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STEPHAN RUSCHEWEYH and LUIS SALINAS

On starlike functions of order $\lambda \in [\frac{1}{2}, 1)$

Dedicated to Professor Z. Lewandowski
on the occasion of his 70th birthday

Abstract. Let zf be starlike of order $\lambda \in [\frac{1}{2}, 1)$, and denote by $s_n(f,z)$ the n-th partial sum of the Taylor expansion of f about the origin. We then prove that

$$\frac{s_n(f,z)}{f(z)} \prec (1-z)^{2\lambda-2}, \quad n \in \mathbb{N}.$$

Applications to Gegenbauer polynomial sums are mentioned, and a new concept of "stable" functions is briefly discussed.

1. Introduction. Let $\mathcal{A}_0$ be the analytic functions f in the unit disk $\mathbb{D}$, normalized by $f(0) = 1$. If $f \in \mathcal{A}$ we say that zf is starlike of order $\lambda < 1$, in symbols $zf \in \mathcal{S}_\lambda$, if

$$\operatorname{Re} \frac{(zf(z))'}{f(z)} > \lambda, \quad z \in \mathbb{D}.$$

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Extremal elements (in various ways) in $\mathcal{S}_\lambda$ are the functions $zf_\lambda = \frac{z}{(1-z)^{2-2\lambda}}$. For $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{A}_0$ we set $s_n(f,z) := \sum_{k=0}^{n} a_k z^k$, the nth partial sum of f. Then it is known (compare [6]) that

$$\frac{s_n(f,z)}{f(z)} \prec \frac{1}{f_{\frac{1}{2}}} \; (= 1 - z), \; zf \in \mathcal{S}_{\frac{1}{2}} \quad n \in \mathbb{N}, \tag{1.1}$$

where $\prec$ stands for the subordination in the unit disk. This result was the crucial ingredient in the extension of the Kakeya-Enestroem Theorem in [6], and therefore it is of interest to ask in which sense it can be generalized. Trivial examples show that (1.1) does not hold for any value $\lambda < \frac{1}{2}$. Starlike functions of order $\lambda \in (\frac{1}{2}, 1)$ have apparently never found much interest in the past, probably since they seem to be comparably narrow classes. Our main result here is that, indeed, (1.1) remains valid when $\frac{1}{2}$ is replaced by λ in that range.

Theorem 1.1. *Let $\lambda \in [\frac{1}{2}, 1)$, $zf \in \mathcal{S}_\lambda$. Then*

$$\frac{s_n(f,z)}{f(z)} \prec \frac{1}{f_\lambda}, \quad n \in \mathbb{N}, \tag{1.2}$$

A consequence of (1.1) was the following result from [6]: *Let $zf \in \mathcal{S}_{\frac{1}{2}}$, but $f(\cdot) \neq f_{\frac{1}{2}}(x\cdot), |x| = 1$. Then for any system $\mu_1, ..., \mu_n$ of non-negative numbers, $\sum_{k=1}^{n} \mu_k = 1$, we have*

$$\sum_{k=1}^{n} \mu_k s_k(f,z) \neq 0, \quad z \in \overline{\mathbb{D}}. \tag{1.3}$$

Our Theorem 1.1 adds to this:

Corollary 1.2. *Let $zf \in \mathcal{S}_\lambda$ for some $\lambda \in [\frac{1}{2}, 1)$. Then, with the notation from above,*

$$\left| \arg \sum_{k=1}^{n} \mu_k s_k(f,z) \right| \leq 2\pi(1-\lambda), \quad z \in \overline{\mathbb{D}}. \tag{1.4}$$

In particular, we have for $zf \in \mathcal{S}_\lambda$, $\lambda \in [\frac{3}{4}, 1)$:

$$\operatorname{Re} s_n(f,z) > 0, \quad z \in \mathbb{D}, \; n \in \mathbb{N}. \tag{1.5}$$

Natural candidates to which these results can be applied are the Gegenbauer polynomial sums

$$P_n^\lambda(z,x) := s_n((1 - 2xz + z^2)^{-\lambda}, z)$$

for $0 < \lambda < \frac{1}{2}$, $-1 \leq x \leq 1$, since

$$\frac{z}{(1-2xz+z^2)^\lambda} = z \sum_{k=0}^{\infty} C_k^\lambda(x) z^k \in \mathcal{S}_{1-\lambda}.$$

C_k^λ are the Gegenbauer polynomials of degree k and order λ. Koumandos [1] proved that the P_k^λ are non-vanishing in the closed unit disk for $0 < \lambda < \frac{1}{2}$. This result, however, is already contained in (1.3) (with $\mu_n = 1$), and can now be sharpened using Corollary 1.2. In particular, we get

Corollary 1.3. *For* $0 < \lambda \leq \frac{1}{4}$, $n \in \mathbb{N}$, $-1 \leq x \leq 1$ *we have*

$$\operatorname{Re} \sum_{k=0}^{n} C_k^\lambda(x) z^k > 0, \quad z \in \mathbb{D}\ .$$

This estimate is not sharp with respect to the upper bound for λ. It follows from the theory of starlike functions that the correct upper bound λ_0 will come from the cases $x = 1$, and numerical experiments indicate that it will be obtained from large n. A reasonable estimate seems to be $\lambda_0 < .35$, but no proof for this is available, and the determination of λ_0 seems to be a technically hard problem.

Since $zf \in \mathcal{S}_\lambda$ if and only if $zf^\alpha \in \mathcal{S}_{1+\alpha(\lambda-1)}$, we obtain yet another form of this generalization of (1.1):

Corollary 1.4. *Let* $zf \in \mathcal{S}_{\frac{1}{2}}, n \in \mathbb{N}$, *and* $0 < \lambda \leq 1$. *Then*

$$\frac{s_n(f^\lambda, z)^{\frac{1}{\lambda}}}{f(z)} \prec 1 - z. \tag{1.6}$$

Taking the limit $\lambda \to 0$ results in the somewhat surprising relation

$$\frac{e^{s_n(\log(f), z)}}{f(z)} \prec 1 - z, \quad n \in \mathbb{N}, \tag{1.7}$$

valid for any $zf \in \mathcal{S}_{\frac{1}{2}}$. For $f = f_{\frac{1}{2}} = (1-z)^{-1}$ this turns into

$$\left|(1-z) e^{\sum_{k=1}^{n} \frac{z^k}{k}} - 1\right| \leq 1, \quad z \in \mathbb{D},\ n \in \mathbb{N}. \tag{1.8}$$

Relation (1.8) is well-known: it is the standard estimate for the *elementary factors* which is instrumental in the theory of the Weierstraß product representation of entire functions (cf. [4, Lemma 15.8]). Therefore, Theorem 1.1 turns out to be just a generalization of this classical inequality of function theory.

Corollary 1.4 can be slightly amended. Using the notations $r_n(f, z) := f(z) - s_{n-1}(f, z)$ and $\mathcal{C}$ for the set of normalized convex univalent functions in $\mathbb{D}$ we arrive at

Theorem 1.5. *Let $g \in \mathcal{C}$, and $h \prec g$. Then*

$$r_n\left(\int_0^z \frac{h(t)}{t}\,dt, z\right) \prec \int_0^z \frac{g(t)}{t}\,dt, \quad n \in \mathbb{N}. \tag{1.9}$$

A simple application of this result is the following general inequality:

Corollary 1.6. *For all $n \in \mathbb{N},\ \tau \geq 1,\ \theta \in \mathbb{R}$ we have*

$$\sum_{k=n}^{\infty} \frac{\cos k\theta}{k^\tau} \geq \sum_{k=1}^{\infty} \frac{(-1)^k}{k^\tau}.$$

This follows from the fact that the functions $g = \sum_{k=1}^{\infty} k^{1-\tau} z^k$ belong to $\mathcal{C}$ (Lewis [3]). We use $h = g$ here.

In the following section we give the proofs for our results, while in the concluding Sect. 3 we introduce and briefly discuss a new concept of "stable" functions.

2. Proofs. We first prove Theorem 1.1 for the cases $f = f_\lambda$. Setting $\alpha := 2 - 2\lambda,\ \alpha \in (0, 1]$, we arrive at an equivalent statement:

Lemma 2.1. *For $\alpha \in (0, 1],\ n \in \mathbb{N},\ z \in \mathbb{D}$ we have*

$$\left|(1-z)\left(s_n\left(\frac{1}{(1-z)^\alpha}, z\right)\right)^{\frac{1}{\alpha}} - 1\right| \leq 1. \tag{2.1}$$

Note that the partial sums involved are non-vanishing by (1.3), so that the expressions under the exponents are non-ambiguous if we assume (and we shall do so in the sequel) that the exponentiated expressions evaluate to 1 in the origin.

We further remark that (2.1) remains valid for $-1 \leq \alpha < 0$ as well. As in our present context we do not have use for that relation, whose proof is somewhat more delicate than for the range presently under consideration, we do not go into the details here. But we refer to Sect. 3.

Proof. For α, n fixed set

$$g_\alpha(z) := \frac{1}{(1-z)^\alpha}, \quad h(z) := 1 - (1-z)s_n(g_\alpha, z)^{\frac{1}{\alpha}}.$$

The relations

$$s_n(g_\alpha, z)' = s_{n-1}(g_\alpha', z), \quad zs_n(g_\alpha, z)' = s_n(zg_\alpha', z),$$

are of a formal nature, and have nothing to do with the special form of g_α. They are readily verified and imply

$$(1-z)s_n(g_\alpha, z)' = s_{n-1}(g_\alpha', z) - s_n(zg_\alpha', z) = s_{n-1}((1-z)g_\alpha', z) - n\frac{(\alpha)_n}{n!}z^n. \tag{2.2}$$

Furthermore we make use of the differential equation

$$g_\alpha + \frac{(1-z)}{\alpha} g'_\alpha = 0.$$

Now using (2.2) and other proper rearrangements we get the following representation

$$\begin{aligned}
h'(z) &= s_n(g_\alpha, z)^{\frac{1}{\alpha}-1} \left(s_n(g_\alpha, z) - \frac{1-z}{\alpha} s_n(g_\alpha, z)' \right) \\
&= s_n(g_\alpha, z)^{\frac{1}{\alpha}-1} \left(s_{n-1}(g_\alpha, z) + \frac{(\alpha)_n}{n!} z^n - s_{n-1}\left(\frac{(1-z)}{\alpha} g'_\alpha, z\right) + n\frac{(\alpha)_n}{\alpha n!} z^n \right) \\
&= s_n(g_\alpha, z)^{\frac{1}{\alpha}-1} \left(s_{n-1}\left(g_\alpha - \frac{1-z}{\alpha} g'_\alpha, z\right) + \frac{(\alpha+1)_n}{n!} z^n \right) \\
&= \frac{(\alpha+1)_n}{n!} z^n s_n(g_\alpha, z)^{\frac{1}{\alpha}-1}.
\end{aligned}$$

Since the Taylor coefficients of g_α are all positive, we find $|s_n(g_\alpha, z)| \leq s_n(g_\alpha, |z|)$ and consequently

$$|h'(z)| \leq h'(|z|), \quad z \in \mathbb{D}.$$

Using $h(0) = 0$ and $h(1) = 1$, we now get

$$|h(z)| = \left| \int_0^z h'(t)dt \right| \leq \int_0^1 |h'(tz)|dt \leq \int_0^1 h'(t)dt = 1, \quad z \in \mathbb{D}, \tag{2.3}$$

the assertion. □

A function zf with $f \in \mathcal{A}_0$ is said to be pre-starlike of order $\lambda < 1$ if the function $zf * zf_\lambda$ is in $\mathcal{S}_\lambda$. Here, and in the sequel the operator $*$ indicates the Hadamard product (convolution) of two analytic functions in $\mathbb{D}$. We shall need the following lemma.

Lemma 2.2. *Let $\lambda < 1$, F pre-starlike of order λ, $G \in \mathcal{S}_\lambda$, and H analytic in $\mathbb{D}$. Then*

$$\frac{F * (GH)}{F * G}(\mathbb{D}) \subset \operatorname{co}(H(\mathbb{D})). \tag{2.4}$$

Here $\operatorname{co}(A)$ stands for the convex hull of a set A. Lemma 2.2 was proved, in various steps of increasing generality, by Ruscheweyh [5], Ruscheweyh & Sheil-Small [8], Suffridge [11], Sheil-Small [10], Lewis [2], and Ruscheweyh [7].

Proof. (Theorem 1.1) There exists a unique F pre-starlike of order λ with $zf = F * zf_\lambda$. Using this and Lemma 2.2 we get

$$\frac{s_n(f,z)}{f(z)} = \frac{F(z) * \left(zf_\lambda(z)\dfrac{s_n(f_\lambda,z)}{f_\lambda(z)}\right)}{F(z) * zf_\lambda(z)} \in \operatorname{co}\left(\frac{s_n(f_\lambda)}{f_\lambda}(\mathbb{D})\right). \tag{2.5}$$

The function $1/f_\lambda$ is in $\mathcal{A}_0$ and convex univalent in $\mathbb{D}$. Furthermore, its range contains the set on the right hand side of (2.5) by Lemma 2.1. Hence we get

$$\frac{s_n(f,z)}{f(z)} \prec \frac{1}{f_\lambda}, \quad n \in \mathbb{N},$$

the assertion. □

Proof. (Corollary 1.2) Taking convex combinations of the left hand side of (1.2) we find

$$\sum_{k=1}^{n} \mu_k s_k(f,z) = \left(\frac{1-w(z)}{1-z}\right)^{2-2\lambda}, \quad z \in \mathbb{D},$$

where w is analytic in $\mathbb{D}$ with $|w(z)| \le |z|$. The relation (1.4) follows readily, and so does (1.5) for $\lambda \geq \frac{3}{4}$. □

For the proof of Theorem 1.5 we shall need the following lemma.

Lemma 2.3 (Ruscheweyh & Stankiewicz [9]). *Let* $f_1, f_2 \in \mathcal{C}$, *and* $h_j \prec f_j$, *for* $j = 1, 2$. *Then* $h_1 * h_2 \prec f_1 * f_2$.

Proof. (Theorem 1.5) The relation (1.8) is equivalent to

$$r_n\left(\log\frac{1}{1-z}\right) \prec \log\frac{1}{1-z}, \quad n \in \mathbb{N}.$$

The result follows from Lemma 2.3 using the facts that $\log\frac{1}{1-z} \in \mathcal{C}$, and

$$h * \log\frac{1}{1-z} = \int_0^z \frac{h(t)}{t}\,dt. \quad \square$$

3. Stable functions.

Inspired by Theorem 1.1 and related results for other functions we call a function $F \in \mathcal{A}_0$ *n-stable with respect to* $G \in \mathcal{A}_0$ if

$$\frac{s_n(F,z)}{F(z)} \prec \frac{1}{G(z)} \tag{3.1}$$

holds for some $n \in \mathbb{N}$. In particular, we call F *n-stable* if it is n-stable with respect to itself. If F is *n-stable* (w.r.t. G) for every n then we call it just *stable* (w.r.t. G).

Our results show that the functions $f_\lambda,\ \lambda \in [\frac{1}{2}, 1)$ are stable, while f/z is stable with respect to f_λ for any $f \in \mathcal{S}_\lambda$ in that same range. It is an interesting question to find other examples of stable functions. For instance, the extension to $\alpha \in [-1, 0)$ in Lemma 2.1 reveals that $(1-z)^\alpha$ is stable for $-1 \leq \alpha \leq 1$. Another function which is known to be stable is $f(z) = \sqrt{\frac{1+z}{1-z}}$, and it is conjectured that the same holds for $\left(\frac{1+z}{1-z}\right)^\alpha,\ 0 < \alpha < \frac{1}{2}$. These questions and some related ones will be discussed in a forthcoming paper.

References

[1] Koumandos, S., *On certain Gegenbauer polynomial sums*, preprint.

[2] Lewis, J.L., *Convolutions of starlike functions*, Indiana Univ. Math. J. **27** (1978), 671–688.

[3] ———, *Convexity of a certain series*, J. Lond. Math. Soc. (2) **27** (1983), 435–446.

[4] Rudin, W., *Real and Complex Analysis*, 2nd ed., McGraw-Hill, New York, 1974.

[5] Ruscheweyh, St., *Über die Faltung schlichter Funktionen*, Math. Z. **128** (1972), 85–92.

[6] ———, *On the Kakeya-Eneström theorem and Gegenbauer polynomial sums*, SIAM J. Math. Anal. **9** (1978), 682–686.

[7] ———, *Linear operators between classes of prestarlike functions*, Comment. Math. Helv. **52** (1977), 497–509.

[8] ———, T.B. Sheil-Small, *Hadamard products of schlicht functions and the Pólya-Schoenberg conjecture*, Comment. Math. Helv. **48** (1973), 119–135.

[9] ———, J. Stankiewicz, *Subordination under convex univalent functions*, Bull. Polish Acad. Sci. Math. **33** (1985), 499-502.

[10] Sheil-Small, T.B., *The Hadamard product and linear transformations of classes of analytic functions*, J. Anal. Math. **34** (1978), 204–239.

[11] Suffridge, T.J., *Starlike functions as limits of polynomials*, Lecture Notes in Math., 505, Springer, 1973/74, pp. 164–202.

Mathematisches Institut
Universität Würzburg
D-97074 Würzburg, Germany
e-mail: ruscheweyh@mathematik.uni-wuerzburg.de

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Departamento de Informática
Universidad Técnica F. Santa María
Valparaíso, Chile
e-mail: lsalinas@inf.utfsm.cl