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On the Möbius distance

Dedicated to Professor Zdzisław Lewandowski
on the occasion of his 70th birthday

Abstract. In this paper we try to initiate an analytic approach of an extension of Schwarz's Lemma to holomorphic functions defined on multiply connected domains as studied by Ahlfors and Grunsky.

0. Introduction. The Möbius resp. the Carathéodory (pseudo)distance has its origin in Complex Analysis of $\mathbb{C}^n$ (cf. [2]). Let $G \subset \mathbb{C}$ be a finitely-connected bounded domain in the complex plane $\mathbb{C}$. By $\mathbf{H}_1(G)$ we denote the set of holomorphic functions $f : G \to \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Then the Möbius (pseudo)distance is given by

$$c_G^*(z,w) := \sup\left\{\left|\frac{f(z)-f(w)}{1-f(z)\overline{f(w)}}\right| : f \in \mathbf{H}_1(G)\right\} \qquad (z,w \in G)$$

and the Carathéodory (pseudo)distance is defined as (cf. [2])

$$c_G(z,w) = \tanh^{-1}(c_G^*(z,w)).$$

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In the case under consideration of a plane domain G both are not only pseudodistances but really metrics.

Without loss of generality we may assume $w = 0 \in G$. By the SCHWARZ-PICK-Lemma we have

$$c(z) := c_G^*(z,0) = \sup\{|f(z)| : f \in \mathbf{H}_1(G) \text{ and } f(0) = 0\}.$$

For fixed $z \in G$ MONTELs Theorem gives us the existence of some $f \in \mathbf{H}_1(G)$ with $f(0) = 0$ and $f(z) = c(z)$. In this case we call f an *extremal function* with respect to z.

In his articles [4] and (mainly in) [5] GRUNSKY characterized these extremal functions, but the proofs are hard to read and harmonic functions are essentially involved. The same is true concerning an article of AHLFORS [1] which deals with the associated problem to maximize $|f'(0)|$ in the class $\mathbf{H}_1^0(G) := \{f \in \mathbf{H}_1(G) : f(0) = 0\}$. This has also been treated by GRUNSKY in his first paper on these extremal problems. Later on, GRUNSKY published his textbook [6], which also includes the mentioned proof.

Of course it would be desirable to derive the results purely analytically. In this paper we introduce two propositions which could give us the key. Unfortunately we are not yet able to prove them purely analytically in full generality, at least not without refering to GRUNSKYs work. But they imply the main property of the extremal functions (in both cases), namely to be proper maps of degree N of the N-connected domain G onto $\mathbb{D}$.

Every finitely-connected bounded plane domain is conformally equivalent to some circular domain (i.e. each boundary component is a circle or a single point), cf. [3]. Therefore it is enough to consider some N-connected, bounded circular domain $G \subset \mathbb{C}$ with $0 \in G$ whose complement components are not singletons.

1. Call for a proof.

Proposition 1. *Let G be some N-connected complex domain with $0 \in G$ and $f \in \mathbf{H}_1^0(G)$ a function which is not a proper map of degree N of G on $\mathbb{D}$. Then, for each $z_1 \in G \setminus \{0\}$, there is some meromorphic function $\Phi = \Phi_{G,f,z_1} : G \to \mathbb{C} \cup \{\infty\} =: \overline{\mathbb{C}}$ with the following properties:*

1. $f \cdot \Phi \in \mathbf{H}_1^0(G)$,
2. $|\Phi(z_1)| > 1$.

Remarks:

1. As already mentioned in the introduction, we do not know any complete proof for this proposition without using the mentioned result of GRUNSKY (which obviously does imply it; take $\Phi := \frac{\Theta}{f}$ where the extremal function Θ is taken respect to z_1).

2. For all $z_1 \in G$ sufficiently close to such a boundary portion, which is mapped into $\mathbb{D}$, one can give a purely analytic proof using the "spike functions" $1 + (1 - z)^\lambda$ (with small positive λ), which allow to give $f(G)$ a local peak. We omit the details.
3. In the simple case $N = 1$ we can prove and improve it in the following way: let B denote the Blaschke produkt sharing the zeros of f in $G = \mathbb{D}$ except 0. Then $\Phi := \dfrac{1}{B}$ has the desired properties simultaneously for all $z_1 \in G$.

Concerning the problem of maximizing the derivative in 0 in the class $\mathbf{H}_1^0(G)$ the following plays the analogous role.

Proposition 2. *Let G and $f \in \mathbf{H}_1^0(G)$ be as above. Then there is some meromorphic function $\phi = \phi_{G,f} : G \to \mathbb{C} \cup \{\infty\}$ with the following properties:*

1. $f \cdot \phi \in \mathbf{H}_1^0(G)$,
2. $|\phi(0)| > 1$.

Remarks:
1. Note the last condition rules that $|(f\phi)'(0)| = |f'(0)\phi(0)| > |f'(0)|$ if ϕ has no pole in 0. The latter can only occur if $f'(0) = 0$.
2. If Proposition 2 can be proved, then it is enough to consider the statement of Proposition 1 under the weaker assumption that f is not a proper map of G onto $\mathbb{D}$, without any assumption of the degree. It is shown below that this suffices to obtain the full result in any case.

2. Some properties of the extremal functions. The following corollaries are consequences of Proposition 1.

Corollary 1. *If G is a N-connected domain in $\overline{\mathbb{C}}$, then each extremal function in $\mathbf{H}_1^0(G)$ is a proper map $G \to \mathbb{D}$ of degree N.*

It is well known that this Corollary can also be expressed in the following way:

Corollary 2. *Let G be as above. If $z_1 \in G$ and f is an extremal function in $\mathbf{H}_1^0(G)$ with respect to z_1, then $|f(z_k)| \to 1$ for every sequence $z_k \in G$ with $z_k \to \zeta \in \partial G$ and f takes the value 0 exactly N times (counting multiplicity).*

The Reflection Principle gives:

Corollary 3. *Each extremal function in $\mathbf{H}_1^0(G)$ has a holomorphic extension on $\overline{G}$.*

Proposition 1 admits the following geometric observation on the location of extremal points.

Theorem 1. *Let $f \in \mathbf{H}_1^0(G)$ and denote by U the union of all open disks $D \subset G$ containing a zero $z_0 \neq 0$ of f. Then f is not extremal for any $z_1 \in U$.*

Proof. Take Φ as the Möbius transformation which maps the disk $D \subset U$ on $\overline{\mathbb{C}} \setminus \mathbb{D}$ having a pole in z_0. Then we obtain $f \cdot \Phi \in \mathbf{H}_1^0(G)$ and $|f \cdot \Phi| > |f|$ on D. $\square$

Remark. If G is a slit domain with all slits on the real axis, then a function which is extremal with respect to some $z \in \mathbb{C}$ must have all zeros in the open lower (resp. upper) half plane if $\Im z > 0$ (resp. < 0). If f is extremal in ∞ then all the zeros of f are real.

Theorem 2. *Let $z_1 \in G$. There is one and only one $f \in \mathbf{H}_1^0(G)$ which is extremal with respect to z_1 and $f(z_1) > 0$.*

We will use an idea due to Heins (cf. the supplement to [5]) to show the uniqueness of the extremal functions.

If $f, g \in \mathbf{H}_1^0(G)$ are extremal with respect to z_1, then $h := \dfrac{f+g}{2} \in \mathbf{H}_1^0(G)$ and h is again extremal with respect to z_1. Corollary 1 says that h is a proper map from G onto $\mathbb{D}$. But obviously $|h(\zeta)| = 1$ can only happen if $f(\zeta) = g(\zeta)$ for all $\zeta \in \partial\mathbb{D}$. So, by Cauchy's Integral Formula, we conclude $f \equiv g$ in G.

3. The infinitesimal case. About sixty years ago Ahlfors [1] and Grunsky [4], [5] (see also [3], ch. XI, §3) studied independently the problem to maximize $|F'(0)|$ in the class $\mathbf{H}_1^0(G)$, where G is as above some fixed N-connected domain in the complex plane containing the origin. They proved that this maximum is attained for some proper map $F : G \to \mathbb{D}$ of degree N. Using a suitable rotation we may provide $F'(0) > 0$. With this normalization F is uniquely determined. This follows from the same argument as we have already used to prove Theorem 2. Let such a function be fixed throughout this section. We will show that F is the limit of a sequence of functions in $f_n \in \mathbf{H}_1^0(G)$ which are extremal with respect to points $z_n \in G \setminus \{0\}$ tending to 0. (Remark: to obtain this result it is not necessary to know the degree of properness of the f_n; below we will determine this degree as a consequence of the next result.)

We can find a subsequence of (f_n) which converges locally uniformly. For the sake of simplicity we assume that f_n tends to f locally uniformly on G. Then $f \in \mathbf{H}_1^0(G)$ and we wish to prove $|f'(0)| = F'(0)$. Note that this will imply that *each* subsequence of the genuine sequence (f_n) has the behaviour.

If this would be not true, then we could find some $\varepsilon > 0$ with $|f'(0)|+4\varepsilon = |F'(0)|$. Let us fix some neighborhood U of 0 with

$$\Big|\frac{F(\zeta)-F(0)}{\zeta-0} - F'(0)\Big| = \Big|\frac{F(\zeta)}{\zeta} - F'(0)\Big| < \varepsilon \quad (\zeta \in U)$$

as well as some n_0 with

$$|f_n'(0) - f'(0)| < \varepsilon \quad (n \geq n_0)$$

and also (note that $f_n(0) = 0$)

$$\Big|f_n'(0) - \frac{f_n(\zeta)}{\zeta}\Big| < \varepsilon \quad (n \geq n_0,\, \zeta \in U).$$

So we can conclude

$$|f'(0)| + 3\varepsilon \leq \Big|\frac{F(\zeta)}{\zeta}\Big| \quad (\zeta \in U)$$

and moreover

$$\Big|\frac{f_n(\zeta)}{\zeta}\Big| - \left(\Big|f'(0) - f_n'(0)\Big| + \Big|f_n'(0) - \frac{f_n(\zeta)}{\zeta}\Big|\right) + 3\varepsilon \leq \Big|\frac{F(\zeta)}{\zeta}\Big| \; (\zeta \in U, n \geq n_0).$$

This gives

$$\Big|\frac{f_n(\zeta)}{\zeta}\Big| < \Big|\frac{f_n(\zeta)}{\zeta}\Big| + \varepsilon \leq \Big|\frac{F(\zeta)}{\zeta}\Big| \quad (\zeta \in U, n \geq n_0)$$

and so we obtain $|f_n(\zeta)| < |F(\zeta)|$ for all $\zeta \in U \setminus \{0\}$ and $n \geq n_0$. But this contradicts the extremality of f_n with respect to $z_n \in U$.

We summarize:

Theorem 3. *If z_n is a sequence in $G \setminus \{0\}$ tending to 0 and $f_n \in \mathbf{H}_1^0(G)$ denotes the extremal function with respect to z_n, then $f = \lim f_n$ exists and $|f'(0)| = \max\{|h'(0)| : h \in \mathbf{H}_1^0(G)\}$ (i.e. $f = c \cdot F$ with some constant c of modulus 1).*

Because it is known from the work of Grunsky and Ahlfors that $F : G \to \mathbb{D}$ is a proper map of degree N and the functions $f_n : G \to \mathbb{D}$ are proper, the Argument Principle gives

Lemma 1. *There is a neighborhood $U \subset G$ of 0 such for each $z_1 \in U$ the associated extremal function f is a proper map of degree N.*

4. Comparing the extremal functions. We consider a sequence $z_n \in G \setminus \{0\}$ tending to some point $z_0 \in G \setminus \{0\}$ and we wish to compare the extremal functions $f_k \in \mathbf{H}_1^0(G)$ with respect to z_k, $k = 0, 1, 2, \ldots$.

Let us assume that the sequence (f_n) converges locally uniformly in G, otherwise we take a suitable subsequence, and let $f = \lim f_n$. The maximality of $f_0(z_0)$ gives $f(z_0) \leq f_0(z_0)$. But $f(z_0) < f_0(z_0)$ would contradict, by a continuity argument and the locally uniform convergence of (f_n), the extremality of the functions f_n. So $f(z_0) = f_0(z_0)$ and, by Theorem 2, we obtain $f \equiv f_0$. So we have proved:

Theorem 4. *Let G be some N-connected domain in the complex plane and let $f_w : G \to \mathbb{D}$ denote the extremal function with respect to $w \in G \setminus \{0\}$. Then f_w is continuous in w with respect to the topology of locally uniform convergence on $\mathbf{H}_1^0(G)$.*

By local repetition of the argument of Lemma 1 along a path from 0 to some given point w, Theorem 3 leads to

Theorem 5. *Let G be as above. Then Proposition 2 implies that for each $w \in G \setminus \{0\}$ the extremal function $f_w : G \to \mathbb{D}$ with respect to w is a proper map of degree N.*

Lemma 2. *An open set in the euclidean space $\mathbb{R}^m$ cannot be covered by countably many, pairwise disjoint closed sets.*

It is obviously enough to prove this in the case $m = 1$. But "strongly" decreasing open intervals

$$O_1 \supset \overline{O_2} \supset O_2 \supset \overline{O_3} \supset O_3 \ldots$$

must have at least one common point.

From this fact we finally derive as a consequence of Theorem 4:

Corollary 4. *Let $w_1 \in G$ be given and define $C(w_1)$ as the set of all $w \in G$ such that $f_w = f_{w_1} e^{i\alpha}$ for some real $\alpha = \alpha(w)$. Then $C(w_1) = G$ or there exist uncountably many distinct sets $C(w_j)$.*

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