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Connections and torsions on TT^*M

Dedicated to Professor Ivan Kolář
on the occasion of his 65-th birthday

Abstract. The connections on the bundle $TT^*M \to T^*M$ are investigated and the results concerning liftings of connections are summarized. General torsions of a connection are defined as the Frölicher–Nijenhuis brackets of the associated horizontal projection and natural affinors on this bundle. All general torsions on TT^*M are derived. Specially, the torsions of linear connections and lifted classical linear connections are described geometrically.

1. The bundle TT^*M. The research on the geometry of the bundle TT^*M is of considerable importance. It yields not only one second order bundle, but according to Modugno and Stefani, [12], there exists a geometrical isomorphism between the bundles TT^*M and T^*TM for every manifold M. From the categorial point of view this is a natural equivalence between bundle functors TT^* and T^*T defined on the category $\mathcal{M}f_m$ of m-dimensional smooth manifolds and smooth mappings. Moreover, if we take into account a classical geometrical construction of a natural equivalence

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between TT^* and T^*T^*, we see that our considerations include the second order bundles TT^*M, T^*TM and T^*T^*M. However, the functor TT is not of this type. It is defined on the whole category $\mathcal{M}f$ of smooth manifolds and smooth mappings and it is product preserving and there is no natural equivalence between TT and T^*T. The fundamental paper of Kolář and Radziszewski [7], includes details concerning the natural transformations of second tangent and cotangent functors. There is also a motivation for studying the properties of bundles TT^*M, T^*TM and T^*T^*M in some problems of the analytical mechanics.

The bundle TT^*M disposes of the following bundle structures: $TT^*M \to T^*M$, $TT^*M \to TM$, $TT^*M \to M$. Given some local coordinates x^i on M, let us denote by x^i, p_i the induced coordinates on T^*M and $x^i, p_i, X^i = dx^i, P_i = dp_i$ the induced coordinates on TT^*M. Then the projections of mentioned structures work in this way: $(x^i, p_i, X^i, P_i) \mapsto (x^i, p_i)$, $(x^i, p_i, X^i, P_i) \mapsto (x^i, X^i)$, $(x^i, p_i, X^i, P_i) \mapsto (x^i)$.

2. Connections on TT^*M.

2.1. General connections on TT^*M. Let $Y \to M$ be an arbitrary fiber bundle, $\dim M = m$, $\dim Y = m + n$. Let $i, j, \dots = 1, \dots, m$, $p, q, \dots = 1, \dots, n$ and let (x^i, y^p) be some local coordinates on Y. We define a *general connection* as a section $\Gamma: Y \to J^1Y$ of the first jet prolongation of Y. A general connection Γ can be identified with the associated horizontal projection denoted by the same symbol Γ, which is a special (1,1)–tensor field on Y. It has the coordinate expression

$$dy^p = \Gamma^p_i(x, y)dx^i.$$

Especially, on $TT^*M \to T^*M$ it yields the coordinate expression of Γ as

$$\begin{aligned} dX^i &= D^i_j(x, p, X, P)dx^j + E^{ij}(x, p, X, P)dp_j \\ dP_i &= F_{ij}(x, p, X, P)dx^j + G^j_i(x, p, X, P)dp_j. \end{aligned}$$

2.2. Linear connections on TT^*M. Let $E \to M$ be an arbitrary vector bundle. Then the first jet prolongation $J^1E \to M$ is also a vector bundle. A general connection $\nabla: E \to J^1E$ is said to be a *linear connection* if ∇ is a vector bundle morphism. In the case $E = TM$ we obtain the well-known concept of the *classical linear connection* on M.

We obtain directly the coordinate expression of a linear connection ∇ on E as

$$dy^p = \nabla^p_{qi}(x)y^q dx^i$$

and so we have on $TT^*M \to T^*M$ a linear connection in a form

$$\begin{aligned} dX^i &= K^i_{jk}(x,p)X^j dx^k + L^{ij}_k(x,p)P_j dx^k + M^{ik}_j(x,p)X^j dp_k \\ &\quad + N^{ijk}(x,p)P_j dp_k \\ dP_i &= P_{ijk}(x,p)X^j dx^k + Q^j_{ik}(x,p)P_j dx^k + R^k_{ij}(x,p)X^j dp_k \\ &\quad + S^{jk}_i(x,p)P_j dp_k, \end{aligned}$$

and we call it the classical linear connection on T^*M, too.

2.3. Liftings of general connections. Let F, G be a natural bundles over m-dimensional manifolds, $m+n = \dim F\mathbb{R}^m$ and let H be a natural bundle over $(m+n)$-dimensional manifolds. We denote $C^\infty GM$ and $C^\infty H(FM)$ the spaces of local sections of $GM \to M$ and $H(FM) \to FM$, respectively. Elements of these spaces are called *geometric* G*–* and H*–objects*.

A *lifting* to F of geometric G–objects from M to geometric H–objects on FM is a family $\Lambda = \{\Lambda_M\}$ of mappings $\Lambda_M\colon C^\infty GM \to C^\infty H(FM)$ satisfying the following conditions:

(i) If $s \in C^\infty GM$ is defined on an open subset $U \subset M$ then $\Lambda_M(s) \in C^\infty H(FM)$ is defined on $FU \subset FM$.

(ii) *(The naturality condition)* For every embedding $\varphi\colon M \to N$, if objects $s_1 \in C^\infty GM$, $s_2 \in C^\infty GN$ are φ-related, then $\Lambda_M(s_1) \in C^\infty H(FM)$, $\Lambda_M(s_2) \in C^\infty H(FN)$ are $F\varphi$-related.

We say that a lifting $\Lambda = \{\Lambda_M\}$ to F satisfies the regularity conditions if

(iii) *(The regularity condition)* If $s_t \in C^\infty GM$ is a smooth family of local fields of geometric objects on M, then $\Lambda_M(s_t) \in C^\infty H(FM)$ is also a smooth family of local fields of geometric objects on FM.

The condition (i) and (ii) imply immediately

(iv) *(The locality condition)* If $s_1, s_2 \in C^\infty GM$ are objects such that $s_{1|U} = s_{2|U}$ for some open subset $U \subset M$, then $\Lambda_M(s_1)_{|FU} = \Lambda_M(s_2)_{|FU}$.

Let $r \in \mathbb{N} \cup \{\infty\}$ is the smallest number for which $j^r_x s_1 = j^r_x s_2$ implies $\Lambda_M(s_1)_{|F_xM} = \Lambda_M(s_2)_{|F_xM}$ for every point $x \in M$ and every two sections $s_1, s_2 \in C^\infty GM$ defined on its neighborhoods. Then Λ is said to be *of order* r. (The implication $j^\infty_x s_1 = j^\infty_x s_2 \Rightarrow \Lambda_M(s_1)_{|F_xM} = \Lambda_M(s_2)_{|F_xM}$ always holds, see [4].)

The problem of classifications of liftings of order $r < \infty$ and satisfying the regularity condition is possible to reduce to classifications of equivariant mappings

$$\lambda\colon F_0\mathbb{R}^m \times J^r_0 G\mathbb{R}^m \to (HF)_0\mathbb{R}^m$$

satisfying $dp_H \circ \lambda = p_1$, where $p_1\colon F_0\mathbb{R}^m \times J^r_0 G\mathbb{R}^m \to F_0\mathbb{R}^m$ is the standard projection onto the first factor and $dp_H\colon (HF)_0\mathbb{R}^m \to F_0\mathbb{R}^m$ is the projection for the natural bundle H. (There is a bijective correspondence between them, see [5].)

If $Y \to M$ is an arbitrary fiber bundle, there are three canonical structures of a fibered manifold on FY, namely $FY \to M$, $FY \to FM$ and $FY \to Y$. In [5] are studied liftings of a general connections to these bundles.

If we are concerned with the case of liftings to $FY \to FM$, especially for $Y = TM$ and $F = T^*$ (all natural transformations $TT^* \to T^*T$ are already described in [7]), we can only state that any natural operator transforming general connections on $Y \to M$ into general connections on $FY \to FM$ is nowhere to be found for any concrete non-product-preserving functor F up to now.

2.4. Liftings of linear connections. In this subsection we recall the problem of lifting of a classical torsion-free linear connection on a manifold M (i.e. a torsion-free linear connection on TM) into a classical linear connection on the cotangent bundle T^*M. (i.e. a linear connection on TT^*M). We remark that the admittance of non-zero torsion complicates this problem very much.

The *classical lifts* of such a type were first considered by Yano and Patterson in [14], [15]. Let ∇ be a classical torsion-free linear connection on M with the coordinate expression $dX^i = \nabla^i_{jk}(x)X^j dx^k$, where x^i, $X^i = dx^i$ are some coordinates on TM.

First we define the complete lift of ∇ to T^*M (x^i, p_i are the corresponding coordinates on T^*M). We consider a (0,2)–tensor field g on T^*M with components

$$\begin{aligned} g_{ij} &= 2p_k \nabla^k_{ij} \\ g^j_i &= \delta^j_i \\ g^i_j &= \delta^i_j \\ g^{ij} &= 0. \end{aligned}$$

Clearly, g is symmetric and regular, i.e. g is a pseudo-Riemannian metric, $(ds)^2 = 2dx^i(dp_i + p_k\nabla^k_{ij}dx^j)$. We call g the *Riemann extension* of ∇ and denote it by ∇^R. Let ∇^C be the Levi-Civita connection determined by the Riemann extension ∇^R. We call ∇^C the *complete lift* of ∇ to T^*M. The coordinate expression of ∇^C is

$$\begin{aligned} dX^i &= \nabla^i_{jk}X^j dx^k \\ dP_i &= p_m(\nabla^m_{jk,i} - \nabla^m_{ij,k} - \nabla^m_{ik,j} - 2\nabla^m_{il}\nabla^l_{jk})X^j dx^k \\ &\quad - \nabla^k_{ij}X^j dp_k - \nabla^j_{ik}P_j dx^k. \end{aligned}$$

Second we define the horizontal lift of ∇. The *horizontal lift* ∇^H of ∇ to

T^*M is a unique classical linear connection on T^*M satisfying

$$\begin{aligned}\nabla^H_{\omega^V}\theta^V &= 0\\ \nabla^H_{\omega^V}Y^H &= 0\\ \nabla^H_{X^H}\theta^V &= (\nabla_X\theta)^V\\ \nabla^H_{X^H}Y^H &= (\nabla_X Y)^H,\end{aligned}$$

where ω^V, θ^V are vertical lifts of 1-forms ω, θ and X^H, Y^H are horizontal lifts of vector fields X, Y with respect to ∇. A direct evaluation yields the following coordinate expression of ∇^H

$$\begin{aligned}dX^i &= \nabla^i_{jk}X^j dx^k\\ dP_i &= p_m(-\nabla^m_{ij,k} - \nabla^m_{lj}\nabla^l_{ik} - \nabla^m_{il}\nabla^l_{jk})X^j dx^k - \nabla^k_{ij}X^j dp_k - \nabla^j_{ik}P_j dx^k.\end{aligned}$$

In [9] it was proved:

Proposition 1. *All natural operators transforming a classical torsion-free linear connection on a manifold M into a classical linear connection on the cotangent bundle T^*M are the sum of a classical (e.g. complete or horizontal) lift with the 21–parameter family*

$$\begin{aligned}dX^i &= (c_1\delta^i_j p_k + c_2\delta^i_k p_j)X^j dx^k\\ dP_i &= (c_7 p_i p_j p_k + (c_4 + c_6)p_i p_l\nabla^l_{jk} + (c_3 - c_2)p_j p_l\nabla^l_{ik}\\ &\quad + (c_5 - c_1)p_k p_l\nabla^l_{ij} + c_8 p_l\mathbf{R}^l_{ijk} + c_9 p_l\mathbf{R}^l_{kij} + c_{10}p_i\mathbf{R}^l_{jkl} + c_{11}p_i\mathbf{R}^l_{klj}\\ &\quad + c_{12}p_j\mathbf{R}^l_{kli} + c_{13}p_j\mathbf{R}^l_{lik} + c_{14}p_k\mathbf{R}^l_{ijl} + c_{15}p_k\mathbf{R}^l_{lij} + c_{16}\mathbf{R}^l_{jikl}\\ &\quad + c_{17}\mathbf{R}^l_{jlik} + c_{18}\mathbf{R}^l_{kijl} + c_{19}\mathbf{R}^l_{klij} + c_{20}\mathbf{R}^l_{lijk} + c_{21}\mathbf{R}^l_{lkij})X^j dx^k\\ &\quad + (c_3\delta^k_i p_j + c_4\delta^k_j p_i)X^j dp_k + (c_5\delta^j_i p_k + c_6\delta^j_k p_i)P_j dx^k,\end{aligned}$$

which is formed upon a natural difference tensor, where $\mathbf{R}^i_{jkl}$, $\mathbf{R}^i_{jklm}$ are the canonical coordinates of the curvature space ($\mathbf{R}^i_{jkl}$ are skew-symmetric in the last two subscripts.).

This family is in [9] interpreted geometrically. Let us remark that if $c_9 = 1$ and all other coefficients are zero, we obtain just the difference between the complete lift and the horizontal lift.

3. Torsions of connections on TT^*M.

3.1. The classical torsion. On the vector bundle $TT^*M \to T^*M$ we can define the torsion τ of the linear connection Γ on TT^*M by the classical formula

$$\tau(\mathcal{X},\mathcal{Y}) = \Gamma_{\mathcal{X}}\mathcal{Y} - \Gamma_{\mathcal{Y}}\mathcal{X} - [\mathcal{X},\mathcal{Y}],$$

where we denote the covariant differentiation with respect to Γ by the symbol of the connection itself and where $\mathcal{X} = X^i \frac{\partial}{\partial x^i} + P_i \frac{\partial}{\partial p_i}$, $\mathcal{Y} = Y^i \frac{\partial}{\partial x^i} + Q_i \frac{\partial}{\partial p_i}$ are vector fields on T^*M. The coordinate expression of the torsion τ is

$$\begin{aligned}(\tau(\mathcal{X},\mathcal{Y}))^i &= (K^i_{jk} - K^i_{kj})X^j Y^k + (L^{ij}_k - M^{ij}_k)P_j Y^k \\ &\quad + (M^{ik}_j - L^{ik}_j)X^j Q_k + (N^{ijk} - N^{ikj})P_j Q_k \\ (\tau(\mathcal{X},\mathcal{Y}))_i &= (P_{ijk} - P_{ikj})X^j Y^k + (Q^j_{ik} - R^j_{ik})P_j Y^k \\ &\quad + (R^k_{ij} - Q^k_{ij})X^j Q_k + (S^{jk}_i - S^{kj}_i)P_j Q_k.\end{aligned}$$

In particular, this yields the well-known results for torsions of the complete lift and the horizontal lift (see Yano, Ishihara, [13]).

Proposition 2. *The complete lift ∇^C is torsion-free, i.e.*

$$\begin{aligned}(\tau(\mathcal{X},\mathcal{Y}))^i &= 0 \\ (\tau(\mathcal{X},\mathcal{Y}))_i &= 0\end{aligned}$$

and the torsion of the horizontal lift ∇^H has the coordinate expression

$$\begin{aligned}(\tau(\mathcal{X},\mathcal{Y}))^i &= 0 \\ (\tau(\mathcal{X},\mathcal{Y}))_i &= -p_l \mathbf{R}^l_{ijk} X^j Y^k,\end{aligned}$$

where $\mathbf{R}$ *represents the curvature tensor.*

3.2. Natural affinors and the Frölicher-Nijenhuis bracket. By an *affinor* A on a manifold M we mean (1,1)–tensor field, which we can consider as a linear morphism $L: TM \to TM$ over id_M. In general, an affinor represents a vector valued 1-form. Specially, an affinor representing a vertical valued 1-form is called the *vertical affinor*.

A *natural affinor* on a natural bundle F over m–manifolds is a system of affinors $A_M: TFM \to TFM$ for every m–manifold M satisfying

$$TFf \circ A_M = A_N \circ TFf$$

for every local diffeomorphism $f: M \to N$.

We remark that Kolář and Modugno determined in [6] all natural affinors for an arbitrary Weil bundle and in addition for T^*M. Kurek, [8], described all natural affinors for $T^{r*}M$ and Doupovec, [1] described all natural affinors for TT^*M.

Let A, B be (1,1)–tensor fields on M. The Frölicher-Nijenhuis bracket $[A, B]$ is defined by

$$\begin{aligned}[A,B](\mathcal{X},\mathcal{Y}) &= [A\mathcal{X}, B\mathcal{Y}] + [B\mathcal{X}, A\mathcal{Y}] + AB[\mathcal{X},\mathcal{Y}] + BA[\mathcal{X},\mathcal{Y}] \\ &\quad - A[\mathcal{X}, B\mathcal{Y}] - A[B\mathcal{X},\mathcal{Y}] - B[\mathcal{X}, A\mathcal{Y}] - B[A\mathcal{X},\mathcal{Y}],\end{aligned}$$

where $\mathcal{X}$, $\mathcal{Y}$ are vector fields on M. One sees directly that the Frölicher-Nijenhuis bracket represents a (1,2)–tensor field on M satisfying

$$[A,B](\mathcal{X},\mathcal{Y}) = -[A,B](\mathcal{Y},\mathcal{X})$$

and which is expressed in coordinates by

$$([A,B](\mathcal{X},\mathcal{Y}))^i = (a^l_j\partial_l b^i_k + b^l_j\partial_l a^i_k - a^i_l\partial_j b^l_k - b^i_l\partial_j a^l_k)\mathcal{X}^j \wedge \mathcal{Y}^k,$$

where a^i_j and b^i_j are coordinates of A and B, respectively. Obviously, for the identity affinor $1_{FM} = id_{TFM}$, as well as for its constant multiples, we have

$$[A, k1_{FM}] = [k1_{FM}, B] = 0$$

for every A, B and that is why we will not consider such affinors.

In this situation, the Frölicher-Nijenhuis bracket $[\Gamma, A]$, where Γ is a general connection and A is a natural affinor, is called the *torsion* of Γ of type A. In [3], [6], [10], [11] are completely described torsions of connections on a number of Weil bundles. Moreover, in [2], [6] are described torsions on T^*M, $T^{r*}M$, $T^{(r)}M$ which are not a Weil bundles.

In [6] it is used a formula for the finding of torsions in the case of vertical affinors. We state a general formula. Consider an arbitrary fibered manifold $Y \to M$, an affinor $\varphi\colon Y \to TY \otimes T^*Y$ and a general connection Γ on Y. The coordinate form of the horizontal projection of Γ is

$$\delta^i_j \frac{\partial}{\partial x^i} \otimes dx^j + F^p_i \frac{\partial}{\partial y^p} \otimes dx^i.$$

A section $\varphi\colon Y \to TY \otimes T^*Y$ has the coordinate expression

$$\varphi^i_j(x,y)\frac{\partial}{\partial x^i}\otimes dx^j + \varphi^i_p(x,y)\frac{\partial}{\partial x^i}\otimes dy^p + \varphi^p_i(x,y)\frac{\partial}{\partial y^p}\otimes dx^i + \varphi^p_q(x,y)\frac{\partial}{\partial y^p}\otimes dy^q.$$

Lemma. *The Frölicher-Nijenhuis bracket $[\Gamma, \varphi]$ has the coordinate expression*

$$
\begin{aligned}
&(F^p_i \frac{\partial \varphi^k_j}{\partial y^p} - \varphi^k_p \frac{\partial F^p_j}{\partial x^i})\frac{\partial}{\partial x^k} \otimes dx^i \wedge dx^j \\
&+ (\frac{\partial \varphi^k_i}{\partial y^p} + F^q_i \frac{\partial \varphi^k_p}{\partial y^q} + \varphi^k_q \frac{\partial F^q_i}{\partial y^p})\frac{\partial}{\partial x^k} \otimes dx^i \wedge dy^p \\
&+ (\frac{\partial \varphi^p_j}{\partial x^i} + \varphi^k_i \frac{\partial F^p_j}{\partial x^k} - F^p_k \frac{\partial \varphi^k_j}{\partial x^i} + F^q_i \frac{\partial \varphi^p_j}{\partial y^q} + \varphi^q_i \frac{\partial F^p_j}{\partial y^q} \\
&- \varphi^p_q \frac{\partial F^q_j}{\partial x^i})\frac{\partial}{\partial y^p} \otimes dx^i \wedge dx^j \\
&+ (\frac{\partial \varphi^p_q}{\partial x^i} - \varphi^j_q \frac{\partial F^p_i}{\partial x^j} - F^p_j \frac{\partial \varphi^j_q}{\partial x^i} + F^p_j \frac{\partial \varphi^j_i}{\partial y^q} + F^r_i \frac{\partial \varphi^p_q}{\partial y^r} - \varphi^r_q \frac{\partial F^p_i}{\partial y^r} \\
&+ \varphi^p_r \frac{\partial F^r_i}{\partial y^q})\frac{\partial}{\partial y^p} \otimes dx^i \wedge dy^q.
\end{aligned}
$$

Proof. We applied the coordinate expression of the Frölicher-Nijenhuis bracket $[A, B]$ in our concrete situation. □

Of course, in the case $\varphi^i_j = \varphi^i_p = 0$ we obtain the same formula as Kolář and Modugno in [6] for vertical affinors.

3.3. All natural affinors on TT^*M. All natural affinors on TT^*M are described by Doupovec in [1], where it is possible to find also their geometrical interpretations. Under the usual identification of the affinors on TT^*M with linear maps $L\colon TTT^*M \to TTT^*M$, we obtain this form of a affinor on TTT^*M

$$\begin{aligned}
dx^i &= \kappa^i_j(x,p,X,P)dx^j + \kappa^{ij}(x,p,X,P)dp_j + \hat\kappa^i_j(x,p,X,P)dX^j \\
&\quad + \hat\kappa^{ij}(x,p,X,P)dP_j \\
dp_i &= \lambda_{ij}(x,p,X,P)dx^j + \lambda^j_i(x,p,X,P)dp_j + \hat\lambda_{ij}(x,p,X,P)dX^j \\
&\quad + \hat\lambda^j_i(x,p,X,P)dP_j \\
dX^i &= \mu^i_j(x,p,X,P)dx^j + \mu^{ij}(x,p,X,P)dp_j + \hat\mu^i_j(x,p,X,P)dX^j \\
&\quad + \hat\mu^{ij}(x,p,X,P)dP_j \\
dP_i &= \nu_{ij}(x,p,X,P)dx^j + \nu^j_i(x,p,X,P)dp_j + \hat\nu_{ij}(x,p,X,P)dX^j \\
&\quad + \hat\nu^j_i(x,p,X,P)dP_j.
\end{aligned}$$

Now, we can formulate the Doupovec's result in the following form.

Proposition 3. *All natural affinors on TT^*M constitute a 11–parameter family determined as a linear combination of A_i, $i = 1, \dots, 11$. The coordinate expressions of the generators are*

$$\begin{aligned}
A_1\colon dx^i &= dx^i & A_2\colon dx^i &= 0 \\
dp_i &= dp_i & dp_i &= 0 \\
dX^i &= dX^i & dX^i &= dx^i \\
dP_i &= dP_i & dP_i &= dp_i
\end{aligned}$$

$$\begin{aligned}
A_3\colon dx^i &= 0 & A_4\colon dx^i &= 0 \\
dp_i &= 0 & dp_i &= 0 \\
dX^i &= p_j dx^j X^i & dX^i &= (p_j dX^j + dp_j X^j)X^i \\
dP_i &= p_j dx^j P_i & dP_i &= (p_j dX^j + dp_j X^j)P_i
\end{aligned}$$

$$A_5{:}\,dx^i = 0 \qquad dp_i = 0 \qquad dX^i = (p_j dX^j + P_j dx^j)X^i \qquad dP_i = (p_j dX^j + P_j dx^j)P_i$$

$$A_6{:}\,dx^i = 0 \qquad dp_i = 0 \qquad dX^i = 0 \qquad dP_i = p_j dx^j p_i$$

$$A_7{:}\,dx^i = 0 \qquad dp_i = 0 \qquad dX^i = 0 \qquad dP_i = (p_j dX^j + dp_j X^j)p_i$$

$$A_8{:}\,dx^i = 0 \qquad dp_i = 0 \qquad dX^i = 0 \qquad dP_i = (p_j dX^j + P_j dx^j)p_i$$

$$A_9{:}\,dx^i = 0 \qquad dp_i = p_j dx^j p_i \qquad dX^i = 0 \qquad dP_i = p_j dx^j P_i$$

$$A_{10}{:}\,dx^i = 0 \qquad dp_i = (p_j dX^j + dp_j X^j)p_i \qquad dX^i = 0 \qquad dP_i = (p_j dX^j + dp_j X^j)P_i$$

$$A_{11}{:}\,dx^i = 0 \qquad dp_i = (p_j dX^j + P_j dx^j)p_i \qquad dX^i = 0 \qquad dP_i = (p_j dX^j + P_j dx^j)P_i.$$

We see that A_1 represents the identity of TT^*M, A_2, A_3, A_4, A_5 represent vertical affinors with respect to the projection $TT^*M \to T^*M$, A_9, A_{10}, A_{11} represent vertical affinors with respect to the projection $TT^*M \to TM$, A_6, A_7, A_8 represent vertical affinors with respect to both the projections $TT^*M \to T^*M$, $TT^*M \to TM$.

We aim only at the generators A_2, A_3 (as the representative of the triple A_3, A_4, A_5), A_6 (as the representative of the triple A_6, A_7, A_8) and A_9 (as the representative of the triple A_9, A_{10}, A_{11}). The geometrical interpretation of generators entitled us to do such a selection.

3.4. General torsions. The general expression for the Frölicher–Nijenhuis bracket enables us to obtain new results concerning torsions for all abovementioned generators.

I. A_2: We have $\mu^i_j = \delta^i_j$, $\nu^j_i = \delta^j_i$ and all other functions from 3.3 are zero.

A direct evaluation yields the coordinate expression of torsion $\tau_2 = [\Gamma, A_2]$

$$dX^i = \frac{\partial D^i_k}{\partial X^j} dx^j \wedge dx^k + (\frac{\partial E^{ik}}{\partial X^j} - \frac{\partial D^i_j}{\partial P_k}) dx^j \wedge dp_k + \frac{\partial E^{ik}}{\partial P_j} dp_j \wedge dp_k$$
$$dP_i = \frac{\partial F_{ik}}{\partial X^j} dx^j \wedge dx^k + (\frac{\partial G^k_i}{\partial X^j} - \frac{\partial F_{ij}}{\partial P_k}) dx^j \wedge dp_k + \frac{\partial G^k_i}{\partial P_j} dp_j \wedge dp_k.$$

If Γ is a linear connection, then we obtain

$$dX^i = K^i_{jk} dx^j \wedge dx^k + (M^{ik}_j - L^{ik}_j) dx^j \wedge dp_k + N^{ijk} dp_j \wedge dp_k$$
$$dP_i = P_{ijk} dx^j \wedge dx^k + (R^k_{ij} - Q^k_{ij}) dx^j \wedge dp_k + S^{jk}_i dp_j \wedge dp_k$$

and this coincides with the pullback $\sigma^*(\tau(\mathcal{X}, \mathcal{Y}))$ of the classical torsion τ given by $\sigma^*: TT^*M \to VTT^*M$, where $\sigma: TT^*M \to T^*M$ is the canonical projection.

We see that the complete lift ∇^C of a classical torsion-free connection ∇ on M is torsion-free in our new sense as well. If Γ is a lifted linear connection expressed as the sum of ∇^C with a natural difference tensor from 21–parameter family determined in Proposition 1, then we obtain

$$dX^i = (c_1 - c_2) p_j dx^i \wedge dx^j$$
$$\begin{aligned} dP_i = ((c_1 - c_2 + c_3 - c_5) p_j p_l \nabla^l_{ik} + (2c_8 - c_9) p_l (\nabla^l_{ij,k} + \nabla^l_{mk} \nabla^m_{ij}) \\ + (-c_{10} - c_{11}) p_i (\nabla^l_{lj,k} + \nabla^l_{mk} \nabla^m_{lj}) + (c_{12} + c_{14}) p_j \mathbf{R}^l_{kli} \\ + (c_{13} + c_{14} - c_{15}) p_j \mathbf{R}^l_{lik} + (c_{16} - c_{18}) \mathbf{R}^l_{jikl} + (c_{17} - c_{19}) \mathbf{R}^l_{jlik} \\ + c_{20} \mathbf{R}^l_{lijk} + c_{21} \mathbf{R}^l_{lkij}) dx^j \wedge dx^k \\ + (c_5 - c_3) p_j dx^j \wedge dp_i + (c_6 - c_4) p_i dx^j \wedge dp_j. \end{aligned}$$

II. A_3: We have $\mu^i_j = X^i p_j$, $\nu_{ij} = P_i p_j$ and all other functions from 3.3 are zero. A direct evaluation yields the coordinate expression of torsion $\tau_3 = [\Gamma, A_3]$

$$\begin{aligned} dX^i = p_j (X^l \frac{\partial D^i_k}{\partial X^l} + P_l \frac{\partial D^i_k}{\partial P_l} - D^i_k) dx^j \wedge dx^k \\ + (p_j (X^l \frac{\partial E^{ik}}{\partial X^l} + P_l \frac{\partial E^{ik}}{\partial P_l} - E^{ik}) - \delta^k_j X^i) dx^j \wedge dp_k \\ dP_i = p_j (X^l \frac{\partial F_{ik}}{\partial X^l} + P_l \frac{\partial F_{ik}}{\partial P_l} - F_{ik}) dx^j \wedge dx^k \\ + (p_j (X^l \frac{\partial G^k_i}{\partial X^l} + P_l \frac{\partial G^k_i}{\partial P_l} - G^k_i) - \delta^k_j P_i) dx^j \wedge dp_k. \end{aligned}$$

If Γ is a linear connection, then we obtain

$$dX^i = -X^i dx^j \wedge dp_j$$
$$dP_i = -P_i dx^j \wedge dp_j$$

with the following geometrical interpretation. Let $A=(x,p,X,P,\xi,\pi,\Xi,\Pi)$, $B=(x,p,X,P,\eta,\theta,\mathrm{H},\Theta)\in TTT_0^*\mathbf{R}^m$. There are two canonical projections $\zeta,\chi\colon TTT^*M\to TT^*M$, $\zeta(A)=\zeta(B)=(X,P)$, $\chi(A)=(\xi,\pi)$, $\chi(B)=(\eta,\theta)$. We use the canonical injection $\iota=\iota_{TT^*M}\colon TT^*M\to TTT^*M$, $\iota(x,p,X,P)=(x,p,X,P,0,0,X,P)$ and we evaluate the images of χ. Then we see that we have obtained

$$(\langle\eta,\pi\rangle-\langle\xi,\theta\rangle)\iota(\zeta)$$

and we discover immediately, that the torsion τ_3 does not depend on Γ, if Γ is linear.

III. A_6: We have $\nu_{ij}=p_ip_j$ and all other functions from 3.3 are zero. A direct evaluation yields the coordinate expression of torsion $\tau_6=[\Gamma,A_6]$

$$dX^i = p_jp_l(\frac{\partial D^i_k}{\partial P_l})dx^j\wedge dx^k + p_jp_l(\frac{\partial E^{ik}}{\partial P_l})dx^j\wedge dp_k$$
$$dP_i = p_jp_l(\frac{\partial F_{ik}}{\partial P_l})dx^j\wedge dx^k + p_jp_l(\frac{\partial G^k_i}{\partial P_l}-\delta^k_j p_i-\delta^k_j p_j)dx^j\wedge dp_k.$$

If Γ is a linear connection, then we obtain

$$dX^i = p_jp_lL^{il}_k dx^j\wedge dx^k + p_jp_lN^{ilk}dx^j\wedge dp_k$$
$$dP_i = p_jp_lQ^l_{ik}dx^j\wedge dx^k + p_jp_lS^{lk}_i dx^j\wedge dp_k - p_idx^j\wedge dp_j - p_jdx^j\wedge dp_i$$

with the following geometrical interpretation. We consider the canonical projections $\sigma\colon TT^*M\to T^*M$, $\tau\colon TT^*M\to TM$. First we take the short subtracted terms. The first one can be interpreted analogously as the term in 3.4.II, but in distinction from it we multiply a vertical vector $\iota_{TT^*M}\circ\iota_{T^*M}(x,p)=(x,p,0,p,0,0,0,p)=:\tilde\iota(x,p)$, where $(x,p)=\sigma\circ\zeta=\sigma\circ\chi(A)=\sigma\circ\chi(B)=:\tilde\sigma$. The interpretation of the second one requires also images $\tau\circ\chi(A),\tau\circ\chi(B)$ which are joining to the evaluation together with $\tilde\sigma$, and the verticalization of $\chi(A)$ and $\chi(B)$. (We write $\tilde\pi=V\chi(A),\tilde\theta=V\chi(B)$.) The main part is a lift of $\iota_{T^*M}(x,p)$ with respect to Γ multiplied by the last evaluation. Then we see that we have obtained

$$\Gamma(\iota(\tilde\sigma))(\langle\xi,p\rangle-\langle\eta,p\rangle)+(\langle\eta,\pi\rangle-\langle\xi,\theta\rangle)\tilde\iota(\tilde\sigma)+\langle\eta,p\rangle\iota(\tilde\pi)-\langle\xi,p\rangle\iota(\tilde\theta).$$

If Γ is an arbitrary lifted linear connection in the sense stated in Proposition 1, then we obtain

$$\begin{aligned} dX^i &= 0 \\ dP_i &= p_j p_l \nabla^l_{ik} dx^j \wedge dx^k - p_i dx^j \wedge dp_j - p_j dx^j \wedge dp_i. \end{aligned}$$

IV. A_9: We have $\lambda_{ij} = p_i p_j$, $\nu_{ij} = P_i p_j$ and all other functions from 3.3 are zero. A direct evaluation yields the coordinate expression of torsion $\tau_9 = [\Gamma, A_9]$

$$\begin{aligned} dX^i =& (p_l p_j \frac{\partial D^i_k}{\partial p_l} + P_l p_j \frac{\partial D^i_k}{\partial P_l}) dx^j \wedge dx^k \\ &+ (p_l p_j \frac{\partial E^{ik}}{\partial p_l} + P_l p_j \frac{\partial E^{ik}}{\partial P_l} + p_j E^{ik} + \delta^k_j p_l E^{il}) dx^j \wedge dp_k \\ dP_i =& (p_l p_j \frac{\partial F_{ik}}{\partial p_l} + P_l p_j \frac{\partial F_{ik}}{\partial P_l} - p_j F_{ik}) dx^j \wedge dx^k \\ &+ (p_l p_j \frac{\partial G^k_i}{\partial p_l} + P_l p_j \frac{\partial G^k_i}{\partial P_l} + p_j G^k_i + \delta^k_j (p_l G^l_i - P_i)) dx^j \wedge dp_k. \end{aligned}$$

If Γ is a linear connection, then we obtain

$$\begin{aligned} dX^i =& (p_l p_j (\frac{\partial K^i_{mk}}{\partial p_l} X^m + \frac{\partial L^{im}_k}{\partial p_l} P_m) + L^{il}_k P_l p_j) dx^j \wedge dx^k \\ &+ (p_l p_j (\frac{\partial M^{ik}_m}{\partial p_l} X^m + \frac{\partial N^{imk}}{\partial p_l} P_m) + M^{ik}_l X^l p_j + 2N^{ilk} P_l p_j) dx^j \wedge dp_k \\ &+ p_l (M^{il}_m X^m + N^{iml} P_m) dx^j \wedge dp_j \\ dP_i =& (p_l p_j (\frac{\partial P_{imk}}{\partial p_l} X^m + \frac{\partial Q^m_{ik}}{\partial p_l} P_m) - P_{ilk} X^l p_j) dx^j \wedge dx^k \\ &+ (p_l p_j (\frac{\partial R^k_{im}}{\partial p_l} X^m + \frac{\partial S^{mk}_i}{\partial p_l} P_m) + R^k_{il} X^l p_j + 2S^{kl}_i P_l p_j) dx^j \wedge dp_k \\ &+ (p_l (R^l_{im} X^m + S^{lm}_i P_m) - P_i) dx^j \wedge dp_j. \end{aligned}$$

The geometrization of torsions formed by the Frölicher–Nijenhuis bracket, in which the projection for Γ and the verticality for A are different, is very complicated and we assume that there is not any utilization for them.

So, we can summarize.

Proposition 4. *There are 10 general torsions τ_i of connections on $TT^*M \to T^*M$ related to the natural affinors A_i, $i = 2, \dots, 11$.*

Proof. The torsions τ_2, τ_3, τ_6, τ_9 related to the natural affinors A_2, A_3, A_6, A_9, respectively, are already described. The finding of the remaining torsions is quite analogous. Moreover, the ideas of geometrization of them in the case of a linear (lifted, respectively) connection are also preserved. □

We hope that here stated approach to general torsions will provide a rather clearer view to general torsions defined as the Frölicher–Nijenhuis brackets of Γ and arbitrary natural affinors.

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