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On the growth of the derivative of Q_p functions

Abstract. In this paper we investigate some properties of the derivative of functions in the Q_p spaces. We first show that $T(r, f')$, the Nevanlinna characteristic of the derivative of a function $f \in Q_p$, $0 < p < 1$, satisfies

$$\int_0^1 (1-r)^p \exp\bigl(2T(r,f')\bigr)\,dr < \infty,$$

and that this estimate is sharp in a very strong sense, extending thus a similar result of Kennedy for functions in the Nevanlinna class.

We also obtain several results concerning the radial growth of the derivative of Q_p functions.

1. Introduction and statements of results. Let Δ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. The Nevanlinna characteristic of an analytic function f

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in Δ is defined by

$$T(r,f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{+}\left|f(re^{i\theta})\right|d\theta,\quad 0\le r<1.$$

The Nevanlinna class N consists of functions f analytic in Δ such that

$$\sup_{0\le r<1}T(r,f)<\infty.$$

It is well known that the condition $f\in N$ does not imply $f'\in N$. This was first proved by O. Frostman [11], who showed the existence of a Blaschke product whose derivative is not of bounded characteristic. Subsequently many other examples have been given. Kennedy [17] obtained the sharp bound on the growth of $T(r,f')$ for $f\in N$. Namely, he proved that if $f\in N$, then

$$\int_0^1(1-r)\exp\big(2T(r,f')\big)dr<\infty, \tag{1}$$

and showed that this result is sharp in the sense that if ϕ is a positive increasing function in $(0,1)$ which satisfies certain "regularity conditions" and is such that

$$\int_0^1(1-r)\exp\big(2\phi(r)\big)dr<\infty,$$

then there exists $f\in N$ such that $T(r,f')>\phi(r)$ for all r sufficiently close to 1.

Since $T(r,f')$ is an increasing function of r, (1) easily implies for $f\in N$

$$\log\frac{1}{1-r}-T(r,f')\longrightarrow\infty \text{ as } r\to 1. \tag{2}$$

For $0<p<\infty$ the following spaces are defined:

$$Q_p=\Big\{f \text{ analytic in } \Delta:\sup_{a\in\Delta}\iint_{\Delta}\left|f'(z)\right|^2g(z,a)^p\,dxdy<\infty\Big\},$$
$$Q_{p,0}=\Big\{f \text{ analytic in } \Delta:\lim_{|a|\to 1}\iint_{\Delta}\left|f'(z)\right|^2g(z,a)^p\,dxdy=0\Big\},$$

where $g(z,a)$ is the Green function of Δ, given by

$$g(z,a)=\log\left|\frac{1-\overline{a}z}{z-a}\right|.$$

These spaces were introduced by R. Aulaskari and P. Lappan in [3] while looking for new characterizations of Bloch functions. They proved that for $p > 1$,

$$Q_p = \mathcal{B}, \qquad \text{and} \qquad Q_{p,0} = \mathcal{B}_0.$$

Recall that the Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$ consist, respectively, of those functions f analytic in Δ for which (see [1] for more information on these spaces)

$$\sup_{z\in\Delta}\big(1-|z|^2\big)\big|f'(z)\big| < \infty, \qquad \text{and} \qquad \lim_{|z|\to 1}\big(1-|z|^2\big)\big|f'(z)\big| = 0.$$

In fact, Q_p spaces put under the same frame a number of important spaces of functions analytic in Δ. We have, using one of the many characterizations of the spaces $BMOA$ and $VMOA$ (see, e.g., [6,12]):

$$Q_1 = BMOA, \qquad \text{and} \qquad Q_{1,0} = VMOA.$$

We refer to [2,5,4,9] for more properties of Q_p spaces.It is shown in [5], that Q_p spaces increase with increasing p,

$$Q_p \subset Q_q \subset BMOA, \qquad 0 < p < q < 1, \tag{3}$$

all the inclusions being strict.

The first object of this paper is to study the possibility of extending Kennedy's results to Q_p spaces. First of all, let us notice that the function f constructed by Kennedy to show the sharpness of (1) was given by a power series with Hadamard gaps, i.e., of the form

$$f(z) = \sum_{k=0}^{\infty} c_k z^{n_k}, \qquad \frac{n_{k+1}}{n_k} \geq \lambda > 1,$$

and such that $\sum |c_k|^2 < \infty$. Such a function belongs to $BMOA$ (see [6, p. 25]) and, even more, to $VMOA$. Since $VMOA \subset BMOA \subset H^p \subset N$, $0 < p < \infty$, (we refer to [8] for the theory of H^p spaces,) it follows that (1) is sharp for $VMOA = Q_{1,0}$ and, hence, for $BMOA = Q_1$ and for all H^p spaces with $0 < p < \infty$. On the other hand, we remark that Girela [13] showed that (1) can be improved for the Dirichlet class $\mathcal{D}$, consisting of all analytic functions in Δ with a finite Dirichlet integral, i.e., such that

$$\iint_\Delta \big|f'(z)\big|^2 dxdy < \infty.$$

It is worth noticing that $\mathcal{D} \subset Q_{p,0}$ for all $p > 0$, the inclusion being strict, see [5].

Now we turn to Q_p spaces with $p > 1$. As said before, $Q_p = \mathcal{B}$ and $Q_{p,0} = \mathcal{B}_0$ for all $p > 1$. We have the following trivial estimate:

$$f \in \mathcal{B} \quad \Longrightarrow \quad T(r, f') \le \log \frac{1}{1-r} + \mathrm{O}\ (1), \quad \text{as } r \to 1.$$

Girela [14] proved that this is sharp in the sense that there exists $f \in \mathcal{B}$ such that

$$\log \frac{1}{1-r} - T(r, f') = \mathrm{O}\ (1), \quad \text{as } r \to 1,$$

and, consequently,

$$\int_0^1 (1-r) \exp\bigl(2T(r, f')\bigr) dr = \infty.$$

Hence, neither (1) nor (2) is true for the Bloch space.

On the other hand, if $f \in \mathcal{B}_0$ then it trivially satisfies (2). However, Girela [14] proved that there exists $f \in \mathcal{B}_0$ which does not satisfy (1).

Hence, it remains to consider Q_p spaces with $0 < p < 1$. We can prove the following results.

Theorem 1. *If $f \in Q_p$, $0 < p < 1$, then*

$$\int_0^1 (1-r)^p \exp\bigl(2T(r, f')\bigr) dr < \infty. \tag{4}$$

Corollary. *If $f \in Q_p$, $0 < p < 1$, then*

$$\frac{p+1}{2} \log \frac{1}{1-r} - T(r, f') \underset{r \to 1}{\longrightarrow} \infty. \quad \square \tag{5}$$

The following theorem shows the sharpness of Theorem 1.

Theorem 2. *Let $0 < p < 1$, and let ϕ be a positive increasing function in $(0, 1)$ satisfying:*

(i) $(1-r)^{\frac{p+1}{2}} \exp \phi(r)$ *decreases as r increases in $(0, 1)$;*

(ii) $\phi(r) - \phi(\rho) \to \infty$, *as* $\dfrac{1-r}{1-\rho} \to 0$;

(iii) $\displaystyle\int_0^1 (1-r)^p \exp\bigl(2\phi(r)\bigr) dr < \infty.$

Then there exists a function $f \in Q_p$ such that, for all r sufficiently close to 1,

$$T(r, f') > \phi(r). \tag{6}$$

Now we turn our attention to study the radial growth of the derivative of Q_p functions. If $p > 1$ and $f \in Q_p = \mathcal{B}$ then, trivially,

$$\left|f'(re^{i\theta})\right| = \mathrm{O}\Big((1-r)^{-1}\Big), \qquad \text{as } r \to 1, \text{ for every } \theta \in \mathbb{R}.$$

This is the best that can be said. Indeed, if $q \in \mathbb{N}$ is sufficiently large, there is $C_q > 0$ such that

$$f(z) = C_q \sum_{k=0}^{\infty} z^{q^k}, \qquad z \in \Delta,$$

satisfies $f \in \mathcal{B}$ and

$$\left|f'(z)\right| \geq \frac{1}{1-|z|^2} \qquad \text{if} \quad 1 - \frac{1}{q^k} \leq |z| \leq 1 - \frac{1}{q^{k+\frac{1}{2}}},$$

(see [19]) which implies

$$\limsup_{r \to 1} (1-r^2)\left|f'(re^{i\theta})\right| \geq 1, \qquad \text{for every } \theta.$$

If $f \in BMOA$, then it has a finite non-tangential limit $f(e^{i\theta})$ for almost every $\theta \in \mathbb{R}$, so, by a result of Zygmund [22, p. 181], it follows that for almost every θ,

$$\left|f'(re^{i\theta})\right| = \mathrm{o}\Big((1-r)^{-1}\Big), \quad \text{as } r \to 1. \tag{7}$$

This result is also sharp in the sense that the right hand side of (7) cannot be substituted by $\mathrm{O}\big((1-r)^{-\alpha}\big)$ for any $\alpha < 1$. Indeed, if

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^{2^k}, \qquad z \in \Delta,$$

then, since f is given by a power series with Hadamard gaps in H^2, we have $f \in BMOA$. Also, by Lemma 1 [22, p. 197], the fact $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$ implies

$$\int_0^1 \left|f'(re^{i\theta})\right| dr = \infty, \qquad \text{for every } \theta \in \mathbb{R}. \tag{8}$$

Consequently, we have proved the following

Proposition 1. *There exists $f \in BMOA$ such that, for any $\alpha < 1$ and any θ*

$$|f'(re^{i\theta})| \neq O\Big((1-r)^{-\alpha}\Big), \qquad \text{as } r \to 1.$$

However, an estimate which is much stronger than (7) is true for the Dirichlet space $\mathcal{D}$. Seidel and Walsh [20, Thm. 6] proved that if $f \in \mathcal{D}$ then, for a.e. θ,

$$|f'(re^{i\theta})| = \mathrm{o}\Big((1-r)^{-1/2}\Big), \qquad \text{as } r \to 1, \tag{9}$$

and Girela [13] proved that this is sharp in a very strong sense.

Now, we shall consider these questions for Q_p spaces, $0 < p \leq 1$. We can prove the following results.

Theorem 3. *If $f \in Q_p$, $0 < p \leq 1$, then for a.e. θ,*

$$|f'(re^{i\theta})| = o\Big((1-r)^{-(p+1)/2}\Big), \quad \text{as } r \to 1. \tag{10}$$

Theorem 4. *Let $0 < p \leq 1$, and let ϕ be a positive increasing function in $(0,1)$ such that*

$$\int_0^1 (1-r)^p \phi^2(r)\,dr < \infty. \tag{11}$$

Then there exists $f \in Q_p$ such that, for every θ,

$$\limsup_{r \to 1^-} \frac{|f'(re^{i\theta})|}{\phi(r)} = \infty. \tag{12}$$

We remark that Theorem 4 for $p = 1$ represents an improvement of Proposition 1.

Finally, let us mention that the techniques used in this work are related to those used by Kennedy [17] and by Girela [13]. Also, we will adopt the convention that C will always denote a positive constant, independent of r, which may be different on other occasion.

2. Proofs of Theorems 1 and 2.

Let $f \in Q_p$, with $0 < p < 1$. By Jensen's inequality, we have

$$\begin{aligned}
\exp\big(2T(r,f')\big) &= \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\log^+|f'(re^{i\theta})|\,d\theta\right) \\
&\leq \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\Big(1+|f'(re^{i\theta})|^2\Big)d\theta\right) \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(1+|f'(re^{i\theta})|^2\Big)d\theta.
\end{aligned}$$

Multiplying by $(1-r)^p$ and integrating, we obtain

$$\int_0^1 (1-r)^p \exp\bigl(2T(r,f')\bigr)dr \le \frac{1}{2\pi}\int_0^1\int_{-\pi}^{\pi}(1-r)^p\Bigl(1+\bigl|f'(re^{i\theta})\bigr|^2\Bigr)d\theta dr.$$

We now refer to [4, Thm. 1.1], where it is shown that a function f is in Q_p, $0<p\le 1$, if and only if $d\mu(z) = \bigl(1-|z|\bigr)^p\bigl|f'(z)\bigr|^2 dxdy$ is a p-Carleson measure. A p-Carleson measure is a finite Borel measure μ in Δ for which there exists a constant $c>0$ such that for all intervals I of the form $I=(\theta_0,\theta_0+h)$, $\theta_0\in\mathbb{R}$ and $0<h<1$, we have

$$\mu\bigl(S(I)\bigr)\le ch^p,$$

where $S(I)$ is the classical Carleson square,

$$S(I)=\{re^{i\theta} : \theta_0<\theta<\theta_0+h,\ 1-h<r<1\}.$$

All this tells us that the term on the right hand side of the above inequality is finite, and therefore Theorem 1 follows. $\square$

To prove Theorem 2, take $0<p<1$, and ϕ as in the statement. Since ϕ is increasing, (iii) implies

$$\begin{aligned}\infty &> \int_0^1 (1-r)^p\exp\bigl(2\phi(r)\bigr)dr \ge \sum_{k=1}^{\infty}\int_{1-2^{-k}}^{1-2^{-(k+1)}}(1-r)^p\exp\bigl(2\phi(r)\bigr)dr\\ &\ge \sum_{k=1}^{\infty}2^{-(k+1)(p+1)}\exp\bigl(2\phi(1-2^{-k})\bigr)\\ &= 2^{-(p+1)}\sum_{k=1}^{\infty}2^{-k(p+1)}\exp\bigl(2\phi(1-2^{-k})\bigr).\end{aligned}$$

So (see for instance [18, Dini's Thm, p. 297] there exists an increasing sequence $\{\alpha_k\}$ of integers greater than 2, such that

$$\sum_{k=1}^{\infty}\alpha_k^2 2^{-k(p+1)}\exp\bigl(2\phi(1-2^{-k})\bigr)<\infty, \tag{13}$$

and

$$\alpha_k \longrightarrow \infty\ ,\quad \alpha_{k+1}/\alpha_k\longrightarrow 1\quad \text{as } k\to\infty\ . \tag{14}$$

Observe that condition (13) implies

$$\sum_{k=1}^{\infty}\alpha_k^{p+1}2^{-k(p+1)}\exp\bigl(2\phi(1-2^{-k})\bigr)<\infty. \tag{15}$$

Define now

$$n_1 = 1, \qquad n_{k+1} = \alpha_k n_k, \quad k = 1, 2, \dots . \tag{16}$$

Clearly $n_{k+1} > 2^k$ for $k \geq 1$ and by (i) we obtain

$$\alpha_k^{p+1} n_{k+1}^{-(p+1)} \exp\big(2\phi(1 - n_{k+1}^{-1})\big) \leq \alpha_k^{p+1} 2^{-k(p+1)} \exp\big(2\phi(1 - 2^{-k})\big),$$

which, together with (15) and (16), yields

$$\sum_{k=1}^{\infty} n_k^{-(p+1)} \exp\big(2\phi(1 - n_{k+1}^{-1})\big) < \infty. \tag{17}$$

For each $k = 1, 2, \dots$, set

$$c_k = 10\, n_k^{-1} \exp\big(\phi(1 - n_{k+1}^{-1})\big), \tag{18}$$

and define the function

$$f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}, \qquad z \in \Delta. \tag{19}$$

The way in which n_k and c_k have been chosen shows that f is a power series with Hadamard gaps defined in Δ. So in order to see that $f \in Q_p$, we will use the following result proved in [5].

Theorem A. *If $0 < p \leq 1$, and $f(z) = \sum_{k=0}^{\infty} c_k z^{n_k}$ is a power series with Hadamard gaps, then*

$$f \in Q_p \iff f \in Q_{p,0} \iff \sum_{k=0}^{\infty} 2^{k(1-p)} \sum_{\{j: n_j \in I_k\}} |c_j|^2 < \infty, \tag{20}$$

where $I_k = \{n \in \mathbb{N} : 2^k \leq n < 2^{k+1}\}$, $k = 0, 1, \dots$.

For each $j \in \mathbb{N}$, let $k(j)$ be the unique non-negative integer such that $2^{k(j)} \leq n_j < 2^{k(j)+1}$. Bearing in mind this and (17), we have

$$\begin{aligned}
\sum_{k=0}^{\infty} 2^{k(1-p)} \sum_{\{j: n_j \in I_k\}} |c_j|^2 &= \sum_{j=1}^{\infty} 2^{k(j)(1-p)} |c_j|^2 \\
&= 10^2 \sum_{j=1}^{\infty} 2^{k(j)(1-p)} n_j^{-2} \exp\big(2\phi(1 - n_{j+1}^{-1})\big) \\
&\leq 10^2 \sum_{j=1}^{\infty} n_j^{-(p+1)} \exp\big(2\phi(1 - n_{j+1}^{-1})\big) < \infty.
\end{aligned}$$

Hence, $f \in Q_p$.

Next, we show that f satisfies (6). Observe that for $k \geq 2$ and $|z| = 1 - \frac{1}{n_k}$,

$$\begin{aligned} |f'(z)| \geq |zf'(z)| &= \Big|\sum_{j=1}^{\infty} c_j n_j z^{n_j}\Big| \\ &\geq c_k n_k |z|^{n_k} - \sum_{j=1}^{k-1} c_j n_j |z|^{n_j} - \sum_{j=k+1}^{\infty} c_j n_j |z|^{n_j} \\ &\geq c_k n_k \Big(1 - \frac{1}{n_k}\Big)^{n_k} - \sum_{j=1}^{k-1} c_j n_j - \sum_{j=k+1}^{\infty} c_j n_j \Big(1 - \frac{1}{n_k}\Big)^{n_j} \\ &= \text{(I)} - \text{(II)} - \text{(III)}. \end{aligned}$$

Since the sequence $(1 - \frac{1}{n})^n$ increases with n, and $n_k \geq 2$,

$$\text{(I)} \geq \frac{1}{4} c_k n_k. \tag{21}$$

Now, in order to estimate (II) and (III), we will use the following lemma stated in [17, p. 339].

Lemma 1. *If $\{s_k\}$ is a sequence of positive numbers and $s_k/s_{k+1} \to 0$ as $k \to \infty$, then,*

$$\sum_{j=1}^{k-1} s_j = o(s_k), \quad \text{and} \quad \sum_{j=k+1}^{\infty} s_j^{-1} = o(s_k^{-1}) \quad \text{as } k \to \infty.$$

Notice that by (18), (ii), (16), and (14),

$$\frac{c_k n_k}{c_{k+1} n_{k+1}} = \exp\big(\phi(1 - n_{k+1}^{-1}) - \phi(1 - n_{k+2}^{-1})\big) \to \infty \quad \text{as } k \to \infty,$$

so by the lemma,

$$\text{(II)} = \text{o}(c_k n_k), \qquad \text{as } k \to \infty. \tag{22}$$

Now using the elementary inequality $(1 - x)^n < 2(nx)^{-2}$, valid for $0 < x < 1$ and $n \geq 1$, we obtain

$$\text{(III)} \leq 2n_k^2 \sum_{j=k+1}^{\infty} \frac{c_j}{n_j}. \tag{23}$$

But also, by (18), (16), (i), and (14),

$$\frac{n_k/c_k}{n_{k+1}/c_{k+1}} = \frac{1}{\alpha_k^2}\,\frac{\exp\phi(1-n_{k+2}^{-1})}{\exp\phi(1-n_{k+1}^{-1})} \le \frac{1}{\alpha_k^2}\Big(\frac{n_{k+2}}{n_{k+1}}\Big)^{\frac{p+1}{2}}$$
$$= \frac{1}{\alpha_k^{(3-p)/2}}\Big(\frac{\alpha_{k+1}}{\alpha_k}\Big)^{\frac{p+1}{2}} \to 0,$$

so by (23) and the lemma again,

$$\text{(III)} = \mathrm{o}(c_k n_k), \qquad \text{as } k\to\infty. \tag{24}$$

Therefore, by (21), (22), and (24), there exists k_0 such that for all $k \ge k_0$,

$$\big|f'(z)\big| > \frac{1}{8}c_k n_k > \exp\phi\big(1-\frac{1}{n_{k+1}}\big), \qquad |z| = 1-\frac{1}{n_k}.$$

Thus, for $k \ge k_0$,

$$T\big(1-\frac{1}{n_k}, f'\big) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+\Big|f'\Big(\big(1-\frac{1}{n_k}\big)e^{i\theta}\Big)\Big|d\theta > \phi\big(1-\frac{1}{n_{k+1}}\big).$$

Now, if $r \ge 1-(n_{k_0})^{-1}$, take $k \ge k_0$ such that $1-(n_k)^{-1} \le r < 1-(n_{k+1})^{-1}$. Since T and ϕ are increasing functions of r, we obtain

$$T(r,f') \ge T\big(1-\frac{1}{n_k}, f'\big) > \phi\big(1-\frac{1}{n_{k+1}}\big) \ge \phi(r).$$

This completes the proof of Theorem 2. □

2. Proofs of Theorems 3 and 4. We start proving Theorem 3. Let $f \in Q_p$. Set

$$F_r(\theta) = \max_{0\le\rho\le r}\big|f'(\rho e^{i\theta})\big|^2, \qquad 0<r<1,\ \theta\in\mathbb{R}.$$

By the Hardy-Littlewood Maximal Theorem,

$$\int_{-\pi}^{\pi} F_r(\theta)d\theta \le C\int_{-\pi}^{\pi}\big|f'(re^{i\theta})\big|^2 d\theta, \qquad 0<r<1.$$

Since $g(z,0) = \log\frac{1}{|z|}$ and $f \in Q_p$, we have

$$\int_0^1\int_{-\pi}^{\pi} F_r(\theta)\Big(\log\frac{1}{r}\Big)^p r\,d\theta dr \le C\int_0^1\int_{-\pi}^{\pi}\big|f'(re^{i\theta})\big|^2 g(re^{i\theta},0)^p r\,d\theta dr < \infty.$$

Hence we deduce that

$$\int_0^1 F_r(\theta)\Big(\log\frac{1}{r}\Big)^p r\,dr < \infty, \qquad \text{a.e. } \theta,$$

which yields, by means of the equivalence $\log\frac{1}{r} \sim (1-r)$ as $r \to 1$,

$$\lim_{r\to 1}\int_r^1 F_s(\theta)(1-s)^p ds = 0, \qquad \text{a.e. } \theta.$$

Since F is an increasing function of r, we have for a.e. θ

$$\big|f'(re^{i\theta})\big|^2\frac{(1-r)^{p+1}}{p+1} \le F_r(\theta)\int_r^1 (1-s)^p ds \le \int_r^1 F_s(\theta)(1-s)^p ds \underset{r\to 1}{\longrightarrow} 0$$

and (10) follows. $\square$

Proof of Theorem 4. We may assume without loss of generality that $\phi(r) \nearrow \infty$ as $r \nearrow 1$. Also, it suffices to prove that there exist $f \in Q_p$ and $C > 0$ such that for every θ

$$\limsup_{r\to 1}\frac{\big|f'(re^{i\theta})\big|}{\phi(r)} \ge C. \tag{25}$$

The reason for this is that if ϕ is a positive increasing function in $(0,1)$ satisfying (11), then it is possible to find ϕ_1, positive and increasing in $(0,1)$ with $\lim_{r\to 1}\phi_1(r) = \infty$, and such that

$$\int_0^1 (1-r)^p\phi^2(r)\,\phi_1^2(r)dr < \infty.$$

Clearly, if there are $f \in Q_p$ and $C > 0$ satisfying (25) for every θ, with ϕ replaced by $\phi\,\phi_1$, then the same f satisfies equation (12) for every θ.

With these assumptions we may start the proof. Take a sequence $\{r_k\} \nearrow 1$, with $r_1 > 1/4$, which satisfies

$$r_{k+1} - r_k > \frac{1}{2}(1-r_k), \quad \text{for all } k, \tag{26}$$

$$\phi(r_{k+1})/\phi(r_k) \to \infty \quad \text{as } k \to \infty, \tag{27}$$

$$\frac{(1-r_{k+1})^{\frac{3-p}{2}}}{(1-r_k)^2} = \mathrm{O}(1), \quad \text{as } k \to \infty. \tag{28}$$

It follows from (26) that for all k

$$1 - r_{k+1} < \frac{1}{2}(1-r_k) < r_{k+1} - r_k. \tag{29}$$

Bearing this in mind, observe that for all $k \in \mathbb{N}$

$$\int_{r_k}^{r_{k+1}} (1-r)^p dr = \frac{1}{1+p}\left((1-r_k)^{1+p} - (1-r_{k+1})^{1+p}\right) \geq \frac{1-2^{-(1+p)}}{1+p}(1-r_k)^{1+p}.$$

Since ϕ is increasing, (11) implies

$$\begin{aligned} \sum_{k=1}^{\infty}(1-r_k)^{1+p}\phi^2(r_k) &\leq \frac{1+p}{1-2^{-(1+p)}} \sum_{k=1}^{\infty} \int_{r_k}^{r_{k+1}} (1-r)^p \phi^2(r_k) dr \\ &\leq \frac{1+p}{1-2^{-(1+p)}} \sum_{k=1}^{\infty} \int_{r_k}^{r_{k+1}} (1-r)^p \phi^2(r) dr \\ &\leq \frac{1+p}{1-2^{-(1+p)}} \int_0^1 (1-r)^p \phi^2(r) dr < \infty. \end{aligned} \tag{30}$$

Now, for each k, let n_k be the unique non-negative integer such that

$$n_k \leq \frac{1}{1-r_k} < n_k + 1.$$

This implies, together with the facts that $\{r_k\}$ is increasing and $r_1 \geq 1/4$,

$$1 - \frac{1}{n_k} \leq r_k < 1 - \frac{1}{n_k+1}, \qquad \text{and} \qquad \frac{1}{4} < n_k(1-r_k) \leq 1. \tag{31}$$

Define now

$$f(z) = \sum_{k=1}^{\infty} (1-r_k)\phi(r_k) z^{n_k}.$$

By (30), f is analytic in Δ. Moreover, f is a power series with Hadamard gaps. Indeed, by the definition of n_k and by (29),

$$\frac{n_{k+1}}{n_k} \geq \frac{\frac{1}{1-r_{k+1}} - 1}{\frac{1}{1-r_k}} = \frac{1-r_k}{1-r_{k+1}} - (1-r_k) > 2 - \frac{3}{4} > 1, \quad \text{all } k.$$

We now check that f is in Q_p. To this end we use Theorem A. For each j, let $k(j)$ be the unique non-negative integer such that

$$2^{k(j)} \leq n_j < 2^{k(j)+1}.$$

In this situation, we have by (31) and (30),

$$\begin{aligned} \sum_{k=0}^{\infty} 2^{k(1-p)} \sum_{2^k \leq n_j < 2^{k+1}} (1-r_j)^2 \phi^2(r_j) &= \sum_{j=1}^{\infty} 2^{k(j)(1-p)} (1-r_j)^2 \phi^2(r_j) \\ \leq \sum_{j=1}^{\infty} n_j^{1-p} (1-r_j)^2 \phi^2(r_j) &\leq \sum_{j=1}^{\infty} (1-r_j)^{1+p} \phi^2(r_j) < \infty. \end{aligned}$$

This shows that $f \in Q_p$.

Next, to show that f satisfies (25), it suffices to find a constant $C > 0$ and $k_0 \in \mathbb{N}$ such that

$$\frac{|f'(r_k e^{i\theta})|}{\phi(r_k)} \geq C \text{ for every } \theta \text{ and all } k \geq k_0.$$

If $|z| = r_k$ $(k \geq 2)$ then, (31) and $r_k^{n_j} \leq 1$ imply

$$\begin{aligned}
|f'(z)| \geq |zf'(z)| &= \Big|\sum_{j=1}^{\infty} n_j(1-r_j)\phi(r_j)z^{n_j}\Big| \\
&\geq n_k(1-r_k)\phi(r_k)r_k^{n_k} - \sum_{j\neq k} n_j(1-r_j)\phi(r_j)r_k^{n_j} \\
&\geq \frac{1}{4}\phi(r_k)\Big(1-\frac{1}{n_k}\Big)^{n_k} - \sum_{j=1}^{k-1}\phi(r_j) - \sum_{j=k+1}^{\infty}\phi(r_j)\Big(1-\frac{1}{n_k+1}\Big)^{n_j} \\
&= \text{(I)} - \text{(II)} - \text{(III)}.
\end{aligned}$$

The procedure now is basically the same as in the proof of Theorem 2. Since the sequence $(1-\frac{1}{n})^n$ increases with n and $n_k \geq 2$, we have (I) $\geq C\phi(r_k)$. Now, by (27) and Lemma 1 we obtain (II) $= \mathrm{o}\big(\phi(r_k)\big)$. Finally, as in (23), we deduce

$$\text{(III)} \leq 2(n_k+1)^2 \sum_{j=k+1}^{\infty} \frac{\phi(r_j)}{n_j^2}.$$

But by (31), (28) and (30),

$$\begin{aligned}
\frac{n_j^2/\phi(r_j)}{n_{j+1}^2/\phi(r_{j+1})} &\leq \frac{16}{\phi(1/4)}\,\frac{(1-r_{j+1})^2\phi(r_{j+1})}{(1-r_j)^2} \\
&= \frac{16}{\phi(1/4)}\,\frac{(1-r_{j+1})^{\frac{3-p}{2}}}{(1-r_j)^2}\,(1-r_{j+1})^{\frac{1+p}{2}}\phi(r_{j+1}) \underset{j\to\infty}{\longrightarrow} 0
\end{aligned}$$

so by Lemma 1,

$$\sum_{j=k+1}^{\infty}\frac{\phi(r_j)}{n_j^2} = \mathrm{o}\Big(\frac{\phi(r_k)}{n_k^2}\Big),$$

which implies (III) $= \mathrm{o}\big(\phi(r_k)\big)$. This completes the proof of Theorem 4. □

4. Remarks.

Remark 1. The estimate given in Theorem 3 allows us to say something about the radial variation of functions in the Q_p spaces. We start recalling some definitions. For a function f analytic in the unit disk Δ and $\theta \in [-\pi, \pi]$, the quantity

$$V(f,\theta) = \int_0^1 \left|f'(re^{i\theta})\right| dr,$$

denotes the radial variation of f along the radius $[0, e^{i\theta}]$, i.e., the length of the image of this radius under the mapping f. The exceptional set $E(f)$ associated to f is then defined as

$$E(f) = \left\{e^{i\theta} \in \partial\Delta : V(f,\theta) = \infty\right\}.$$

Since $\int_0^1 (1-r)^{-(p+1)/2} dr$ is finite if and only if $p < 1$, then an immediate consequence of Theorem 3 is the following

Theorem 5. *If $f \in Q_p$, $0 < p < 1$, then the exceptional set $E(f)$ has linear measure 0.*

Observe that nothing of this kind can be stated for Q_p with $p \geq 1$. Indeed, as we have noticed above before Proposition 1, if $f(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^{2^k}$, then $f \in BMOA = Q_1$ and $V(f,\theta) = \infty$ for every θ.

On the other hand, for functions in the Dirichlet class $\mathcal{D} \equiv Q_0$ there is a more precise result due to Beurling [7].

Theorem B. *If $f \in \mathcal{D}$, then the exceptional set $E(f)$ has a zero logarithmic capacity.*

We refer to [10,16,21] for the definition and basic results about capacities and Hausdorff measures. We do not know whether the conclusion of Theorem B is true for Q_p, $0 < p < 1$. However, something can be said. For $0 < p < 1$, let $\mathcal{D}_p$ be the space of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, analytic in Δ such that

$$\sum_{n=1}^{\infty} n^{1-p} |a_n|^2 < \infty.$$

Zygmund proved the following result (see [16, Ch. 4]).

Theorem C. *If $f \in \mathcal{D}_p$, $0 < p < 1$, then the exceptional set $E(f)$ has zero p-capacity. Conversely, if E is a set of zero p-capacity, then there is $f \in \mathcal{D}_p$ whose exceptional set contains E.*

It is not difficult to see that $f \in Q_p$, $0 < p < 1$ implies $f \in \mathcal{D}_p$. In fact, if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in Q_p$, $0 < p < 1$, there exists $C > 0$ such that

$$\iint_\Delta \left|f'(z)\right|^2 g^p(z,a)\, dx\, dy < C, \qquad \text{for all } a \in \Delta.$$

In particular, we have for $a = 0$, using properties of the Beta function and Stirling's formula for the Gamma function: $\Gamma(t+1) \sim t^t e^{-t}(2\pi t)^{1/2}$,

$$\begin{aligned}\infty > \int_0^1 \int_{-\pi}^{\pi} \left|f'(re^{i\theta})\right|^2 \log^p \frac{1}{r} r dr d\theta &= \sum_{n=1}^{\infty} n^2|a_n|^2 \int_0^1 r^{2n-1} \log^p \frac{1}{r} dr \\ &\geq \sum_{n=1}^{\infty} n^2|a_n|^2 \int_0^1 r^{2n-1}(1-r)^p dr = \sum_{n=1}^{\infty} n^2|a_n|^2 \mathrm{B}(2n, p+1) \\ &= \sum_{n=1}^{\infty} n^2|a_n|^2 \frac{\Gamma(2n)\Gamma(p+1)}{\Gamma(2n+p+1)} \approx \sum_{n=1}^{\infty} n^{1-p}|a_n|^2.\end{aligned}$$

Therefore, an immediate consequence of Zygmund's result is the following

Theorem 6. *If $f \in Q_p$, $0 < p < 1$, then the exceptional set $E(f)$ has zero p-capacity.*

However, we do not know whether for a given set E of null p-capacity there is $f \in Q_p$ whose exceptional set contains E.

Remark 2. From Beurling's result (Theorem B), it follows that any $f \in \mathcal{D}$ has non-tangential limit everywhere except for a set of null logarithmic capacity, and then

$$\left|f'(re^{i\theta})\right| = \mathrm{o}\big((1-r)^{-1}\big) \text{ as } r \to 1, \tag{32}$$

whenever $e^{i\theta}$ is a point at which f has a finite non-tangential limit.

This implies that for $f \in \mathcal{D}$ the estimate (32) holds for every $\theta \in (-\pi, \pi]$, except for a set of null logarithmic capacity. Girela [15] showed that this estimate is sharp in a very strong sense. In our case, using Theorem 6 and (32), we obtain a similar result for Q_p, $0 < p < 1$, although we do not know whether it is sharp in the sense given by Girela.

Theorem 7. *If $f \in Q_p$, $0 < p < 1$, then*

$$\left|f'(re^{i\theta})\right| = o\big((1-r)^{-1}\big) \text{ as } r \to 1,$$

for every $\theta \in (-\pi, \pi]$, except for a set of null p-capacity.

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