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Optimization problems for convex functions

Dedicated to Professor Zdzisław Lewandowski
on the occasion of his 70th birthday

Abstract. Assume that A, B are non-empty convex subsets of a real linear space and let $f : A \to \mathbb{R}$ be a given convex function. When B is determined by a finite number of convex constraints, there are known necessary and sufficient conditions for $p \in A \cap B$ to be a solution of the constrained problem $f(p) = \min f(A \cap B)$ considered as the unconstrained problem for a suitable Lagrange function over the set A. The purpose of this article, except a short presentation of the mentioned convex programming, is to discuss in detail a quite different problem of maximizing f over the set $A \cap B$.

1. Basic concepts. Let X be a real linear space and let $[x; y]$ (resp. $(x; y)$) denote the closed (resp. open) line segment joining x, $y \in X$. A subset A of X is said to be *plane* (resp. *convex*) if $\ell(x; y) \subset A$ for all $x, y \in A$, $x \neq y$ (resp. $[x; y] \subset A$ for all $x, y \in A$), where $\ell(x; y)$ denotes the straight line through the points x and y. Since the intersection of a family of plane (resp. convex) sets is again plane (resp. convex), we define the *affine* (resp.

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convex) *hull* of $B \subset X$, written $\mathrm{af}(B)$ (resp. $\mathrm{co}(B)$), to be the smallest plane (resp. convex) set containing B:

$$\mathrm{af}(B) = \left\{ \sum_{j=1}^{n} \lambda_j x_j \ : \ \lambda_j \in \mathbb{R},\ x_j \in B,\ \sum_{j=1}^{n} \lambda_j = 1,\ n = 1, 2, \ldots \right\},$$

$$\mathrm{co}_n(B) = \left\{ \sum_{j=1}^{n} \lambda_j x_j \ : \ \lambda_j \geq 0,\ x_j \in B,\ \sum_{j=1}^{n} \lambda_j = 1 \right\},$$

$$\mathrm{co}(B) = \bigcup_{n=1}^{\infty} \mathrm{co}_n(B).$$

Clearly, $\ell(x;y) = \mathrm{af}(\{x,y\})$ for $x \neq y$ and $[x;y] = \mathrm{co}_2(\{x,y\}) = \mathrm{co}(\{x,y\})$. By Carathéodory's theorem [5, 14 (th. 6), 15, 16 p. 73], if $\varnothing \neq B \subset \mathbb{R}^n$, then $\mathrm{co}(B) = \mathrm{co}_{n+1}(B)$ and every point of the set $[\partial\, \mathrm{co}(B)] \cap \mathrm{co}(B)$ can be expressed as a convex combination of at most n points of B. Moreover, if B has at most n components, then $\mathrm{co}(B) = \mathrm{co}_n(B)$.

When $A \subset X$ is a non-empty convex set, we will consider the families $\mathrm{Conv}(A)$, $\mathrm{Qconv}(A)$ and $\mathrm{Aff}(A)$ of all *convex, quasi-convex* and *affine* real-valued *functions* defined on A. By definition, a function $f : A \to \mathbb{R}$ is said to be in $\mathrm{Conv}(A)$ (resp. $\mathrm{Qconv}(A)$) if $f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$ (resp. $\leq \max\{f(x), f(y)\}$) for all $x, y \in A$ and $0 < \lambda < 1$. Furthermore, $\mathrm{Aff}(A) = \mathrm{Conv}(A) \cap [-\mathrm{Conv}(A)]$. An application of Kuratowski-Zorn's Lemma shows that every function $f \in \mathrm{Aff}(A)$ is the restriction of a functional $x' + c$ to the set A, where x' is in X', the algebraic dual of X, and $c \in \mathbb{R}$. However, there are compact convex sets A in every infinite dimensional Hilbert space X and continuous $f \in \mathrm{Aff}(A)$ that have no continuous extension to a member of $\{x^* + c \ : \ x^* \in X^*,\ c \in \mathbb{R}\}$, where X^* is the topological dual of X. Geometrically speaking, $f \in \mathrm{Conv}(A)$ (resp. $f \in \mathrm{Qconv}(A)$) if and only if the set $\{(x,t) \ : \ t \geq f(x),\ x \in A\}$ is convex in $X \times \mathbb{R}$ (resp. $\{x \in A \ : \ f(x) \leq t\}$ is convex for every $t \in \mathbb{R}$). Moreover, every $f \in \mathrm{Conv}(A)$ is continuous on each open line segment contained in A (with respect to one-dimensional Euclidean topology), and it is generally false for members of $\mathrm{Qconv}(A)$. A function $f : A \to \mathbb{R}$ is said to be *concave* (resp. *quasi-concave*) iff $-f \in \mathrm{Conv}(A)$ (resp. $-f \in \mathrm{Qconv}(A)$). Thus all the problems for concavity one can consider in terms of convexity. Observe that if $f \in \mathrm{Aff}(A)$ and $\Phi \in \mathrm{Conv}(\mathbb{R})$ (or only $\Phi \in \mathrm{Conv}(f(A))$), then $\Phi \circ f \in \mathrm{Conv}(A)$.

Let $\varnothing \neq A \subset X$. We will say that p belongs to the *intrinsic core* of A (or to the *relative algebraic interior* of A), written $p \in \mathrm{icr}(A)$, if for each $x \in \mathrm{af}(A) \setminus \{p\}$ there is a point $y \in (p;x)$ such that $[p;y] \subset A$. When A is convex, then

$$\mathrm{icr}(A) = \{p \in X \ : \ \forall_{x \in A \setminus \{p\}}\ \exists_{y \in A}\ \ p \in (x;y)\}.$$

It is common known that in any infinite dimensional linear space X there are non-empty convex sets A with $\operatorname{icr} A = \varnothing$, for instance

$$A = \{\sum_{\alpha\in I} \lambda_\alpha e_\alpha \,:\, \lambda_\alpha \geq 0 \text{ for } \alpha \in I \subset \Lambda,\ \operatorname{card}(I) < \infty\},$$

where $\{e_\alpha \,:\, \alpha \in \Omega\}$ is a given Hamel basis for X.

A finite set $\{x_0, x_1, \dots, x_n\} \subset X$ is affinely independent if the set $\{x_1 - x_0, \dots, x_n - x_0\}$ is linearly independent. The convex hull of such a set is called an *n-simplex* with vertices $x_0, x_1, \dots, x_n$. Clearly, each point of the n-simplex $S(x_0, x_1, \dots, x_n)$ with vertices $x_0, x_1, \dots, x_n$ is uniquely expressed as a convex combination of its vertices: if $x \in S(x_0, x_1, \dots, x_n)$, then $x = \sum_{j=0}^{n} \lambda_j(x) x_j$ with unique $0 \leq \lambda_j(x) \leq 1$, $\sum_{j=0}^{n} \lambda_j(x) = 1$. The coefficients $\lambda_j(x)$ are called the *barycentric coordinates* of x. For the n-simplex $S(x_0, x_1, \dots, x_n)$,

$$\operatorname{icr}(S(x_0, x_1, \dots, x_n)) = \left\{ \sum_{j=0}^{n} \lambda_j x_j \,:\, \lambda_j > 0,\ \sum_{j=0}^{n} \lambda_j = 1 \right\}.$$

Suppose now that X is a linear topological space. When X is complex, then X is also a real linear topological space if we admit only multiplication by real scalars. Let $A \subset X$. By $\overline{A}$, ∂A, $\operatorname{int}(A)$, $\partial_{\mathrm{af}}(A)$ and rel-int(A) we denote the closure of A, the boundary of A, the interior of A, the relative boundary of A and the relative interior of A, both the last mentioned with respect to $\overline{\mathrm{af}}(A)$. If $A \subset X$ is convex, then rel-int$(A) \subset \operatorname{icr}(A)$ with equality instead of inclusion whenever rel-int$(A) \neq \varnothing$. Of course, there are locally convex Hausdorff spaces containing infinite-dimensional compact convex subsets A with rel-int$(A) = \varnothing \neq \operatorname{icr} A$. However, every non-empty convex set $A \subset \mathbb{R}^n$ has a non-empty relative interior and hence rel-int$(A) = \operatorname{icr}(A)$. The same holds for all closed convex subsets A of every Banach space (which is of second category).

In the theory of convex programming there are problems having a strictly algebraic character. Namely, assume that A, B are non-empty convex subsets of a real linear space X and let $f \in \operatorname{Conv}(A)$. Consider the minimum of $f(A \cap B)$, i.e. the problem of minimizing $f(x)$ for $x \in A$ subject to the constraint $x \in B$, which is usually written as a system of simultaneous convex constraints:

$$x \in A\,,\ f_j(x) \leq 0\,, j = 1, \dots, n+s\,,$$

with given $f_1, \dots, f_n \in \operatorname{Conv}(A)$ and $f_{n+1}, \dots, f_{n+s} \in \operatorname{Aff}(A)$. If p is a point of local minimum for $f|_{A\cap B}$ (with respect to all line segments

$[p;q] \subset A \cap B$), then p is a global one. In fact, for a given $q \in A \cap B$ and a sufficiently small $t > 0$ we have

$$f(p) \le f((1-t)p + tq) \le (1-t)f(p) + tf(q)\,, \quad \text{i.e. } f(p) \le f(q)\,.$$

Extremum problems (local, global, existence, calculation) are not new in mathematics. However, the demand of economics as well as a common use of personal computers has made every numerical solving of such problems to be an important method.

2. Minima of convex functions. We will touch only a few aspects of the convex programming. For the convenience of the reader we adapt from [1, 9, 14, 15] the typical two results that have applications concerning necessary and sufficient optimality conditions known as the Kuhn-Tucker theorems.

Proposition 1. *Let n, s be non-negative integers and let A be a non-empty convex subset of a real linear space. Choose arbitrary $f_0, \dots, f_n \in \mathrm{Conv}(A)$ and, provided $s \ge 1$, non-zero functions $f_{n+1}, \dots, f_{n+s} \in \mathrm{Aff}(A)$. Consider*

$$\begin{aligned} A_k =& \{x \in A : f_j(x) < 0 \text{ for } k \le j \le n,\ f_j(x) \le 0 \text{ for } n+1 \le j \le n+s\}, \\ B_k =& \{x \in A : f_j(x) < 0\,,\ \ j = k, \dots, n+s\} \end{aligned}$$

and

$$C[k] \equiv \textit{there exist non-negative numbers } \lambda_j,\ 0 \le j \le n+s, \ \textit{such that}$$
$$\sum_{j=0}^{k} \lambda_j > 0 \ \textit{and} \ \inf\Big(\sum_{j=0}^{n+s} \lambda_j f_j\Big)(A) \ge 0.$$

Here, $A_k = \{x \in A : f_j(x) < 0\,,\ \ j = k, \dots, n\}$ if $s = 0$, and $A_{n+1} = \{x \in A : f_j(x) \le 0\,,\ j = n+1, \dots, n+s\}$ if $s \ge 1$.

Under the above notation we have

(*i*) *If $A_0 = \varnothing$, then $C[n+s]$.*

(*ii*) *If $C[k]$ holds for some $k \in \{1, \dots, n\}$, then $A_0 = \varnothing$.*

(*iii*) *Suppose $B_k \ne \varnothing$ for some $k \in \{1, \dots, n+1\}$. Then $A_0 = \varnothing$ if and only if $C[k-1]$.*

(*iv*) *Let $s \ge 1$, $A_k \ne \varnothing$ for some $k \in \{1, \dots, n\}$ and suppose $B_{n+1} \ne \varnothing$ or $A_{n+1} \cap \mathrm{icr}(A) \ne \varnothing$. Then $A_0 = \varnothing$ if and only if $C[k-1]$.*

Remark 1. The proof of such general result is enough simple. The point (i) is a consequence of the separation theorem for the following convex subsets of $\mathbb{R}^{n+s+1}$:

$$\begin{aligned} U =& \{(f_0(x) + \varepsilon_0, \dots, f_n(x) + \varepsilon_n, f_{n+1}(x), \dots, f_{n+s}(x)) : x \in A,\ \varepsilon_j > 0, \\ & j = 0, \dots, n\} \end{aligned}$$

and

$$V = \{(\zeta_0, \dots, \zeta_{n+s}) \, : \, \zeta_j \le 0 \, , \ \ j = 0, \dots, n+s\}$$

that are disjoint if and only if $A_0 = \varnothing$. In the proof of (iii)–(iv) we observe a special form of the set V so that there is a hyperplane $H = \{(\zeta_0, \dots, \zeta_{n+s}) : \sum_{j=0}^{n+s} \lambda_j \zeta_j = 0\}$ separating U from V with $C[n+s]$ and $U \setminus H \neq \varnothing$. In fact, if $U \subset H$, then H cannot be of the form $\{(\zeta_0, \dots, \zeta_{n+s}) \, : \, \zeta_j = 0\}$, where $j \in \{0, \dots, n+s\}$. Thus one can turn H about the origin preserving separated U and V.

Remark 2. For given $f, f_1, \dots, f_n \in \operatorname{Conv}(A)$ and, provided $s \ge 1$, for non-zero $f_{n+1}, \dots, f_{n+s} \in \operatorname{Aff}(A)$, the general convex programming problem is to decide whether any point $p \in A$ is a solution in the sense that

$$f(p) = \min\{f(x) \, : \, x \in A, \ f_j(x) \le 0 \, , \ \ j = 1, \dots, n+s\}. \tag{1}$$

Put $f_0 = f - f(p)$. In the notation of Proposition 1, if $A_0 = \varnothing \neq A_1$, then $\inf f_0(A_1) \ge 0$ and also $\inf f_0(\{x \in A \, : \, f_j(x) \le 0 \, , \ \ j = 1, \dots, n+s\}) \ge 0$. Indeed, if $f_0(x_0) < 0$ for some $x_0 \in A$ with $f_j(x_0) \le 0$, $j = 1, \dots, n+s$, then for every $x_1 \in A_1$ and $0 < t < 1$ we have $(1-t)x_0 + tx_1 \in A_1$, and hence

$$0 \le f_0((1-t)x_0 + tx_1) \le (1-t)f_0(x_0) + tf_0(x_1) \to f_0(x_0) < 0$$

as $t \to 0^+$, a contradiction. We have thus established:

If a point $p \in A$ with $f_j(p) \le 0$, $j = 1, \dots, n+s$, is a solution of (1), then $A_0 = \varnothing$.

If a point $p \in A$ with $f_j(p) \le 0$, $j = 1, \dots, n+s$, satisfies $C[0]$ or $A_0 = \varnothing \neq A_1$, then p is a solution of (1).

This way Proposition 1 implies

Theorem 1 (Kuhn-Tucker). *With the notation of Proposition 1, let $f \in \operatorname{Conv}(A)$, $f_0 = f - f(p)$ and suppose that one of the following three conditions holds:*

(i) $B_1 \neq \varnothing$,

(ii) $s \ge 1$, $A_1 \neq \varnothing$ *and* $B_{n+1} \neq \varnothing$,

(iii) $s \ge 1$, $A_1 \neq \varnothing$ *and* $A_{n+1} \cap \operatorname{icr} A \neq \varnothing$.

Then p is a solution of (1) if and only if $C[0]$ holds and $p \in \{x \in A : f_j(x) \le 0 \, , \ \ j = 1, \dots, n+s\}$. In the necessary condition we may assume that $\lambda_j f_j(p) = 0$ for all $j = 1, \dots, n+s$.

Proposition 2. *Let $n \geq 0$, $s \geq 1$ and let A be a non-empty convex subset of a real linear space. Take arbitrary $f_0, \dots, f_n \in \mathrm{Conv}(A)$, $f_{n+1}, \dots, f_{n+s} \in \mathrm{Aff}(A)$, and consider the following sets and conditions:*

$$\begin{aligned} A_k &= \{x \in A : f_j(x) < 0 \text{ for } k \leq j \leq n,\ f_j(x) = 0 \text{ for } n+1 \leq j \leq n+s\}, \\ B &= (f_{n+1}, \dots, f_{n+s})(A) \subset \mathbb{R}^s \end{aligned}$$

and

$C[k] \equiv$ *there exist real numbers* λ_j, $0 \leq j \leq n+s$, *such that*

$$\lambda_j \geq 0 \text{ for } 0 \leq j \leq n, \quad \sum_{j=0}^{k} |\lambda_j| > 0 \text{ and } \inf\Big(\sum_{j=0}^{n+s} \lambda_j f_j\Big)(A) \geq 0.$$

Under the above notation

(*i*) *If $A_0 = \varnothing$, then $C[n+s]$.*

(*ii*) *If $C[k]$ holds for some $k \in \{0, \dots, n\}$, then $A_0 = \varnothing$.*

(*iii*) *Suppose that $A_k \neq \varnothing$ for some $k \in \{1, \dots, n\}$ and that* $\mathrm{int}(B)$ *contains the origin of $\mathbb{R}^s$. Then $A_0 = \varnothing$ if and only if $C[k-1]$.*

Remark 3. In the proof of (*i*) we have to separate the convex subsets of $\mathbb{R}^{n+s+1}$: U from Remark 1 and $\{\theta\}$, where θ is the origin of $\mathbb{R}^{n+s+1}$. Both the sets are disjoint if and only if $A_0 = \varnothing$.

Remark 4. Like in Remark 2, for given $f, f_1, \dots, f_n \in \mathrm{Conv}(A)$ and $f_{n+1}, \dots, f_{n+s} \in \mathrm{Aff}(A)$, a necessary (resp. *sufficient*) *condition for $p \in A$ to be a solution of the problem*

(2)
$$f(p) = \min\{f(x) : x \in A, f_j(x) \leq 0,\ 1 \leq j \leq n, f_j(x) = 0,\ n+1 \leq j \leq n+s\}$$

is that

(3) $p \in \{x \in A : f_j(x) \leq 0 \text{ if } 1 \leq j \leq n,\ f_j(x) = 0 \text{ if } n+1 \leq j \leq n+s\}$

and $A_0 = \varnothing$ (resp. (3) *and* $C[0]$), where $f_0 = f - f(p)$, while A_0 and $C[0]$ are defined in Proposition 2.

Hence we conclude

Theorem 2 (Kuhn-Tucker). *With the notation of Proposition 2, let $f \in \mathrm{Conv}(A)$, $f_0 = f - f(p)$, and suppose that $A_1 \neq \varnothing$ and that the set* $\mathrm{int}(B)$ *contains the origin of $\mathbb{R}^s$. Then p is a solution of* (2) *if and only if* (3) *and $C[0]$ hold. For the necessity we may assume that $\lambda_j f_j(p) = 0$ when $1 \leq j \leq n$, and $f_j(p) = 0$ when $n+1 \leq j \leq n+s$.*

3. Some convexity techniques. Let A be a non-empty subset of a real linear space. Denote by $\mathrm{ext}(A)$ the set of all extreme points of A. By definition, $\mathrm{ext}(A) = \{e \in A : \forall_{a,b \in A}(e \in [a;b] \implies e = a \text{ or } e = b)\}$ and $\mathrm{ext}(\mathrm{co}(A)) \subset \mathrm{ext}(A) \subset A$. If A is convex, then $\mathrm{ext}(A) = \{e \in A : A \setminus \{a\} \text{ is convex}\}$. The basic result asserts the relation between compact convex subsets of a locally convex Hausdorff space and their extreme points.

Theorem 3 (Krein-Milman, see [3, 9, 13, 16]). *Suppose X is a linear topological space on which X^* separates points, e.g. X is a locally convex Hausdorff space. If $A \subset X$ is non-empty compact, then $\mathrm{ext}(A) \neq \varnothing$. If moreover A is convex, then $A = \overline{\mathrm{co}}(\mathrm{ext}(A))$ and $\max f(A) = \max f(\mathrm{ext}(A))$ for every continuous $f \in \mathrm{Qconv}(A)$.*

Remark 5.

(i) Suppose $f : A \to \mathbb{R}$ is strictly quasi-convex:

$$f((1-\lambda)x + \lambda y) < \max\{f(x), f(y)\}$$

for all $0 < \lambda < 1,\ x \in A,\ y \in A,\ x \neq y$. Under the assumptions of Theorem 3, if $p \in A$ is a solution in the sense that $f(p) = \max f(A)$, then $p \in \mathrm{ext}(A)$.

(ii) Every finite dimensional subspace of a real linear topological Hausdorff space X is closed and topologically isomorphic to the Euclidean space. If now A is a non-empty compact convex subset of X with $n = \dim A = \dim(\mathrm{af}(A))$, then $A = \mathrm{co}_{n+1}(\mathrm{ext}(A)) = \mathrm{co}(\mathrm{ext}(A))$, the Minkowski-Carathéodory theorem, see [5, 9, 13].

A generalization of the Minkowski-Carathéodory theorem is contained in

Proposition 3. *Let A be a non-empty compact convex subset of X. Consider $\Phi = (f_1, \dots, f_n) : A \to \mathbb{R}^n$, where the functions $f_1, \dots, f_n \in \mathrm{Aff}(A)$ and all are continuous on A. Then*

(*i*) *$\Phi(A)$ is a compact convex subset of $\mathbb{R}^n$ with $\varnothing \neq \mathrm{ext}(\Phi(A)) \subset \Phi(\mathrm{ext}(A))$.*

(*ii*) $\Phi(A) = \mathrm{co}_{n+1}(\mathrm{ext}(\Phi(A))) = \Phi(\mathrm{co}_{n+1}(\mathrm{ext}(A))) = \mathrm{co}(\Phi(\mathrm{ext}(A)))$.

(*iii*) *$\Phi(A) = \Phi(\mathrm{co}_n(\mathrm{ext}(A)))$ when the set $\Phi(\mathrm{ext}(A))$ has at most n-components.*

Remark 6. For $X = \mathbb{R}^n$ and $\Phi = \mathrm{id}_A$, the identity map on A, we get the Minkowski-Carathéodory theorem. The point (i) is an easy consequence of Theorem 3: $\mathrm{ext}(A) \neq \varnothing$, $\mathrm{ext}(\Phi(A)) \neq \varnothing$ and $\mathrm{ext}(\Phi^{-1}(e)) \subset \mathrm{ext}(A)$ for every $e \in \mathrm{ext}(\Phi(A))$.

Remark 7. Assume that $\mathbb{P}$ is the set of all (regular Borel) probability measures on a compact Hausdorff space T. In the real linear space $\mathrm{af}(\mathbb{P}-\mathbb{P})$ of all signed finite measures on T [3, 16, 17], $\exp(\mathbb{P}) = \{\delta_s : s \in T\}$ and $\mathbb{P} = \overline{\mathrm{co}}\{\delta_s : s \in T\}$ in the weak *-topology, where δ_s means the Dirac measure concentrated at s. Let $\varphi : T \to \mathbb{R}$ be continuous and $\tau \in \varphi(T)$. In [2] the authors solved the following problem from the constrained optimization: the sets $A = \{\alpha \in \mathbb{P} : \int_T \varphi d\alpha = \tau\}$ and

$$\overline{\mathrm{co}}\{(1-\lambda)\delta_s + \lambda\delta_t : 0 \le \lambda \le 1,\ s, t \in T,\ (1-\lambda)\varphi(s) + \lambda\varphi(t) = \tau\}$$

are the same (originally $T = [0; 1]$ and $\varphi = \mathrm{id}_T$). A profound extension of this solution is contained in the following proposition that states the case when there is a non-trivial variation in the examined set preserving a given system of affine constraints.

From now we regard X as a locally convex Hausdorff space.

Proposition 4 [18, 20]. *Assume that A is a non-empty compact convex subset of X and that $f_1, \dots, f_n$ are arbitrary continuous members of* $\mathrm{Aff}(A)$. *Consider $\Phi = (f_1, \dots, f_n) : A \to \mathbb{R}^n$. Then for every $a \in A$ either*

(*i*) $a \in \mathrm{co}_{n+1}(\mathrm{ext}(A))$

or

(*ii*) *there is a non-zero $b \in X$ such that for all $-1 \le t \le 1$ we have $a + tb \in A$ and $\Phi(a + tb) = \Phi(a)$.*

Remark 8. For $X = \mathbb{R}^n$ and $\Phi = \mathrm{id}_A$ we get $A = \mathrm{co}_{n+1}(\mathrm{ext}(A))$, once more the Minkowski-Carathéodory theorem. To prove Proposition 4 we have to use Remark 5 and a fact that if $x \in A \setminus \mathrm{co}_k(\mathrm{ext}(A))$, then there is a k-simplex $S \subset A$ such that $x \in \mathrm{icr}(S)$. Therefore, if $a \in A \setminus \mathrm{co}_{n+1}(\mathrm{ext}(A))$, then $a = \sum_{j=0}^{n+1} \lambda_j x_j$ for some $\lambda_j > 0$, $x_j \in A$ with $\lambda_0 + \lambda_1 + \cdots + \lambda_{n+1} = 1$ such that $x_j - x_0$, $j = 1, \dots, n+1$, are linearly independent. Since $\Phi(x_j) - \Phi(x_0)$, $j = 1, \dots, n+1$, are always linearly dependent in $\mathbb{R}^n$, there are real numbers $s_0, s_1, \dots, s_n$ such that $\sum_{j=0}^{n+1} s_j = 0$, $\sum_{j=0}^{n+1} |s_j| = 1$ and $\sum_{j=0}^{n+1} s_j \Phi(x_j) = (0, \dots, 0)$. Define $b = \varepsilon \sum_{j=0}^{n+1} s_j x_j = \varepsilon \sum_{j=1}^{n+1} s_j(x_j - x_0)$, where $0 < \varepsilon < \min\{\lambda_j : 0 \le j \le n+1\} / \max\{|s_j| : 0 \le j \le n+1\}$.

Remark 9. Suppose A, B are given non-empty compact convex subsets of X, $f \in \mathrm{Qconv}(A)$ and $A \cap B \neq \varnothing$. If f is continuous on A, then for every $C \subset X$ with

$$\mathrm{ext}(A \cap B) \subset C \subset A \cap B \tag{4}$$

we have $\max f(A \cap B) = \max f(C)$. Thus the main maximization problem is how to describe a set C satisfying (4), as small as possible, knowing only the set $\mathrm{ext}(A)$ and constraints determining the set B.

The next results are direct consequences of Proposition 4.

Theorem 4 [18, 20]. . *Assume A is a non-empty compact convex subset of X and $f_1, \dots, f_n$ are continuous members of* $\mathrm{Aff}(A)$. *If $\Phi = (f_1, \dots, f_n)$ and W is a non-empty compact convex subset of $\Phi(A)$, then*

$$\mathrm{ext}(\Phi^{-1}(W)) \subset A_1 \cup A_2 \subset \Phi^{-1}(W) \cap \mathrm{co}_{n+1}(\mathrm{ext}(A)),$$

where $A_1 = \Phi^{-1}(W) \cap \mathrm{ext}(A)$ and

$$A_2 = \Big\{ x = \sum_{j=1}^{n+1} \lambda_j e_j \ : \\ \lambda_j \geq 0, e_j \in \mathrm{ext}(A), \sum_{j=1}^{n+1} \lambda_j = 1, \Phi(e_j) \neq \Phi(e_s) \text{ for } j \neq s, \Phi(x) \in \partial W \Big\}.$$

Theorem 5 [19, 20]. *Let A be a non-empty compact convex subset of X. Consider the set $Z = \{\lambda x \ : \ \lambda \geq 0, \ x \in A\}$ and a linear continuous map $\Phi \ : \ X \to \mathbb{R}^n$. If $(0, \dots, 0) \notin \Phi(A)$, then*

(*i*) *Z is a closed convex cone in X,*

(*ii*) *for every compact convex set $W \subset \Phi(Z)$ the set $(\Phi|_Z)^{-1}(W)$ is compact convex and*

$$\mathrm{ext}((\Phi|_Z)^{-1}(W)) \subset B \subset (\Phi|_Z)^{-1}(\partial W),$$

where

$$B = \big\{ x = \sum_{j=1}^{n} \lambda_j e_j : \\ \lambda_j \geq 0, \ e_j \in \mathrm{ext}(A), \ \Phi(e_j) \neq \Phi(e_s) \text{ for } j \neq s, \ \Phi(x) \in \partial W \big\}.$$

In the above representation we do not claim that $\lambda_1 + \dots + \lambda_n = 1$.

Theorem 6 [6, 12, 20]. *Suppose $\varphi \ : \ X \to \mathbb{C}$ is positively homogeneous (i.e. $\varphi(\lambda x) = \lambda \varphi(x)$ for all $\lambda \geq 0$ and $x \in X$), $\mathfrak{c} \in \mathbb{C} \setminus \{0\}$ and A is a compact convex subset of $\varphi^{-1}(\mathfrak{c})$. Let $\psi \in \mathrm{Aff}(A)$ be continuous with $0 \notin \psi(A)$ and let $B = \{a/\psi(a) \ : \ a \in A\}$. Then*

(*i*) *B is a compact convex subset of X,*

(*ii*) *the map $a \mapsto a/\psi(a)$ is a homeomorphism of A onto B,*

(*iii*) $\mathrm{ext}(B) = \{a/\psi(a) \ : \ a \in \mathrm{ext}(A)\}$.

Consider now a σ-algebra $\mathcal{B}$ in a set T. A countable collection $\{E_j\}$ of members of $\mathcal{B}$ is called a partition of E if $E = \sum_{j=1}^{\infty} E_j$ and $E_j \cap E_s = \varnothing$ whenever $j \neq s$. Let $(Y, \|\cdot\|)$ be a real normed linear space with $\dim Y = k < +\infty$. A vector measure μ on $\mathcal{B}$ with values in Y is then a set function $\mu : \mathcal{B} \to Y$ such that

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E_j) \quad \text{for } E \in \mathcal{B} \text{ and every partition } \{E_j\} \text{ of } E\,. \tag{5}$$

Since μ assumes only finite values, the series (5) converges absolutely (each rearrangement of the series (5) is convergent). Therefore the set function

$$|\mu|(E) = \sup\Big\{ \sum_{j=1}^{\infty} \|\mu(E_j)\| \,:\, \{E_j\} \text{ is a partition of } E \Big\}\,, \quad E \in \mathcal{B}\,, \tag{6}$$

is correctly defined (we may use only finite partitions), for details see [17], where real and complex measures are considered. Since any norm in Y is equivalent to that of the Euclidean k-space, the set function $|\mu|$, so-called the total variation measure of μ, is a non-negative finite measure on $\mathcal{B}$. Denote by $\mathbb{M}_k$ the set of all vector measures on $\mathcal{B}$ with values in Y, and let θ mean the zero measure, i.e. $\theta(A)$ is the zero element of Y for all $A \in \mathcal{B}$.

Theorem 7 [7]. *Let $\varnothing \neq V \subset Y \times \mathbb{R}$. If $\mu_0 \in \operatorname{ext}\{\mu \in \mathbb{M}_k : (\mu(T), |\mu|(T)) \in V\}$, then either $\mu_0 = \theta$ or μ_0 is purely atomic with at most $k+1$ disjoint atoms.*

Theorem 8 [10]. *Fix a non-negative $\mu \in \mathbb{M}_1$ and let μ_A, $A \in \mathcal{B}$, denote the measure defined by the formula: $\mu_A(B) = \mu(A \cap B)$ for all $B \in \mathcal{B}$. For the convex subsets*

$$\{\nu \in \mathbb{M}_1 : \theta \le \nu \le \mu\} \quad \textit{and} \quad \{\nu \in \mathbb{M}_1 : \theta \le \nu \le \mu\,,\ \nu(T) = c\}$$

we have

(*i*) $\operatorname{ext}\{\nu \in \mathbb{M}_1 : \theta \le \nu \le \mu\} = \{\mu_A : A \in \mathcal{B}\}$.

(*ii*) *If μ is non-atomic, then*

$$\operatorname{ext}\{\nu \in \mathbb{M}_1 : \theta \le \nu \le \mu\,,\ \nu(T) = c\} = \{\mu_A : A \in \mathcal{B},\ \mu(A) = c\}\,.$$

(*iii*) *If μ has atoms, $0 \le c \le \mu(T)$, then*

$$\begin{aligned}\operatorname{ext}\{\nu \in \mathbb{M}_1 : \theta \le \nu \le \mu\,,\ \nu(T) = c\} = \{&\mu_A + (c - \mu(A))\mu_D/\mu(D) : \\ &A \in \mathcal{B}\,,\ D \text{ is an atom of } \mu\,,\ A \cap D = \varnothing \text{ and } \mu(A) \le c \le \mu(A \cup D)\}\,.\end{aligned}$$

For other sets of measures and their extreme points see [10–11]. For applications of Theorems 4–6 to holomorphic and harmonic mappings see [6, 8, 12, 19–20].

4. Maxima of convex functions. We start with an application of Theorem 4.

Theorem 9. *Let k, n be non-negative integers, $n \ge 1$, and let A be a non-empty compact convex subset of X. Fix arbitrary continuous $f \in \mathrm{Qconv}(A)$ and, provided $k \ge 1$, continuous $f_1, \dots, f_k \in \mathrm{Qconv}(A)$, and also continuous $f_{k+1}, \dots, f_{k+n} \in \mathrm{Aff}(A)$. For any compact convex subset W of $\Phi(A) = (f_{k+1}, \dots, f_{k+n})(A)$ consider the following convex programming problem*

$$(7)\quad f(p) = \max\{f(x) : x \in A,\ f_j(x) \le 0,\ j = 1, \dots, k,\ \Phi(x) \in W\},\ p \in A\,.$$

(*i*) *Assume $k = 0$. For the problem (7) there is a solution $p \in A_1 \cup A_2$, where A_1, A_2 are defined in Theorem 4. Furthermore, if f is strictly quasi-convex on A, then every solution p of* (7) *belongs to the set $A_1 \cup A_2$.*

(*ii*) *Assume $k \ge 1$. For the problem* (7) *there is a solution $p \in A_{1k} \cup A_{2k}$, where*

$$A_{1k} = \{x \in A_0 \,:\, \Phi(x) \in W\}$$

and

$$A_{2k} = \Big\{x = \sum_{j=1}^{n+1} \lambda_j e_j \,:\, \lambda_j \ge 0,\ e_j \in A_0,\ \sum_{j=1}^{n+1} \lambda_j = 1,\ \Phi(e_j) \ne \Phi(e_s) \text{ for } j \ne s,\ \Phi(x) \in \partial W\Big\}$$

with arbitrary A_0 satisfying

$$\mathrm{ext}\{x \in A \,:\, f_j(x) \le 0,\ j = 1, \dots, k\} \subset A_0 \subset \{e \in \mathrm{ext}(A) \,:\, f_j(e) \le 0,\ j = 1, \dots, k\} \cup \{x \in A \,:\, f_1(x) \cdot \ldots \cdot f_k(x) = 0\}.$$

Moreover, if f is strictly quasi-convex on A, then every solution p of (7) *belongs to the set $A_{1k} \cup A_{2k}$.*

An application of Theorem 5 is contained in

Theorem 10. *Let $Z = \{\lambda x \,:\, \lambda \ge 0,\ x \in A\}$, where A is a non-empty compact convex subset of X. Assume that $\Phi \,:\, X \to \mathbb{R}^n$ is a linear continuous mapping with $(0, \dots, 0) \notin \Phi(A)$. For arbitrary continuous $f \in \mathrm{Qconv}(Z)$ and any compact convex set $W \subset \Phi(Z)$ consider the problem*

$$(8)\qquad f(p) = \max\{f(x) \,:\, x \in Z,\ \Phi(x) \in W\},\ p \in Z\,.$$

Then there is a solution p of (8) belonging to the set B, see Theorem 5. Moreover, if f is strictly quasi-convex on Z, then each solution of (8) is in B.

A direct conclusion from Theorem 7 gives

Theorem 11. *Consider the set*

$$I = \{\mu \in \mathbb{M}_k \,:\, \Phi_\alpha(\mu(T), |\mu|(T)) \geq 0,\ \alpha \in \Lambda\},$$

where $\Phi_\alpha \,:\, Y \times [0;\infty) \to \mathbb{R}$, $\alpha \in \Lambda$, are arbitrarily given. If $\mu_0 \in \operatorname{ext}(I)$, then either $\mu_0 = \theta$ or μ_0 is purely atomic with at most $k+1$ disjoint atoms.

Remark 10. Suppose that I is convex and $f \,:\, I \to \mathbb{R}$ is strictly quasi-convex on I. If there exists $\max f(I) = f(\mu_0)$, $\mu_0 \in I$, then $\mu_0 \in \operatorname{ext}(I)$.

5. Illustrative examples. The classical methods, see e.g. [4], applied to both problems described below do not work well because of involved boundary solutions.

Problem 1. Let

$$A = \{(x,y,z,w) \,:\, x \geq 0,\ y \geq 0,\ z \geq 0,\ w \geq 0,\ x+y+z+w \leq 1\}.$$

Determine all the elements in the set

$$B = \{(x,y,z,w) \in A \,:\, (2x-2y-2z-w)^2 + (x+2y+2z-3w)^2 \leq 1\}$$

of maximal Euclidean norm.

Problem 2. Let

$$Z = \{(x,y,z,w) \,:\, x \geq 0,\ y \geq 0,\ z \geq 0,\ w \geq 0\}.$$

Determine all the elements in the set

$$B = \{(x,y,z,w) \in Z \,:\, 2(y+5z+5w)^2 + 3x - 2y - 3z - 3w \leq 4\}$$

of maximal Euclidean norm.

Solution of Problem 1. Observe first that A is a 4-simplex with vertices $E_0 = (0,0,0,0)$, $E_1 = (1,0,0,0)$, $E_2 = (0,1,0,0)$, $E_3 = (0,0,1,0)$ and $E_4 = (0,0,0,1)$. Consider the linear map Φ from $\mathbb{R}^4$ onto $\mathbb{R}^2$ defined as follows

$$\Phi(x,y,z,w) = (2x-2y-2z-w,\ x+2y+2z-3w).$$

Since $\text{ext}(A) = \{E_0, E_1, E_2, E_3, E_4\}$, we conclude from Proposition 3 that

$$\Phi(A) = \text{co}\{\Phi(E_j) \,:\, j = 0,1,2,3,4\} = \text{co}\{(2,1),(-2,2),(-1,-3)\}$$

so that $B = (\Phi|_A)^{-1}(W)$, where $W = \{(u,v) \,:\, u^2 + v^2 \le 1\} \subset \Phi(A)$. According to Theorem 4, every point $e \in \text{ext}(B)$ has the form: $e = sE_1 + tE_j$ for $j = 2,3,4$ or $e = sE_j + tE_4$ for $j = 2,3$ or else $e = (1-s-t)E_1 + sE_j + tE_4$ for $j = 2,3$, where $s \ge 0$, $t \ge 0$, $s + t \le 1$, and also $\Phi(e) \in \partial W$ except $e = E_0 \in B$. Thus, because of Theorem 9(i), we need to consider the following four cases.

(i) $e = sE_1 + tE_j$, $j = 2,3$. Then $\Phi(e) \in \partial W = \{(\cos\varphi, \sin\varphi) \,:\, -\pi < \varphi \le \pi\}$ iff

$$\begin{aligned}\|(s,t,0,0)\| &= \|(s,0,t,0)\| \\ &= [(13 + 4\sin 2\varphi - 3\cos 2\varphi)/72]^{1/2} \le 0.5\end{aligned}$$

with equality only for $\tan\varphi = 2$, $0 < \varphi < \pi/2$, that is for $s = 2t = 1/\sqrt{5}$.

(ii) $e = sE_1 + tE_4$. Then $\Phi(e) \in \partial W$ iff

$$\|(s,0,0,t)\| = [(3 - 2\sin\varphi + \cos 2\varphi)/10]^{1/2} \le \sqrt{(3+\sqrt{5})/10}$$

with equality only for $\tan\varphi = (1-\sqrt{5})/2$, $-\pi/2 < \varphi < 0$, that is for $s = \sqrt{5+2\sqrt{5}}/5$ and $t = \sqrt{10+2\sqrt{5}}/10$. Here $\sqrt{(3+\sqrt{5})/10} < 0.724$.

(iii) $e = sE_j + tE_4$, $j = 2,3$. Then $\Phi(e) \in \partial W$ iff

$$\begin{aligned}\|(0,s,0,t)\| = \|(0,0,s,t)\| &= [(9 + \sin 2\varphi + 4\cos 2\varphi)/64]^{1/2} \\ &\le \sqrt{9+\sqrt{17}}/8\end{aligned}$$

with equality only for $\tan\varphi = \sqrt{17} - 4$, $-\pi < \varphi < -\pi/2$, that is for $s = \sqrt{5 + 13/\sqrt{17}}/8$ and $t = \sqrt{1 + 1/\sqrt{17}}/4$. Here $\sqrt{9+\sqrt{17}}/8 < 0.453$.

(iv) $e = (1-s-t)E_1 + sE_j + tE_4$, $j = 2,3$. Then $\Phi(e) \in \partial W$ iff

$$\begin{aligned}\|(1-s-t,s,0,t)\| &= \|(1-s-t,0,s,t)\| \\ &= \sqrt{151 + F(\cos\varphi,\sin\varphi)}/19\,,\end{aligned}$$

where $F(u,v) = 4u(4u+7) + 2(1+3u)(-v)$. Since $F(u,v) \le 4u(4u+7) + 2|1+3u| \le 2$ for $-1 \le u \le 0$, $|v| \le 1$, and $F(1,0) = 44$, to find

$\max\{F(u,v)\,:\,u^2+v^2=1\}$ it is enough to consider $0 \le u \le 1$ and $v = -\sqrt{1-u^2}$. The critical points of the function

$$u \mapsto F(u, -\sqrt{1-u^2})\,, \quad 0 < u < 1\,, \tag{9}$$

satisfy the equation

$$L(u) = (14+16u)\sqrt{1-u^2} = 6u^2 + u - 3 = R(u)\,,$$

where L is strictly concave on $[0;1]$, R is strictly convex on $[0;1]$, $R(0) = -3 < L(0) = 14$ and $R(1) = 4 > L(1) = 0$. Thus there is only one critical point u_0 of the function (9), $u_0 = 0.99148\ldots$, $F(0,-1) = 2$, $F(1,0) = 44$ and $F(u_0, -\sqrt{1-u_0^2}) = 44.52537\ldots$. Thus the maximal norm in the current case is equal to $0.735949\ldots$ and is attained only by two elements $(1-s-t, s, 0, t)$, $(1-s-t, 0, s, t)$, with $s = (5-4\cos\varphi+3\sin\varphi)/19$, $t = (6-\cos\varphi-4\sin\varphi)/19$, $\cos\varphi = u_0$ and $\sin\varphi = -\sqrt{1-u_0^2} = -0.13024\ldots$. Because of (i)–(iii), this is the maximal case.

Solution of Problem 2. Observe that $Z = \{(\lambda x, \lambda y, \lambda z, \lambda w)\,:\,\lambda \ge 0, (x,y,z,w) \in A\}$, where $A = \mathrm{co}\{E_1, E_2, E_3, E_4\}$, see the solution of Problem 1. Define

$$\Phi(x,y,z,w) = (y+5z+5w,\; 3x-2y-3z-3w)\,,$$

a linear map from $\mathbb{R}^4$ onto $\mathbb{R}^2$. Clearly, $(0,0) \notin \Phi(A) = \mathrm{co}\{(0,3), (1,-2), (5,-3)\}$, $\Phi(Z) = \{(u,v)\,:\,v \ge -2u,\; u \ge 0\}$ and $B = (\Phi|_Z)^{-1}(W)$, where $W = \{(u,v)\,:\,-2u \le v \le 4-2u^2\,,\; u \ge 0\} \subset \Phi(Z)$. By Theorem 5,

$$\begin{aligned}
\mathrm{ext}(B) \subset{} & \{\lambda E_1\,:\,0 \le \lambda \le 4/3\} \cup \{\lambda E_2\,:\,0 \le \lambda \le 2\} \\
& \cup \left\{\frac{3+\sqrt{809}}{100}E_j\,:\,j = 3,4\right\} \cup \left\{\frac{4+2t-2t^2}{3}E_1 + tE_2\,:\;0 < t < 2\right\} \\
& \cup \left\{\frac{4+3t-50t^2}{3}E_1 + tE_j\,:\,0 < t < \frac{3+\sqrt{809}}{20}\,,\; j = 3,4\right\} \\
& \cup \left\{\frac{10u^2-3u-20}{7}E_2 + \frac{4+2u-2u^2}{7}E_j\,:\right. \\
& \qquad \left.\frac{3+\sqrt{809}}{20} < u < 2\,,\; j = 3,4\right\}.
\end{aligned}$$

Observe that $\frac{3+\sqrt{809}}{100} < 0.315$, $u_0 = \frac{3+\sqrt{809}}{20} > 1.572$, and

(i) $$4 - \left(\frac{4+2t-2t^2}{3}\right)^2 - t^2 = \frac{(2-t)(4t^3+3(2-t)+4)}{9} > 0 \text{ for } 0 < t < 2\,,$$

(ii) $$\left(\frac{4+3t-50t^2}{3}\right)^2 + t^2 < \frac{4.1^2}{9} + 0.315^2 < 2 \quad \text{for } 0 < t < 0.315\,,$$

(iii) $$\frac{(10u^2-3u-20)^2+(4+2u-2u^2)^2}{49} < 4 \quad \text{for } u_0 < u < 2\,,$$

since $u \mapsto h(u) = (10u^2-3u-20)^2+(4+2u-2u^2)^2$ is strictly convex on $[1;2]$. In fact, we have $h''(0) < 0 < h''(1)$, which means that $h'' > 0$ on $[1;2]$. Hence $h(u) < \max\{h(u_0), h(2)\} = 14^2 = h(2)$ for $u_0 < u < 2$, as we have $h(u_0) = (4+2u_0-2u_0^2)^2 = 49u_0^2/25 < 4.9$. Finally, in accordance with Theorem 10, the point $2E_2 = (0,2,0,0)$ is the only element of the set B with maximal norm.

Remark 11. Suppose now that $\mathbb{M}_2$ (resp. $\mathbb{M}_1$) is the collection of all complex (resp. real) Borel measures on a compact metric space T. The classes of measures

$$I_\alpha = \{\mu \in \mathbb{M}_k \,:\, |\mu(T)-1| + |\mu|(T) \le \alpha\}, \quad \alpha \ge 1\,,$$

and

$$U_\alpha = \{\mu \in \mathbb{M}_k \,:\, \mu(T) = 1,\ |\mu|(T) \le \alpha\}, \quad \alpha \ge 1\,,$$

where $k = 1, 2$, are both convex and weak*-compact. In [7] the sets $\mathrm{ext}(I_\alpha)$ and $\mathrm{ext}(U_\alpha)$ have been determined as an application of Theorems 7, 11.

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