ANNALES
UNIVERSITATIS MARIAE CURIE – SKŁODOWSKA
LUBLIN – POLONIA

VOL. L V, 7 SECTIO A 2001

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On maximum modulus of polynomials

Abstract. For a polynomial $p(z)$ of degree n, it is known that

$$|p(Re^{i\theta})| + |q(Re^{i\theta})| \le (R^n + 1)\{\max_{|z|=1} |p(z)|\},$$

$R \ge 1$ and $0 \le \theta \le 2\pi$, where

$$q(z) = z^n \overline{p(1/\overline{z})}.$$

We obtain a refinement, as well as a generalization, of this inequality.

1. Introduction and statement of results. For an arbitrary entire function $f(z)$, let $M(f, r) = \max_{|z|=r} |f(z)|$. For a polynomial $p(z)$ of degree n, it is known ([4, section 5], [1, Lemma]) that

$$|p(Re^{i\theta})| + |q(Re^{i\theta})| \le (R^n + 1)M(p, 1)\,, R \ge 1 \text{ and } 0 \le \theta \le 2\pi, \tag{1.1}$$

where

$$q(z) = z^n \overline{p(1/\overline{z})}. \tag{1.2}$$

In this note, we obtain a refinement, as well as a generalization, of inequality (1.1). More precisely, we prove

1991 *Mathematics Subject Classification.* 30C10.
Key words and phrases. Polynomial of degree n.

Theorem 1. *If $p(z)$ is a polynomial of degree n, $n \geq 3$, then for every positive integer s, we have*

$$\begin{aligned}(1.3)\qquad &\left|p(Re^{i\theta})\right|^{s} + \left|q(Re^{i\theta})\right|^{s} \leq (R^{ns}+1)\{M(p,1)\}^{s} \\ &- \left(\frac{R^{ns}-1}{ns} - \frac{R^{ns-2}-1}{ns-2}\right) \left||p'(0)| - |q'(0)|\right| s\{M(p,1)\}^{s-1}, \\ &R \geq 1 \text{ and } 0 \leq \theta \leq 2\pi.\end{aligned}$$

Remark. For $s = 1$, inequality (1.3) becomes

$$\begin{aligned}|p(Re^{i\theta})| + |q(Re^{i\theta})| &\leq (R^{n}+1)M(p,1) \\ &- \left(\frac{R^{n}-1}{n} - \frac{R^{n-2}-1}{n-2}\right) \left||p'(0)| - |q'(0)|\right|,\end{aligned}$$

and is therefore a refinement of inequality (1.1), as

$$\frac{R^{n}-1}{n} - \frac{R^{n-2}-1}{n-2} \geq 0.$$

Further, by (1.3), we obviously have

$$\left|p(Re^{i\theta})\right|^{s} + \left|q(Re^{i\theta})\right|^{s} \leq (R^{ns}+1)\{M(p,1)\}^{s},$$

suggesting a generalization of inequality (1.1).

2. Lemmas. For the proof of Theorem 1, we require the following lemmas.

Lemma 1. *If $p(z)$ is a polynomial of degree at most n, $n \geq 2$, then for $R > 1$*

$$M(p,R) \leq R^{n}M(p,1) - (R^{n} - R^{n-2})|p(0)|.$$

The coefficient of $|p(0)|$ is best possible for each R.

This lemma is due to Frappier, Rahman and Ruscheweyh, cf. [2, Theorem 2].

Lemma 2. *If $p(z)$ is a polynomial of degree n, then for $|z| = 1$*

$$|p'(z)| + |q'(z)| \leq nM(p,1).$$

This lemma is due to Malik [3, inequality 17].

3. Proof of Theorem 1. The polynomial

$$G(z) = p'(z) + \alpha q'(z),\ |\alpha| = 1$$

is of degree at most $n-1\,(\geq 2)$. Hence, if $|\alpha|=1$, $t\geq 1$ and $0\leq\theta\leq 2\pi$, then applying Lemma 1 followed by Lemma 2, we obtain

$$\begin{aligned}|p'(te^{i\theta})+\alpha q'(te^{i\theta})| &\leq t^{n-1}\max_{|z|=1}|p'(z)+\alpha q'(z)|\\ &\quad-(t^{n-1}-t^{n-3})|p'(0)+\alpha q'(0)|\\ &\leq t^{n-1}nM(p,1)\\ &\quad-(t^{n-1}-t^{n-3})|p'(0)+\alpha q'(0)|\end{aligned}$$

and so

$$\begin{aligned}|p'(te^{i\theta})|+|q'(te^{i\theta})| &\leq nt^{n-1}M(p,1)\\ &\quad-(t^{n-1}-t^{n-3})||p'(0)|-|q'(0)||.\end{aligned} \tag{3.1}$$

Since

$$\begin{aligned}\{p(Re^{i\theta})\}^s-\{p(e^{i\theta})\}^s &= \int_1^R \frac{d}{dt}\{p(te^{i\theta})\}^s dt\\ &= \int_1^R s\{p(te^{i\theta})\}^{s-1}p'(te^{i\theta})e^{i\theta}dt,\end{aligned}$$

we see that

$$|\{p(Re^{i\theta})\}^s-\{p(e^{i\theta})\}^s|\leq s\int_1^R |p'(te^{i\theta})||p(te^{i\theta})|^{s-1}dt,$$

which, by virtue of Lemma 1, implies

$$|\{p(Re^{i\theta})\}^s-\{p(e^{i\theta})\}^s|\leq s\int_1^R |p'(te^{i\theta})|t^{n(s-1)}\{M(p,1)\}^{s-1}dt\,. \tag{3.2}$$

Similarly, we have

$$\begin{aligned}|\{q(Re^{i\theta})\}^s-\{q(e^{i\theta})\}^s| &\leq s\int_1^R t^{n(s-1)}|q'(te^{i\theta})|\{M(q,1)\}^{s-1}dt\\ &= s\int_1^R |q'(te^{i\theta})|\{M(p,1)\}^{s-1}t^{n(s-1)}dt\end{aligned}$$

which together with (3.2) gives

$$
\begin{aligned}
&|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| + |\{q(Re^{i\theta})\}^s - \{q(e^{i\theta})\}^s| \\
&\le s\{M(p,1)\}^{s-1} \int_1^R t^{n(s-1)} \left(|p'(te^{i\theta})| + |q'(te^{i\theta})|\right) dt \\
&\le sn\{M(p,1)\}^s \int_1^R t^{ns-1} dt \\
&- s\{M(p,1)\}^{s-1} \,||p'(0)| - |q'(0)|| \int_1^R \left(t^{ns-1} - t^{ns-3}\right) dt,
\end{aligned}
$$

where at the last step we have used (3.1). Since $|p(e^{i\theta})| = |q(e^{i\theta})| \le M(p,1)$, we obtain

$$
\begin{aligned}
&|p(Re^{i\theta})|^s + |q(Re^{i\theta})|^s \le (R^{ns}+1)\,(M(p,1))^s \\
&- s(M(p,1))^{s-1}\,||p'(0)| - |q'(0)|| \left(\frac{R^{ns}-1}{ns} - \frac{R^{ns-2}-1}{ns-2}\right),
\end{aligned}
$$

which is what we wanted to prove. □

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received February 21, 2000