is the size of inscribed Stolz angles with corners at a and b and the limit is supposed to exist.

Proof. There is a conformal univalent mapping $f(z)$ from D onto the upper half-plane H^+ with the expansion

$$f(z) = (z-a)^{\frac{\pi}{\varphi_a}}(c_1 + c_2(z-a) + \dots)$$

in an angular neighbourhood of a in D or

$$f(z) = (z-b)^{\frac{-\pi}{\varphi_b}}(d_1 + d_2(z-b) + \dots)$$

in an angular neighbourhood of b in D. The image of the arc δ_a is the interval $(0, \Delta_1)$ such that

$$\varepsilon_1^{\frac{\pi}{\varphi_a}}(|c_1| - |c_2|\varepsilon_1 - o(\varepsilon_1))) \leq \Delta_1 \leq \varepsilon_1^{\frac{\pi}{\varphi_a}}(|c_1| + |c_2|\varepsilon_1 + o(\varepsilon_1)).$$

A similar inequality can be derived for the point b and for the image (Δ_2, ∞) of the arc δ_b. The module of the quadrangle D'_ε can be calculated as

$$M(D'_\varepsilon) = \frac{\mathbf{K}'}{\mathbf{K}}\left(\frac{\sqrt{\Delta_2 - \Delta_1}}{\sqrt{\Delta_2}}\right) = \frac{\mathbf{K}}{\mathbf{K}'}\left(\sqrt{\frac{\Delta_1}{\Delta_2}}\right),$$

where $\mathbf{K}(k)$ and $\mathbf{K}'(k)$ are complementary complete elliptic integrals.

We deduce that

$$\frac{\Delta_2}{\Delta_1} \in \left(\frac{\varepsilon_2^{-\frac{\pi}{\varphi_b}}(|d_1| - |d_2|\varepsilon_2 - o(\varepsilon_2))}{\varepsilon_1^{\frac{\pi}{\varphi_a}}(|c_1| + |c_2|\varepsilon_1 + o(\varepsilon_1))},\ \frac{\varepsilon_2^{-\frac{\pi}{\varphi_b}}(|d_1| + |d_2|\varepsilon_2 + o(\varepsilon_2))}{\varepsilon_1^{\frac{\pi}{\varphi_a}}(|c_1| - |c_2|\varepsilon_1 - o(\varepsilon_1))}\right).$$

Moreover, we have the following asymptotic behaviour

$$\lim_{k\to 0}\left(\frac{\mathbf{K}'}{\mathbf{K}}(k) - \frac{2}{\pi}\log\frac{4}{k}\right) = 0.$$

Therefore,

$$\lim_{\varepsilon_{1,2}\to 0}\left(\frac{1}{M(D'_\varepsilon)} + \frac{1}{\varphi_a}\log\varepsilon_1 + \frac{1}{\varphi_b}\log\varepsilon_2\right) = \frac{1}{\pi}\log\left|\frac{d_1}{c_1}\right| + \frac{2}{\pi}\log 4 = m(D,a,b) + \frac{2}{\pi}\log 4.$$

Taking into account $m(H^+, 0, \infty) = 0$ we obtain Lemma 4.1. □

Now we pose the problem of the extremal partition of S_0. With any domain of I type we associate the reduced module $m(D,a)$ with respect to the puncture a and with each domain of II type we associate the reduced module of the biangle D with corners at one or two boundary points $m(D,a,b)$.

Some of domains $(D_1,\dots,D_m)$ (not all of them) can degenerate. In this case assume the reduced module to be vanishing.

Let γ_j be of the first type for $j=1,\dots,k$ and of the second one for $j=k+1,\dots,m$. The general theorem by G. Kuz'mina [4] and E. Emel'yanov [2] (see also [14]) asserts that any collection of non-overlapping admissible doubly connected parabolic domains and simply connected domains $\{D_j\}$ of I-II types associated with admissible system $(\gamma_1,\dots,\gamma_m)$ satisfies the following inequality

$$\sum_{j=1}^{k} t_j^2 m(D_j,a_j) - \sum_{j=k+1}^{m} t_j^2 m(D_j,a_j,b_j) \leq \sum_{j=1}^{k} t_j^2 m(D_j^*,a_j) - \sum_{j=k+1}^{m} t_j^2 m(D_j^*,a_j,b_j) \tag{4.1}$$

with the equality sign only for $D_j = D_j^*$. Here $m(D_j,a_j)$ is the reduced module of the simply connected domain $D_j\cup\{a_j\}$ with respect to the puncture a_j of S_0, $m(D_j,a_j,b_j)$ is the reduced module of the biangle D_j with respect to the boundary points a_j and b_j of S_0 (possibly $a_j=b_j$).

Each D_j^* is either a circular domain or a strip domain in the trajectory structure of the unique quadratic differential $\varphi(\zeta)d\zeta^2$ associated with the problem about the extremal partition posed (see [4], [2] for the details). If D_j^* is a circular domain, then there is a conformal mapping $g_j(\zeta)$, $\zeta\in D_j^*$ satisfying the differential equation

$$t_j^2\left(\frac{g_j'(\zeta)}{g_j(\zeta)}\right)^2 = -4\pi^2\varphi(\zeta),$$

that maps D_j^* onto the punctured disk $0<|w|<\exp(2\pi m(D_j^*,a_j))$. If D_j^* is a strip domain, then there is a conformal mapping $g_j(\zeta)$, $\zeta\in D_j^*$ satisfying the differential equation

$$t_j^2\left(\frac{g_j'(\zeta)}{g_j(\zeta)}\right)^2 = 4\pi^2\varphi(\zeta),$$

that maps D_j^* onto the biangle $\mathbb{C}\setminus[0,\infty)$ with corners 0 and ∞.

The critical trajectories of $\varphi(\zeta)d\zeta^2$ split S_0 into at most m circular domains and strip domains $\{D_j\}$ associated respectively with homotopy classes of curves (some of D_j can degenerate).

Let $S_0 = \mathbb{C} \setminus \{0, a\}$, $a > 0$ be the twice-punctured complex plane. We consider on S_0 the admissible system (γ_1, γ_2) of I and II type respectively where $\gamma_1 = \{w : |w| = 1/\varepsilon\}$ and $\gamma_2 = \{w : |w - a| = a\}$, so that ε is sufficiently small. Let $\mathfrak{D}$ be the set of all pairs (D_1, D_2) consisting of a doubly connected parabolic domain and a simply connected domain of homotopy type (γ_1, γ_2). Then the problem of the extremal partition of S_0 (cf. [2,4]]) consists in maximizing the sum $t_1^2 m(D_1, \infty) - t_2^2 m(D_2, 0, 0)$ as $(D_1, D_2) \in \mathfrak{D}$. Without loss of generality, assume $t_1 = t$, $t_2 = 1$, $t \in [0, \infty)$, and denote the maximum of this sum by $M(t, a)$. There is the unique pair (D_1^*, D_2^*) which is extremal in this problem. D_1^* is a circular domain and D_2^* is a strip domain in the trajectory structure of the differential

$$\varphi(z)dz^2 = -A\frac{(z-c)dz^2}{z^2(z-a)}, \quad A > 0, \ c \le 0. \tag{4.2}$$

Here A and c are functions of t. If $t = 0$, then $D_1^* = \emptyset$ and $D_2^* = \mathbb{C}\setminus(-\infty, a]$ is the biangle with two corners with the same support 0. In this case $M(0, a) = \frac{2}{\pi} \log 4a$. If $t \to \infty$, then $D_1^* = \mathbb{C} \setminus [0, a]$. In this case $M(\infty, a) = \frac{1}{2\pi} \log 4/a$.

Theorem 4.1. *Let* $0 < t < \infty$. *Then*

$$m(D_1^*, \infty) = \frac{1}{2\pi} \log \frac{4t^2}{a(1+t^2)} - \frac{1}{\pi t}\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{t}\right),$$

$$m(D_2^*, 0, 0) = \frac{2}{\pi} \log \frac{4a}{1+t^2} + \frac{4t}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{t}\right).$$

Proof. We consider the mapping $u = u(z)$ whose inverse is

$$z = c\frac{a+1+(a-1)\cos u}{(c+1)+(c-1)\cos u}, \tag{4.3}$$

and obtain the representation of the differential φ in terms of the parameter u at regular points

$$\begin{aligned} \varphi(z)dz^2 &= Q(u)du^2 \\ &= \frac{4Ac(a-c)^2(1+\cos u)^2}{((c+1)+(c-1)\cos u)^2((a+1)+(a-1)\cos u)^2}du^2. \end{aligned} \tag{4.4}$$

Here

$$\left|\frac{a+1}{a-1}\right| > 1, \text{ and } \left|\frac{c+1}{c-1}\right| \le 1.$$

Now we study the trajectory structure of this quadratic differential which is a complete square of a linear one. The differential $Q(u)du^2$ has zeros of order 4 at the points $\pi+2\pi k$ which are images of c under the mapping $u(z)$. Furthermore, $u(0)=\pm\eta_k$, so that $\operatorname{Re}\eta_0=0$ in case $a<1$ or $\operatorname{Re}\eta_0=\pi$ in case $a>1$, and

$$\eta_k=\cos^{-1}\frac{1+a}{1-a},\ k=1,2,\dots,n,\dots.$$

For definiteness, assume now $a<1$. Then $u(\infty)=\theta_k=\cos^{-1}(1+c)/(1-c)$, $\theta_0\in(0,\pi)$ and θ_k, η_k are the poles of second order. The points $u(a)=2\pi k$ are regular for this differential.

Consider a fixed branch of the function $u(z)$ which maps $\overline{\mathbb{C}}\setminus[c,a]$ onto the strip $0<\operatorname{Re}u<\pi$. The circular domain $D_1^u=u(D_1^*)$ is bounded by the critical trajectory of $Q(u)du^2$ starting and ending at π enclosing the real point θ_0. The strip domain $D_2^u=u(D_2^*)$ is bounded by the same trajectory, the imaginary axis, and the straight line $\operatorname{Re}u=\pi$.

Let $\zeta_j(u)$, $j=1,2$ be conformal mappings from the domains D_j^u onto the unit disk $\mathbb{D}$ and the biangle $\mathbb{C}\setminus[0,\infty)$ respectively, such that $\zeta_1(\theta_0)=0$ and $\zeta_2(\eta_0)=0$, $\zeta_2(-\eta_0)=\infty$. These functions satisfy in the domains D_1^u the differential equation

$$t\frac{d\zeta_1(u)}{\zeta_1(u)}=2\pi\sqrt{-Q(u)}du, \tag{4.5}$$

and in the domains D_2^u the differential equation

$$\frac{d\zeta_2(u)}{\zeta_2(u)}=2\pi\sqrt{Q(u)}du, \tag{4.6}$$

or in terms of the parameter z

$$t\left(\frac{d\zeta_1(u(z))}{\zeta_1(u(z))}\right)^2=-4\pi^2\varphi(z)dz^2. \tag{4.7}$$

$$\left(\frac{d\zeta_2(u(z))}{\zeta_2(u(z))}\right)^2=4\pi^2\varphi(z)dz^2. \tag{4.8}$$

Letting $z\to\infty$ in (4.7) in the case of $j=1$ or $z\to0$ in (4.8) in the case of $j=2$, we obtain $A=t/4\pi^2$ and $c=-a/t^2$.

Now we compute the reduced module of the circular domain. The part $[\theta_0+\delta,\pi]$ of the orthogonal trajectory of the differential $Q(u)du^2$ for sufficiently small δ has the preimage $[-1/\varepsilon_1,c]$ under the mapping $u(z)$. From (4.3) we derive

$$\delta=\frac{\sqrt{-c}(a-c)}{1-c}\varepsilon_1+O(\varepsilon_1^2). \tag{4.9}$$

These two segments have as their image in the ζ-plane the segment $[\varepsilon e^{i\beta}, e^{i\beta}]$. Without loss of generality assume $\beta = 0$.

Let $z = f_1(\zeta) = A_{-1}/\zeta + A_0 + A_1\zeta + \dots$ be the function from $\mathbb{D}$ onto D_1^*. Then the reduced module of D_1^* can be computed as $m(D_1^*, \infty) = \frac{1}{2\pi} \log 1/|A_1|$. We calculate directly from (4.4)

$$\begin{aligned}\sqrt{-Q(u)} &= 2\sqrt{-cA}\left(\frac{1}{c+1+(c-1)\cos u} - \frac{1}{a+1+(a-1)\cos u}\right)\\ &= \pm\frac{1}{2\pi}\cdot\frac{d}{du}\left(t\log\frac{t\tan\frac{u}{2}-\sqrt{a}}{t\tan\frac{u}{2}+\sqrt{a}} - 2\tan^{-1}\frac{\tan\frac{u}{2}}{\sqrt{a}}\right).\end{aligned} \tag{4.10}$$

We choose the branch of the root such that we have $(+)$ in front of the previous expression. Moreover, $\tan(\theta_0/2) = \sqrt{-c} = \sqrt{a}/t$. Integrating (4.5) along the segments described we derive

$$\begin{aligned}\varepsilon &= \frac{t\tan\frac{\theta_0+\delta}{2} - \sqrt{a}}{t\tan\frac{\theta_0+\delta}{2} + \sqrt{a}}\cdot\exp\left(\frac{2}{t}\left(\frac{\pi}{2} - \tan^{-1}\frac{\tan\frac{\theta_0+\delta}{2}}{\sqrt{a}}\right)\right)\\ &= \frac{(a+t^2)}{4t\sqrt{a}}\exp\left(\frac{2}{t}\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{t}\right)\right)\cdot\delta + O(\delta^2)\end{aligned}$$

and finally using (4.9) we obtain

$$A_1 = \frac{a(1+t^2)}{4t^2}\exp\left(\frac{2}{t}\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{t}\right)\right).$$

Then the module $m(D_1^*, \infty)$ has the form stated in Theorem 4.1.

Next we calculate the reduced module of the biangle D_2^* with respect to its corners with the same support 0. To this end we consider the strip domain D_2^u in the u-plane and the segment of the imaginary axis $[0, \eta_0 - i\delta]$ that belongs to the critical trajectory of the differential $Q(u)du^2$. It has the preimage $[\varepsilon_1, a]$ in z-plane under the mapping $u(z)$. From (4.3) we obtain

$$\delta = \frac{c-a}{c\sqrt{a}(a-1)}\varepsilon_1 + O(\varepsilon_1^2). \tag{4.11}$$

For these two segments there is an image in the ζ-plane $[1, 1/\varepsilon]$ that belongs to the boundary of the biangle $\mathbb{C}\setminus[0,\infty)$ which is the image of the domain D_2^* under the map $\zeta_2(u(z))$. We find that the length of the segments $[\varepsilon, 1]$ and $[1, 1/\varepsilon]$ is equal in the metric $|d\zeta|/|\zeta|$ and, therefore, $1 = \zeta_2(0)$. Let $z = f_2(\zeta) = B_{-1}/\zeta + B_0 + B_1\zeta + \dots$ be the conformal map from $\mathbb{C}\setminus[0,\infty)$ onto D_2^*. Then the reduced module of D_2^* turns out to be

$m(D_2^*,0,0)=\frac{2}{\pi}\log|B_{-1}|$. Here we understand this derivative as one of the angular derivatives at D_2^*.

By (4.10) we have

$$\sqrt{Q(u)}=\pm\frac{i}{2\pi}\cdot\frac{d}{du}\left(t\log\frac{t\tan\frac{u}{2}-\sqrt{a}}{t\tan\frac{u}{2}+\sqrt{a}}-2\tan^{-1}\frac{\tan\frac{u}{2}}{\sqrt{a}}\right). \tag{4.12}$$

Again we choose the branch of the root such that we have (+) in front of the right-hand side of (4.12). We have $\tan(\eta_0/2)=i\sqrt{a}$. Rewrite the equation (4.6) as

$$\frac{d\zeta}{\zeta}=i\frac{d}{du}\left(t\log\frac{t\tan\frac{u}{2}-\sqrt{a}}{t\tan\frac{u}{2}+\sqrt{a}}-2\tan^{-1}\frac{\tan\frac{u}{2}}{\sqrt{a}}\right)du.$$

Since we use the complex tangent, we better transfer the right-hand side using the transform identity

$$\tan^{-1}w=\frac{1}{2i}\log\frac{1+iw}{1-iw}.$$

Then,

$$\frac{d\zeta}{\zeta}=\frac{d}{du}\left(-2t\tan^{-1}\frac{i\sqrt{a}}{t\,\tan\frac{u}{2}}-\log\frac{1+i\frac{\tan\frac{u}{2}}{\sqrt{a}}}{1-i\frac{\tan\frac{u}{2}}{\sqrt{a}}}\right). \tag{4.13}$$

Integrating (4.13) along the segment $[1,1/\varepsilon]$ on the left-hand side and along the vertical segment $[0,\eta_0-i\delta]$, $\operatorname{Im}\eta_0>0$, on the right-hand side, we deduce

$$\begin{aligned}\varepsilon&=\frac{1+i\frac{\tan\frac{\eta_0-i\delta}{2}}{\sqrt{a}}}{1-i\frac{\tan\frac{\eta_0-i\delta}{2}}{\sqrt{a}}}\cdot\exp\left(2t\left(\tan^{-1}\frac{i\sqrt{a}}{t\,\tan\frac{\eta_0-i\delta}{2}}-\frac{\pi}{2}\right)\right)\\&=\frac{1-a}{4\sqrt{a}}\exp\left(2t\left(\tan^{-1}\frac{1}{t}-\frac{\pi}{2}\right)\right)\delta+O(\delta^2).\end{aligned}$$

Finally using (4.9) and substituting $c=-a/t^2$ we obtain

$$|B_1|=\frac{4a}{1+t^2}\exp\left(2t\left(\frac{\pi}{2}-\tan^{-1}\frac{1}{t}\right)\right).$$

This leads to the expressions in Theorem 4.1. The case $a>1$ can be obtained by applying the mapping $w=kz$, where $k>1/a$. This leads to the same expressions. □

Let $S_1 = \mathbb{C} \setminus \{c_1, 0, a_1\}$, $a_1 > 0$, $c_1 < 0$, be the thrice-punctured complex plane. We consider on S_1 the admissible system (γ_1, γ_2) of I and II type respectively where $\gamma_1 = \{w : |w| = 1/\varepsilon\}$ and $\gamma_2 = \{w : |w - a_1| = a_1\}$, so that $r > a_1$ and ε is sufficiently small. Let $\mathfrak{B}$ be the set of all pairs (B_1, B_2) consisting of a doubly connected parabolic domain (or a punctured simply connected domain) and a simply connected domain of homotopy type (γ_1, γ_2). Then the problem of extremal partition of S_1 consists in finding the maximum of the sum $t_1^2 m(B_1, \infty) - t_2^2 m(B_2, 0, 0)$ as $(B_1, B_2) \in \mathfrak{D}$. Without loss of generality, assume $t_1 = t$, $t_2 = 1$, $t \in [0, \infty)$, and denote the maximum of this sum by $\mathcal{M}(t, c_1, a_1)$. There is a unique pair (B_1^*, B_2^*) that is extremal in this problem. B_1^* is a circular domain and B_2^* is a strip domain in the trajectory structure of the differential

$$\psi(z)dz^2 = -A\frac{(z-b)^2 dz^2}{z^2(z-a_1)(z-c_1)}, \;\; A > 0, \;\; b \le 0. \tag{4.14}$$

Here A and b are functions of t. For $t \in [0, \sqrt{\frac{a_1}{-c_1}}]$ the problem can be reduced to that in the previous case with $a = a_1$. If $t \to \infty$, then $B_1^* = \mathbb{C} \setminus [c_1, a_1]$. In this case $\mathcal{M}(\infty, c_1, a_1) = \frac{1}{2\pi} \log 4/(a_1 - c_1)$.

Theorem 4.2. *Let* $\sqrt{\frac{a_1}{-c_1}} \le t < \infty$. *Then*

$$m(B_1^*, \infty) = \frac{1}{2\pi} \log \frac{4}{a_1 - c_1} - \frac{1}{\pi t}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{-c_1}{a_1}}\right),$$

$$m(B_2^*, 0, 0) = \frac{2}{\pi} \log \frac{4a_1 c_1}{a_1 - c_1} + \frac{4t}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{-c_1}{a_1}}\right).$$

Proof. As in the previous theorem, we consider the mapping $u = u(z)$ whose inverse is

$$z = c_1 \frac{a_1 + 1 + (a_1 - 1)\cos u}{(c_1 + 1) + (c_1 - 1)\cos u},$$

and obtain the representation of the differential ψ in terms of the parameter u at regular points

$$\begin{aligned} \psi(z)dz^2 &= \Phi(u)du^2 \\ &= \frac{4A}{c_1}\frac{(c_1(a_1+1) - b(c_1+1) + (c_1(a_1-1) - b(c_1-1))\cos u)^2}{((c+1) + (c-1)\cos u)^2((a+1) + (a-1)\cos u)^2} du^2. \end{aligned} \tag{4.15}$$

Here $\left|\frac{a_1+1}{a_1-1}\right| > 1$, $\left|\frac{c_1+1}{c_1-1}\right| \le 1$, and $\left|\frac{c_1(a_1+1)-b(c_1+1)}{c_1(a_1-1)-b(c_1-1)}\right| > 1$. Now we study the trajectory structure of this quadratic differential which is a square of a linear differential. The differential $Q(u)du^2$ has zeros of order 2 at the points $\pm\gamma_k+2\pi k$ which are the images of $[c_1(a_1+1)-b(c_1+1)]/[c_1(a_1-1)-b(c_1-1)]$ under the mapping $u(z)$. Furthermore, $u(0)=\pm\eta_k$, so that $\operatorname{Re}\eta_0=0$ in case $a_1<1$ or $\operatorname{Re}\eta_0=\pi$ in case $a_1>1$, and

$$\eta_k=\cos^{-1}\frac{1+a}{1-a},\ k=1,2,\dots,n,\dots.$$

For definiteness, assume now $a_1<1$. The case $a_1>1$ can be considered as in Theorem 4.1. Then, $u(\infty)=\theta_k=\cos^{-1}(1+c)/(1-c)$, $\theta_0\in(0,\pi)$ and θ_k, η_k are the poles of second order. The points $u(a_1)=2\pi k$ are regular for this differential.

Consider a fixed branch of the function $u(z)$ which maps $\overline{\mathbb{C}}\setminus[c,a]$ onto the strip $0<\operatorname{Re}u<\pi$. The circular domain $D_1^u=u(D_1^*)$ is bounded by the critical trajectory of $Q(u)du^2$ starting and ending at π, enclosing the real point θ_0. The strip domain $D_2^u=u(D_2^*)$ is bounded by the same trajectory, the imaginary axis, and the straight line $\operatorname{Re}u=\pi$.

Let $\zeta_j(u)$, $j=1,2$ be univalent conformal mappings from the domains B_j^u onto the unit disk $\mathbb{D}$ and the biangle $\mathbb{C}\setminus[0,\infty)$ respectively, such that $\zeta_1(\theta_0)=0$ and $\zeta_2(\eta_0)=0$, $\zeta_2(-\eta_0)=\infty$. These functions satisfy in the domains B_1^u the differential equation

$$t\frac{d\zeta_1(u)}{\zeta_1(u)}=2\pi\sqrt{-\Phi(u)}du, \tag{4.16}$$

and in the domains B_2^u the differential equation

$$\frac{d\zeta_2(u)}{\zeta_2(u)}=2\pi\sqrt{\Phi(u)}du, \tag{4.17}$$

or in terms of the parameter z

$$t\left(\frac{d\zeta_1(u(z))}{\zeta_1(u(z))}\right)^2=-4\pi^2\psi(z)dz^2. \tag{4.18}$$

$$\left(\frac{d\zeta_2(u(z))}{\zeta_2(u(z))}\right)^2=4\pi^2\psi(z)dz^2. \tag{4.19}$$

Letting $z\to\infty$ in (4.18) in the case of $j=1$ or $z\to 0$ in (4.19) in the case of $j=2$ we obtain $A=t/4\pi^2$ and $b=-\sqrt{-a_1c_1}/t$.

As in Theorem 4.1, an analogous calculation gives us

$$\begin{aligned}\sqrt{-\Phi(u)}&=\frac{2\sqrt{A}}{\sqrt{-c_1}}\left(\frac{c_1}{c_1+1+(c_1-1)\ cos\,u}-\frac{b}{a_1+1+(a_1-1)\cos u}\right)\\&=\frac{1}{2\pi}\cdot\frac{d}{du}\left(-t\log\frac{\tan\frac{u}{2}-\sqrt{-c_1}}{\tan\frac{u}{2}+\sqrt{-c_1}}+2\tan^{-1}\frac{\tan\frac{u}{2}}{\sqrt{a_1}}\right).\end{aligned}$$

Taking into account

$$\delta = \frac{(a_1 - c_1)\sqrt{-c_1}}{1 - c_1}\varepsilon_1 + O(\varepsilon_1^2),$$

and integrating (4.16, 4.17) as before, we obtain the expressions in Theorem 4.2. □

Let $U' = \mathbb{D} \setminus \{0, w\}$, $|w| < 1$ be the punctured unit disk. We consider on U' the admissible curve system $(\gamma_1, \gamma_2^{(n)})$, where $\gamma_1 = \{z : |z| = \varepsilon\}$ and $\gamma_2^{(n)}$ belongs to the countable set of arcs with certain homotopy on U' with starting and ending points at 1, enclosing w and such that $\gamma_1 \cap \gamma_2^{(n)} = \emptyset$. By means of $n = 1$ we suppose that $\gamma_2^{(1)}$ is homotopic to the segment $[1, w]$. Here $\varepsilon < |w|$ is sufficiently small. Let $\mathfrak{D}^{(n)}$ be the set of all pairs $(D_1^{(n)}, D_2^{(n)})$ consisting of the domains in U' of homotopy type $(\gamma_1, \gamma_2^{(n)})$. Then the problem on extremal partition of U' consists of maximizing the sum $t^2 m(D_1^{(n)}, 0) - m(D_2^{(n)}, 1, 1)$ as $(D_1^{(n)}, D_2^{(n)}) \in \mathfrak{D}^{(n)}$. The maximum of this sum is denoted by $M_w^{(n)}(t, w)$.

Lemma 4.2. *In the family $\mathfrak{D}^{(n)}$ the inequality*

$$M_w^{(n)}(t, w) \le M_w^{(1)}(t, w)$$

holds for all t and $n = 2, 3, \ldots$.

Proof. Without loss of generality we assume $\operatorname{Im} w > 0$. Now we are going to apply the results about polarization of doubly connected domains as in [13], [15]. To this end we construct the pair of doubly connected domains $({D_1}_\varepsilon^{(n)}, {D_2}_\varepsilon^{(n)})$ where ${D_1}_\varepsilon^{(n)}$ is the extremal circular domain ${D_1^{(n)}}^*$ in the above module problem minus the disk $|z| < \varepsilon$, and ${D_2}_\varepsilon^{(n)}$ is the extremal strip domain ${D_2^{(n)}}^*$ minus the disk $|z - 1| < \varepsilon$ plus the symmetric image of this quadrangle with respect to the circle $|z - 1| = \varepsilon$. Now we apply polarization to the domains $({D_1}_\varepsilon^{(n)}, {D_2}_\varepsilon^{(n)})$ with respect to the real axis for $n \ge 2$. We obtain as a result the pair of non-overlapping doubly connected domains $(\tilde{D}_1^\varepsilon, \tilde{D}_2^\varepsilon)$ with the modules $M({D_1}_\varepsilon^{(n)}) \le M(\tilde{D}_1^\varepsilon)$, $M({D_2}_\varepsilon^{(n)}) \le M(\tilde{D}_2^\varepsilon)$. Moreover, the part of $\tilde{D}_2^\varepsilon$ lying outside the disk $|z - 1| < \varepsilon$ is still symmetric to that inside. That is why the same inequality is true for the module of the quadrangle $\tilde{D}_2^\varepsilon \setminus \{|z - 1| < \varepsilon\}$ inside $\mathbb{D}$. Letting $\varepsilon \to 0$ we obtain the pair of domains (D_1^0, D_2^0) and the inequality $M^{(n)}(t, w) \le t^2 m(D_1^0, 0) - m(D_2^0, 1, 1)$. The pair (D_1^0, D_2^0) is admissible for the family $\mathfrak{D}^{(1)}$. Therefore, $t^2 m(D_1^0, 0) - m(D_2^0, 1, 1) \le t^2 m({D_1^{(1)}}^*, 0) - m({D_2^{(1)}}^*, 1, 1)$. This completes the proof. □

5. Application to an extremal problem. Now let $U_z = \mathbb{D} \setminus \{0, r\}$, $r \in (0,1)$ be the punctured unit disk. We consider on U_z the admissible curve system (γ_1^z, γ_2^z), where $\gamma_1^z = \{z : |z| = \varepsilon\}$ and γ_2^z is an arc with starting and ending points at 1, enclosing r such that $\gamma_1^z \cap \gamma_2^z = \emptyset$. Here $\varepsilon < r$ is sufficiently small. Let $\mathfrak{D}^z$ be the set of all pairs (D_1^z, D_2^z) consisting of the domains in U_z of homotopy type (γ_1^z, γ_2^z). Then the problem of the extremal partition of U_z consists of finding the maximum of the sum $t^2 m(D_1^z, 0) - m(D_2^z, 1, 1)$ as $(D_1^z, D_2^z) \in \mathfrak{D}^z$. We denote the maximum of this sum by $M_z(t, r)$. Under the transformation $Z(z) = (1-z)^2/z$ two extremal domains (D_1^{z*}, D_2^{z*}) in the problem of $M_z(t,r)$ are mapped onto two extremal domains (B_1^*, B_2^*) in the problem of finding $\mathcal{M}(t, c_1, a_1)$ where $c_1 = -4$, $a_1 = (1-r)^2/r$. Taking into account the change of the reduced modules, we derive from Theorem 4.2 that for $t \geq \frac{1-r}{2\sqrt{r}}$

$$m(D_1^{z*}, 0) = \frac{1}{2\pi} \log \frac{4r}{(1+r)^2} - \frac{1}{\pi t}\left(\frac{\pi}{2} - \tan^{-1}\frac{2\sqrt{r}}{1-r}\right), \tag{5.1}$$

$$m(D_2^{z*}, 1, 1) = \frac{4}{\pi} \log \frac{4(1-r)}{1+r} + \frac{4t}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\frac{2\sqrt{r}}{1-r}\right). \tag{5.2}$$

Now we consider the same problem of the extremal partition replacing r by $x \in (0, r)$. Denote by (D_1^{w*}, D_2^{w*}) the extremal pair of domains and let t vary within $[0, \infty)$. By $M_w(t, w)$ we denote the maximum $t^2 m(D_1^{w*}, 0) - m(D_2^{w*}, 1, 1)$. For $t \geq \frac{1-x}{2\sqrt{x}}$ we have the expressions given by (5.1), (5.2). For $0 \leq t \leq \frac{1-x}{2\sqrt{x}}$ we deduce from Theorem 4.1

$$m(D_1^{w*}, 0) = \frac{1}{2\pi} \log \frac{4xt^2}{(1-x)^2(1+t^2)} - \frac{1}{\pi t}\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{t}\right), \tag{5.3}$$

$$m(D_2^{w*}, 1, 1) = \frac{2}{\pi} \log \frac{4(1-x)^2}{x(1+t^2)} + \frac{4t}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{t}\right). \tag{5.4}$$

Lemma 5.1. (i) *Let $\beta \geq 1$ be fixed. For $\frac{1-r}{2\sqrt{r}} \leq t < \infty$ and $w = x \in (0, r)$ the equation*

$$m(D_2^{z*}, 1, 1) + \frac{4}{\pi} \log \beta = m(D_2^{z*}, 1, 1) \tag{5.5}$$

defines the unique solution $x = x(t)$ that belongs to the interval $(0, r)$ for t fixed. The function $x(t)$ is differentiable and increases in t from $x(\frac{1-r}{2\sqrt{r}}) =:$ x_1 which is the solution of the equation

$$\frac{x_1}{(1-x_1)^2} = \frac{1}{\beta^2}\frac{r}{(1-r)^2},$$

to $\lim_{t\to\infty} x(t) = r$.

(ii) *For* $\frac{1-r}{2\sqrt{r}} \le t \le t_0$ *this solution* $x(t)$ *is defined by the equation*

$$\frac{x}{(1-x)^2} = \frac{1}{4\beta^2}\left(\frac{1+r}{1-r}\right)^2 \frac{1}{1+t^2} \exp\left[2t\left(\tan^{-1}\frac{2\sqrt{r}}{1-r} - \tan^{-1}\frac{1}{t}\right)\right] \tag{5.6}$$

as $x \in [x_1, r)$.

For $t_0 \le t < \infty$ *the solution* $x(t)$ *is defined by the equation*

$$\log \beta \cdot \frac{1+x}{1-x} \cdot \frac{1-r}{1+r} = t\left(\tan^{-1}\frac{2\sqrt{r}}{1-r} - \tan^{-1}\frac{2\sqrt{x}}{1-x}\right), \tag{5.7}$$

as $x \in [x_1, r)$.

Here t_0 *is the unique solution of the equation* $P(t) = 0$, *where*

$$P(t) := \log \frac{1-r}{1+r}\beta\frac{\sqrt{1+t^2}}{t} + t\left(\tan^{-1}\frac{1}{t} - \tan^{-1}\frac{2\sqrt{r}}{1-r}\right).$$

Proof. First we consider the case $\frac{1-r}{2\sqrt{r}} \le t \le \frac{1-x}{2\sqrt{x}}$. The equation (5.5) and the formulas (5.2), (5.4) imply the equation (5.6). Since the left-hand side of (5.6) is positive, there is always a unique solution of the equation (5.6). Differentiating both sides of (5.6) with respect to t we obtain

$$x'(t)\frac{1+x}{1-x} = \frac{1}{2\beta}\left(\frac{1+r}{1-r}\right)^2 \frac{1}{1+t^2}\left(\tan^{-1}\frac{2\sqrt{r}}{1-r} - \tan^{-1}\frac{1}{t}\right) \times \exp\left[2t\left(\tan^{-1}\frac{2\sqrt{r}}{1-r} - \tan^{-1}\frac{1}{t}\right)\right].$$

Therefore, $x'(t) > 0$ and the function $x(t)$ increases in $\frac{1-r}{2\sqrt{r}} \le t \le \frac{1-x}{2\sqrt{x}}$.

For $t = \frac{1-r}{2\sqrt{r}}$ we have $x(\frac{1-r}{2\sqrt{r}}) = x_1 < r$. For $t = \frac{1-x}{2\sqrt{x}}$ the equation (5.6) has the form

$$H(x) := \frac{1+x}{1-x} - \frac{1}{\beta}\cdot\frac{1+r}{1-r}\exp\left[\frac{1-x}{2\sqrt{x}}\left(\tan^{-1}\frac{2\sqrt{r}}{1-r} - \tan^{-1}\frac{2\sqrt{x}}{1-x}\right)\right] = 0,$$

where

$$H(r) = \frac{1+r}{1-r}\left(1 - \frac{1}{\beta}\right) \ge 0, \quad \lim_{x\to 0} H(x) = -\infty.$$

Calculating the derivative we obtain

$$H'(x) = \frac{2}{(1-x)^2} + \frac{1}{2\beta}\cdot\frac{1+r}{1-r}\left(\frac{1+x}{2\sqrt{x}}\left(\tan^{-1}\frac{2\sqrt{r}}{1-r} - \tan^{-1}\frac{2\sqrt{x}}{1-x}\right) + \frac{1-x}{4x(1+x)}\right) \times \exp\left[\frac{1-x}{2\sqrt{x}}\left(\tan^{-1}\frac{2\sqrt{r}}{1-r} - \tan^{-1}\frac{2\sqrt{x}}{1-x}\right)\right],$$

which is positive as $x \in (0, r)$. Therefore, the equation $H(x) = 0$ has the unique solution $x = x_2 \in (0, r)$. Moreover,

$$H(x_1) = \frac{\sqrt{(1+r)^2 - 2r(1-\frac{1}{\beta^2})}}{1-r} - \frac{1+r}{1-r} \cdot \frac{1}{\beta} \exp\left[\frac{\beta}{2}\frac{1-r}{\sqrt{r}}\left(\tan^{-1}\frac{2\sqrt{r}}{1-r} - \tan^{-1}\frac{2\sqrt{r}}{\beta(1-r)}\right)\right] < 0,$$

hence, $x_2 > x_1$.

Finally, the equation $t = \frac{1-x(t)}{2\sqrt{x(t)}}$ leads to the equation $P(t) = 0$. To show that t_0 is the unique solution of the latter equation we note that $P(\frac{1-r}{2\sqrt{r}}) = \log\beta > 0$ and $\lim\limits_{t\to\infty} P(t) = -\infty$. Moreover, $P'(t) = -\frac{1}{t} - \tan^{-1}\frac{2\sqrt{r}}{1-r} + \tan^{-1}\frac{1}{t} < 0$. Therefore, there is a unique solution t_0 of the equation $P(t) = 0$.

Now we consider the case $\frac{1-x}{2\sqrt{x}} \le t < \infty$. The equation (5.5) and the formula (5.2) imply the equation (5.7). Define

$$G(x) := \log\beta\frac{1+x}{1-x}\frac{1-r}{1+r} - t\left(\tan^{-1}\frac{2\sqrt{r}}{1-r} - \tan^{-1}\frac{2\sqrt{x}}{1-x}\right).$$

The equation (5.7) is equivalent to the equation $G(x) = 0$. Calculation shows that

$$G(x_1) = \log\frac{\sqrt{(1+r)^2 - 2r(1-\frac{1}{\beta^2})}}{1+r} + t\left(\tan^{-1}\frac{1}{\beta}\frac{2\sqrt{r}}{1-r} - \tan^{-1}\frac{2\sqrt{r}}{1-r}\right) < 0,$$

$G(r) = \log\beta > 0$. Moreover, $G'(x) > 0$. Therefore, the unique solution of the equation (5.7) exists in the interval (x_1, r). Denote it by $x(t)$. Calculating its derivative we obtain

$$x'(t)\left(\frac{1}{1-x^2} + \frac{t}{(1+x)\sqrt{x}}\right) = \tan^{-1}\frac{2\sqrt{r}}{1-r} - \tan^{-1}\frac{2\sqrt{x}}{1-x}.$$

Since $t \ge \frac{1-x}{s\sqrt{x}}$, the function $x(t)$ increases in t. The condition $t = \frac{1-x(t)}{2\sqrt{x(t)}}$ leads to the same equation $P(t) = 0$. Therefore $x(t_0) = x_2$. This completes the proof. $\square$

Now denote by $\mathcal{M}^1(\beta)$ the class of all univalent maps f from $\mathbb{D}$ to $\mathbb{D}$ with the expansion $f(z) = \alpha z + a_2 z^2 + \dots$ such that $f(1) = 1$ with the fixed angular derivative $|f'(1)| = \beta$.

Theorem 5.1. (i) *The boundary curve* Γ^+ *of the range of the system of functionals* $(|f(r)|, |f'(0)|)$ *over the class* $\mathcal{M}^1(\beta)$ *corresponding to the problem of* $\max|f'(0)|$ *as* $|f(r)|$ *is fixed, consists of the points* (x, α). *The part* Γ_1^+ *of* Γ^+ *over the segment* $x \in [x_1, x_2]$ *is given parametrically* $(x(t), \alpha(t))$ *as* $\frac{1-r}{2\sqrt{r}} \le t \le t_0$, *where* $x(t)$ *is defined in Lemma 5.1 by the equation* (5.6), $x(\frac{1-r}{2\sqrt{r}}) = x_1$, $x(t_0) = x_2$, *and*

$$\alpha(t) = \frac{t^2(1+r)^4}{4\beta^2 r(1-r)^2(1+t^2)^2} \exp\left[2(\frac{1}{t} - t)\left(\tan^{-1}\frac{1}{t} - \tan^{-1}\frac{2\sqrt{r}}{1-r}\right)\right].$$

The part Γ_2^+ *of* Γ^+ *over the interval* $[x_2, r)$ *is given explicitly by the formula*

$$\alpha = \frac{x(1+r)^2}{r(1+x)^2} \exp\left[\frac{2\left(\tan^{-1}\frac{2\sqrt{r}}{1-r} - \tan^{-1}\frac{2\sqrt{x}}{1-x}\right)}{\log\frac{\beta(1-r)(1+x)}{(1+r)(1-x)}}\right].$$

(ii) *Each point of the curve* Γ^+ *is given by a unique function from the class* $\mathcal{M}^1(\beta)$. *The point* $(x(\frac{1-r}{2\sqrt{r}}), \alpha(\frac{1-r}{2\sqrt{r}})) = (x_1, \frac{1}{\beta^2})$ *is given by the canonical function* p_{1/β^2} *that satisfies the identity* (1.4).

(iii) *Each point of* Γ^+ *over the interval* $(x_1, x_2]$ *is given by the function that maps the unit disk* $\mathbb{D}$ *onto* $\mathbb{D}$ *slit along the negative real segment* $[-1, \frac{(1-c(t))^2}{c(t)}]$ *and two analytic arcs starting from* $\frac{(1-c(t))^2}{c(t)}$ *at the angles* $\frac{2\pi}{3}$. *This function satisfies the differential equation*

(5.8)
$$\frac{(w+1)^2(w - \frac{(1-c(t))^2}{c(t)})(w - \frac{c(t)}{(1-c(t))^2})dw^2}{w^2(w-x(t))(w-1/x(t))(w-1)^2} = \frac{(z-d(t))^2(z-\overline{d(t)})^2dz^2}{z^2(z-r)(z-1/r)(z-1)^2},$$

$t \in (\frac{1-r}{2\sqrt{r}}, t_0)$. *Here* $d(t)$ *and* $\overline{d(t)}$ *are two conjugate roots of the equation*

$$\frac{(1-d)^2}{d} = -\frac{2(1-r)}{t^2\sqrt{r}}, \quad |d(t)| = 1, \quad c(t) = \frac{-(1-x(t))^2}{t^2 x(t)}.$$

(iv) *Each point of* Γ^+ *over the interval* (x_2, r) *is given by the function that maps the unit disk* $\mathbb{D}$ *onto* $\mathbb{D}$ *slit along two analytic arcs symmetric with respect to the real axis starting orthogonally at the points* $h(t)$ *of* $\mathbb{T}$. *This function satisfies the differential equation*

$$(5.9) \qquad \frac{(w-h(t))^2(w-\overline{h(t)})^2dw^2}{w^2(w-x(t))(w-1/x(t))(w-1)^2} = \frac{(z-d(t))^2(z-\overline{d(t)})^2dz^2}{z^2(z-r)(z-1/r)(z-1)^2},$$

where $h(t)$ and $\overline{h(t)}$ are two conjugate roots of the equation

$$\frac{(1-h)^2}{h} = -\frac{2(1-x(t))}{t^2\sqrt{x(t)}}, \ \ |h(t)| = 1.$$

Proof. We start constructing the extremal maps. Let $t \in [\frac{1-r}{2\sqrt{r}}, t_0]$. Assume $a_1 = \frac{(1-r)^2}{r}$, $c_1 = -4$, $a = \frac{(1-x(t))^2}{x(t)}$. Then the differential equation (5.8) is equivalent to the equation $\psi(Z)dZ^2 = \varphi(W)dW^2$ under the transformations

$$Z = \frac{(1-z)^2}{z}, \ \ W = \frac{(1-w)^2}{w},$$

where the differentials φ and ψ are defined in (4.2) and (4.14). Now construct the functions $f_2(Z)$ and $F_2(W)$ that map the domains B_2^* and D_2^* respectively onto the same strip domain $\mathbb{C} \setminus [0, \infty)$. Due to (4.8) and (4.19) they satisfy the equations

$$\left(\frac{d\, f_2(Z)}{f_2(Z)}\right)^2 = 4\pi^2\varphi(Z)dZ^2, \ \ \left(\frac{d\, F_2(W)}{F_2(W)}\right)^2 = 4\pi^2\psi(W)dW^2.$$

Then we construct the map $w = f^*(z) \equiv W^{-1} \circ F_2^{-1} \circ f_2 \circ Z(z)$ from the domain D_2^{z*} onto D_2^{w*}. This map can be continued analytically by the equation (5.8) into D_1^{z*} through the analytic arc of the trajectory of the right-hand side differential in (5.8) connecting $d(t)$ and $\overline{d(t)}$. Calculating the derivatives and taking into account Lemma 5.1 we deduce that $|f^{*\prime}(0)| = \alpha(t)$, $|f^{*\prime}(1)| = \beta$, $|f^*(r)| = x(t)$. The same we do in the case $t_0 < t < \infty$. In this case

$$\alpha(t) = \frac{x(t)}{r} \cdot \frac{(1+r)^2}{(1+x(t))^2} \cdot \exp\left[\frac{2}{t}\left(\tan^{-1}\frac{2\sqrt{x(t)}}{1-x(t)} - \tan^{-1}\frac{2\sqrt{r}}{1-r}\right)\right],$$

and we can obtain the explicit formula substituting t by Lemma 5.1.

Let f be an arbitrary map from the class $\mathcal{M}^1(\beta)$. Then two extremal domains (D_1^{z*}, D_2^{z*}) in the problem about $M_z(t, r)$ are mapped onto two admissible domains $(f(D_1^{z*}), f(D_2^{z*}))$ in the problem about $M_w^{(n)}(t, f(r))$ for some (n) and

$$\begin{aligned}(5.10)\quad t^2 m(D_1^{z*}, 0) - m(D_2^{z*}, 1, 1) + \frac{t^2}{2\pi}\log|f'(0)| - \frac{4}{\pi}\log\beta \\ = t^2 m(f(D_1^{z*}), 0) - m(fD_2^{z*}), 1, 1) \leq M_w^{(n)}(t, f(r)).\end{aligned}$$

Since $x(t)$ increases in $t \in [0, \infty)$ from x_1 to r, and $x_1 = \min_{f \in \mathcal{M}^1(\beta)} |f(r)|$, there is $t^* \in [0, \infty)$ such that $x(t^*) = |f(r)|$. Denote by $(\tilde{D}_1^{(n)}, \tilde{D}_2^{(n)})$ the extremal pair of domains in the problem of $M_w^{(n)}(t, f(r))$ and by $(\tilde{D}_1, \tilde{D}_2)$ the extremal pair of domains in the problem of $M_w^{(1)}(t, f(r))$. By Lemma 4.2 we have $M_w^{(n)}(t, f(r)) \leq M_w^{(1)}(t, f(r))$. Now we apply circular symmetrization to the domains $(\tilde{D}_1, \tilde{D}_2)$ in the following way. Denote by $\tilde{D}_1^\varepsilon$ the doubly connected domain $\tilde{D}_1 \setminus \{|w| \leq \varepsilon\}$. Now we construct the quadrangle $\tilde{D}_2^\varepsilon$ fixing two arcs of the domain $\tilde{D}_2$ that start from the point 1 lying within the disk $|w-1| < \varepsilon$. Then we apply circular symmetrization in the usual way in J. Jenkins [3] sense to the doubly connected domain $\tilde{D}_1^\varepsilon$ with respect to the positive real axis, and to the quadrangle $\tilde{D}_2^\varepsilon$ with respect to the negative real axis (first we have to construct the symmetric image of $\tilde{D}_2^\varepsilon$ with respect to the unit circle and then apply symmetrization to the doubly connected domain obtained). Denote by (D_1^*, D_2^*) the result of this symmetrization as $\varepsilon \to 0$. Using Lemma 4.1, adding necessary expressions dependent on ε we derive

$$t^2 m(\tilde{D}_1, 0) - m(\tilde{D}_2, 1, 1) \leq t^2 m(\tilde{D}_1^*, 0) - m(\tilde{D}_2^*, 1, 1). \tag{5.11}$$

In its turn, the pair (D_1^*, D_2^*) is admissible in the problem concerning $M_w(t, x(t))$. Therefore,

$$t^2 m(\tilde{D}_1^*, 0) - m(\tilde{D}_2^*, 1, 1) \leq M_w(t, x(t)). \tag{5.12}$$

Taking into account the equality

$$M_w(t, x(t)) = t^2 m(D_1^{z*}, 0) - m(D_2^{z*}, 1, 1) + \frac{t^2}{2\pi} \log \alpha(t) - \frac{4}{\pi} \log \beta,$$

the chain of inequalities (5.10–5.12) leads to the inequality $|f'(0)| \leq \alpha(t)$. The uniqueness of the extremal function f^* follows from the uniqueness of the extremal configuration for the maximum $M_w(t, x(t))$. $\square$

Remark.

1. Since the class $\mathcal{M}^1(\beta)$ is not compact, the point $x(\infty) = r$ is not reachable and the limit function $f(z) \equiv z$ as $t \to \infty$ does not belong to the class $\mathcal{M}^1(\beta)$.

2. Theorem 5.1 gives the sharp lower estimate of $|f(r)|$ over the class $\mathcal{M}^1(\beta)$ with $|f'(0)| = \alpha$ fixed, $\alpha \in [1/\beta^2, 1]$.

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Fachbereich Mathematik
Technische Universtät Berlin
D-10623 Berlin, Germany
e-mail: pommeren@math.tu-berlin.de

received October 15, 1999

Departamento de Matemáticas
Universidad de los Andes
Cll.19, No.1-11, Santafé de Bogotá, Colombia
e-mail: avassill@uniandes.edu.co