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On pseudo-metrics on the space of generalized quasisymmetric automorphisms of a Jordan curve

Dedicated to Professor Hiroki Sato
on the occasion of his 60th birthday

Abstract. We discuss conformally invariant pseudo-metrics on the class of all sense-preserving homeomorphisms of a given Jordan curve by means of the second module of a quadrilateral.

1. Introduction. Given a domain $\Omega \subset \hat{\mathbb{C}}$ and $K \geq 1$, let $\mathrm{QC}(\Omega; K)$ stand for the class of all K-quasiconformal (qc. for short) self-mappings of Ω and let

$$\mathrm{QC}(\Omega) := \bigcup_{K \geq 1} \mathrm{QC}(\Omega; K) \ .$$

Assume that Ω is a Jordan domain bounded by a Jordan curve Γ. A classical result says that each $F \in \mathrm{QC}(\Omega)$ has a homeomorphic extension F^* of the closure $\overline{\Omega} = \Omega \cup \Gamma$ onto itself; cf. [12]. Then the restriction

$$\mathrm{Tr}[F] := F^*_{|\Gamma} \in \mathrm{Hom}^+(\Gamma) \ ,$$

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where $\mathrm{Hom}^+(\Gamma)$ is the class of all sense-preserving homeomorphic self-mappings of Γ. For $K \geq 1$ consider the class

$$\mathrm{Q}(\Gamma; K) := \{\mathrm{Tr}[F] : F \in \mathrm{QC}(\Omega; K)\}$$

and

$$\mathrm{Q}(\Gamma) := \{\mathrm{Tr}[F] : F \in \mathrm{QC}(\Omega)\} \ .$$

From respective properties of quasiconformal mappings (cf. [12]) it follows that the functional

$$\mathrm{K}(f) := \inf\{K \geq 1 : f \in \mathrm{Q}(\Gamma; K)\} \ , \quad f \in \mathrm{Q}(\Gamma)$$

has the following properties

$$\mathrm{K}(f \circ g) \leq \mathrm{K}(f)\mathrm{K}(g) \ , \quad f, g \in \mathrm{Q}(\Gamma) \ ;$$
$$\mathrm{K}(f) = \mathrm{K}(f^{-1}) \ , \quad f \in \mathrm{Q}(\Gamma) \ ;$$
$$\mathrm{K}(f) = 1 \iff f \in \mathrm{Q}(\Gamma; 1) \ , \quad f \in \mathrm{Q}(\Gamma) \ .$$

Hence the functional

$$\tau(f, g) := \frac{1}{2} \log \mathrm{K}(f \circ g^{-1}) \ , \quad f, g \in \mathrm{Q}(\Gamma) \ ;$$

is a pseudo-metric on $\mathrm{Q}(\Gamma)$ called the *Teichmüller pseudo-metric* on $\mathrm{Q}(\Gamma)$. There are several descriptions of the class $\mathrm{Q}(\Gamma)$ without using quasiconformal extensions; cf. e.g. [4], [1], [12], [11], [10], [16] and [15, Introduction]. Throughout this paper we use a description of $\mathrm{Q}(\Gamma)$ in terms of the *second module* $\mathrm{m}(Q)$ of a quadrilateral Q; cf. [15, Definition 1.3]. We recall that a quadrilateral $G(z_1, z_2, z_3, z_4)$ is a Jordan domain $G \subset \hat{\mathbb{C}}$ with distinct points z_1, z_2, z_3, z_4, called vertices, lying on the boundary curve ∂G and ordered according to the positive orientation of ∂G with respect to G; cf. [12, pp. 8-9]. The considerations in [15] justify to call any quadrilateral alternatively a hyperbolic rectangle and write $\mathrm{HR}(\Omega)$ for the class of all quadrilaterals $Q := \Omega(z_1, z_2, z_3, z_4)$ with vertices lying on the boundary curve $\Gamma = \partial\Omega$. Write $\mathrm{HS}(\Omega)$ for the class of all *hyperbolic squares* $\Omega(z_1, z_2, z_3, z_4)$, i.e. all quadrilaterals $Q \in \mathrm{HR}(\Omega)$ such that $\mathrm{m}(Q) = 1$; cf. [15]. If $f \in \mathrm{Hom}^+(\Gamma)$ and $Q := \Omega(z_1, z_2, z_3, z_4)$ is a quadrilateral, then we use the notation $f * Q$ for the quadrilateral $\Omega(f(z_1), f(z_2), f(z_3), f(z_4))$. The smallest $M \in [1; +\infty]$ such that the inequality

$$1/M \leq \mathrm{m}(f * Q) \leq M \tag{0.1}$$

holds for all $Q \in \mathrm{HS}(\Omega)$ is said to be the *generalized quasisymmetric dilatation* of $f \in \mathrm{Hom}^+(\Gamma)$ and is denoted by $\delta(f)$. [15, Thm. 2.2] says that

$$\mathrm{Q}(\Gamma) = \mathrm{GQS}(\Gamma) := \{f \in \mathrm{Hom}^+(\Gamma) : \delta(f) < \infty\} \ ;$$

$$\begin{aligned} \mathrm{GQS}(\Gamma; M) &:= \{f \in \mathrm{Hom}^+(\Gamma) : \delta(f) \leq M\} \\ &\subset \mathrm{Q}\left(\Gamma; \min\{M^{3/2}, 2M - 1\}\right) \ , \quad M \geq 1 \ ; \end{aligned} \tag{0.2}$$

(0.3) $$\mathrm{Q}(\Gamma;K)\subset \mathrm{GQS}(\Gamma;\lambda(K))\ ,\quad K\geq 1\ ,$$

where $\lambda(K) := \Phi_K(1/\sqrt{2})^2\Phi_{1/K}(1/\sqrt{2})^{-2}$ and Φ_K is the familiar Hersch-Pfluger distortion function; cf. [8], [12, pp. 53, 63]. We recall that (for $M \geq 1$) a homeomorphism $f \in \mathrm{Hom}^+(\Gamma)$ is called a *generalized (M-) quasisymmetric homeomorphism of* Γ provided $\delta(f) < \infty$ ($\delta(f) \leq M$).

The main topic in this paper is to construct functionals ρ on $\mathrm{Hom}^+(\Gamma)\times \mathrm{Hom}^+(\Gamma)$ which take values in $[0;+\infty]$ and satisfy all or some of the following six properties:

Property I. *ρ is a pseudo-metric on* $\mathrm{Hom}^+(\Gamma)$*, i.e. for all $f, g, h \in$* $\mathrm{Hom}^+(\Gamma)$*,*

$$\rho(f,g)=\rho(g,f)\quad ,\quad \rho(f,h)\leq \rho(f,g)+\rho(g,h)\quad ,\quad \rho(f,f)=0\ .$$

Property II. *For arbitrary $f, g \in$* $\mathrm{Hom}^+(\Gamma)$*,*

$$\rho(f,g)=0 \iff f\circ g^{-1}\in \mathrm{Q}(\Gamma;1)\ .$$

Property III. *ρ is equivalent to τ on* $\mathrm{Q}(\Gamma)$*, i.e. for any sequence $f_n \in$* $\mathrm{Q}(\Gamma)$*, $n \in \mathbb{N}$, and any $f \in$* $\mathrm{Q}(\Gamma)$*,*

$$(\ \rho(f_n,f)\to 0\quad \textit{as } n\to\infty\ )\iff (\ \tau(f_n,f)\to 0\quad \textit{as } n\to\infty\ )\ .$$

Property IV. *ρ is complete on* $\mathrm{Q}(\Gamma)$*, i.e. for any sequence $f_n \in$* $\mathrm{Q}(\Gamma)$*, $n \in \mathbb{N}$,*

$$(\ \rho(f_n,f_m)\to 0\quad \textit{as } n,m\to\infty\ )\implies (\ \rho(f_n,f)\to 0\quad \textit{as } n\to\infty\ )$$

for some $f \in$ $\mathrm{Q}(\Gamma)$*.*

Property V. *ρ determines the class* $\mathrm{Q}(\Gamma)$*, i.e. there exists $\lambda \in (0;+\infty]$ such that*

$$\mathrm{Q}(\Gamma)=\{f\in \mathrm{Hom}^+(\Gamma):\rho(f,\mathrm{id})<\lambda\}\ ,$$

where id *is the identity self-mapping of* Γ.

Property VI. *ρ is invariant in this sense that for all $f, g, h \in$* $\mathrm{Hom}^+(\Gamma)$*,*

$$\rho(f\circ h,g\circ h)=\rho(f,g)\ .$$

From the theory of quasiconformal mappings it follows easily that $\rho := \tau$ has all the properties (I)-(VI). In this note we construct such pseudo-metrics without using quasiconformal extensions to Ω. An example of such a pseudo-metric is the functional

$$\rho(f,g) := \log\inf\{K \ge 1 : f\circ g^{-1} \in \mathrm{QH}(\Gamma;K)\}\ , \quad f,g\in \mathrm{Hom}^+(\Gamma)\ ,$$

where $\mathrm{QH}(\Gamma;K)$ stands for the class of all K-quasihomographies of Γ, introduced by Zając; cf. [16] for their definition and properties. However, Zając's description involves the distortion function Φ_K, and so it is somewhat complicated in applications. Using the second module $\mathrm{m}(Q)$ of a quadrilateral Q we introduce in Section 1 simpler pseudo-metrics ρ satisfying some of the properties (I)-(VI). They have especially simple representations by means of the cross-ratio in the most essential case for applications, where Γ is the unit circle $\mathbb{T} := \{z\in\mathbb{C} : |z| = 1\}$ and Ω is the unit disk $\mathbb{D} := \{z\in\mathbb{C} : |z| < 1\}$, or Γ is the extended real axis $\hat{\mathbb{R}} := \mathbb{R}\cup\{\infty\}$ and Ω is the upper half plane $\mathbb{C}_+ := \{z\in\mathbb{C} : \mathrm{Im}\, z > 0\}$. The key role in our approach is played by the second module $\mathrm{m}(Q)$ of a quadrilateral Q, the generalized quasisymmetric dilatation $\delta(f)$ of $f\in\mathrm{Hom}^+(\Gamma)$ and their properties developed in [15]. Due to the simplicity of the pseudo-metric d it can be very useful in topics dealing with topological properties of the Teichmüller pseudo-metric τ. We present some results of this type in Section 2. Following considerations from Hamilton's paper [7] we construct in Section 3 a pseudo-metric $\hat{d}$ satisfying all the properties (I)-(VI). In the last section we gather some complementary results and technical tools that support our consideration in Sections 1 and 3.

1. The pseudo-metrics d and d^*. Write $\omega(z,\Omega)[I]$ for the harmonic measure at the point $z\in\Omega$ of the arc $I\subset\Gamma$ with respect to a domain $\Omega\subset\hat{\mathbb{C}}$ bounded by a Jordan curve $\Gamma = \partial\Omega$. Given distinct points $z_1, z_2\in\Gamma$ we denote by $\Gamma(z_1,z_2)$ the open arc from z_1 to z_2 according to the positive orientation of Γ with respect to Ω. By [15, Lemma 1.1] there exists a unique point $\mathrm{c}(Q)\in\Omega$, called the *hyperbolic center* of a quadrilateral $Q := \Omega(z_1,z_2,z_3,z_4)\in\mathrm{HR}(\Omega)$, such that

$$\omega(\mathrm{c}(Q),\Omega)[\Gamma(z_1,z_2)] = \omega(\mathrm{c}(Q),\Omega)[\Gamma(z_3,z_4)]$$

and

$$\omega(\mathrm{c}(Q),\Omega)[\Gamma(z_2,z_3)] = \omega(\mathrm{c}(Q),\Omega)[\Gamma(z_4,z_1)]\ .$$

We recall that the ratio

$$\mathrm{m}(Q) := \frac{\tan\pi\omega(\mathrm{c}(Q),\Omega)[\Gamma(z_1,z_2)]}{\tan\pi\omega(\mathrm{c}(Q),\Omega)[\Gamma(z_2,z_3)]}$$

is said to be the *second module* of Q; cf. [15, Definition 1.3]. If $Q \in \mathrm{HR}(\mathbb{D})$ or $Q \in \mathrm{HR}(\mathbb{C}_+)$, then [15, Lemma 3.1] says that

$$\mathrm{m}(Q) = \frac{[z_2, z_3, z_4, z_1]}{[z_1, z_2, z_3, z_4]} = \frac{1}{[z_1, z_2, z_3, z_4]} - 1 \ , \tag{1.1}$$

where

$$[w_1, w_2, w_3, w_4] := \frac{w_2 - w_3}{w_1 - w_3} \cdot \frac{w_1 - w_4}{w_2 - w_4}$$

is the *cross-ratio* of a quadruple of distinct points $w_1, w_2, w_3, w_4 \in \hat{\mathbb{C}}$. Given $f, g \in \mathrm{Hom}^+(\Gamma)$ we introduce

$$d(f,g) := \sup\left\{\left|\frac{1}{1+\mathrm{m}(f*Q)} - \frac{1}{1+\mathrm{m}(g*Q)}\right| : Q \in \mathrm{HR}(\Omega)\right\} \tag{1.2}$$

and

$$d^*(f,g) := \sup\left\{\left|\frac{1}{1+\mathrm{m}(f*Q)} - \frac{1}{1+\mathrm{m}(g*Q)}\right| : Q \in \mathrm{HS}(\Omega)\right\} \ . \tag{1.3}$$

It is easy to show that d and d^* are pseudo-metrics on $\mathrm{Hom}^+(\Gamma)$. In what follows we describe various properties of d and d^*. For this purpose we widely use results in [15].

Theorem 1.1. *The functional d satisfies the properties* (I), (II), (III), (IV) *and* (VI) *with ρ replaced by d.*

Proof. From (1.2) we easily conclude that the functional d satisfies (I) with $\rho := d$, and hence d is a pseudo-metric on $\mathrm{Hom}^+(\Gamma)$. By [15, Thm. 1.5] the second module m is conformally invariant, i.e. for all $h \in \mathrm{Q}(\Gamma;1)$ and $Q \in \mathrm{HR}(\Omega)$

$$\mathrm{m}(h*Q) = \mathrm{m}(Q) \ . \tag{1.4}$$

If now $f, g \in \mathrm{Hom}^+(\Gamma)$ and $h \in \mathrm{Q}(\Gamma;1)$ satisfy $f = h \circ g$, then by (1.2) and (1.4)

$$d(f,g) = d(f, h\circ g) = d(f,f) = 0 \ . \tag{1.5}$$

Conversely, assume that $f, g \in \mathrm{Hom}^+(\Gamma)$ and $d(f,g) = 0$. Then

$$\mathrm{m}(f*Q) = \mathrm{m}(g*Q) \ , \quad Q \in \mathrm{HR}(\Omega) \ ,$$

and hence

$$\mathrm{m}((f\circ g^{-1})*Q) = \mathrm{m}(Q) \ , \quad Q \in \mathrm{HS}(\Omega) \ .$$

By [15, Thm. 2.2] (or Lemma 4.2) we get $f \circ g^{-1} \in \mathrm{Q}(\Gamma; 1)$, which shows (II). Let $f_n \in \mathrm{Q}(\Gamma)$, $n \in \mathbb{N}$ be a sequence. If $\tau(f_n, f) \to 0$ as $n \to \infty$ for some $f \in \mathrm{Q}(\Gamma)$, then Lemma 4.1 implies

$$d(f_n, f) \le M(K(f_n \circ f^{-1})) \to 0 \quad \text{as } n \to \infty\ . \tag{1.6}$$

Conversely, assume that $d(f_n, f) \to 0$ as $n \to \infty$. Then by (1.2), (1.3) and (4.1),

$$d^*(f_n \circ f^{-1}, \mathrm{id}) \le d(f_n \circ f^{-1}, \mathrm{id}) = d(f_n, f) \to 0 \quad \text{as } n \to \infty\ .$$

Applying now Lemmas 4.4 and 4.5 we get

$$\tau(f_n, f) \to 0 \quad \text{as } n \to \infty\ .$$

Combining this with (1.6) we deduce that d is equivalent to τ, i.e. (III) holds. Assume now $f_n \in \mathrm{Q}(\Gamma)$, $n \in \mathbb{N}$ is a Cauchy sequence in $(Q(\Gamma), d)$. Then Lemma 4.1 shows that the inequality

$$\left|\frac{1}{1+\mathrm{m}((f_n \circ f_{n_0}^{-1}) * Q)} - \frac{1}{1+\mathrm{m}(Q)}\right| \le d((f_n \circ f_{n_0}^{-1}), \mathrm{id}) = d(f_n, f_{n_0}) < 1/4$$

holds for sufficiently large $n_0 \in \mathbb{N}$ and for all $n \in \mathbb{N}$, $n \ge n_0$ and $Q \in \mathrm{HS}(\Omega)$. Therefore, for every $Q \in \mathrm{HS}(\Omega)$,

$$1/3 < \mathrm{m}((f_n \circ f_{n_0}^{-1}) * Q) < 3\ , \quad n \ge n_0\ ,$$

hence by (0.1)

$$\delta((f_n \circ f_{n_0}^{-1}) * Q) < 3\ , \quad n \ge n_0\ ,$$

and finally, by [15, Thm. 2.2], we get

$$\delta(f_n) < \lambda(3^{3/2} K(f_{n_0}))\ , \quad n \ge n_0\ .$$

Lemma 4.3 now shows that there exist $f \in \mathrm{Q}(\Gamma)$ and sequences $g_n \in \mathrm{Q}(\Gamma)$, $n \in \mathbb{N}$ and $n_k \in \mathbb{N}$, $k \in \mathbb{N}$ satisfying (4.8) and (4.9). Let φ be a homeomorphic mapping of $\overline{\Omega}$ onto $\overline{\mathbb{C}_+}$ and conformal on Ω. For every $n \in \mathbb{N}$ set $\tilde{g}_n := \varphi \circ g_n \circ \varphi^{-1}$. Since the second module $\mathrm{m}(Q)$ is conformally invariant, given $Q := \Omega(z_1, z_2, z_3, z_4) \in \mathrm{HS}(\Omega)$ we conclude from (1.1) and (4.9) that

$$\begin{aligned} \mathrm{m}(g_{n_k} * Q) &= \mathrm{m}(\tilde{g}_{n_k} * (\varphi * Q)) \\ &= [\tilde{g}_{n_k} \circ \varphi(z_1), \tilde{g}_{n_k} \circ \varphi(z_2), \tilde{g}_{n_k} \circ \varphi(z_3), \tilde{g}_{n_k} \circ \varphi(z_4)]^{-1} - 1 \\ &\to [\tilde{f} \circ \varphi(z_1), \tilde{f} \circ \varphi(z_2), \tilde{f} \circ \varphi(z_3), \tilde{f} \circ \varphi(z_4)]^{-1} - 1 \\ &= \mathrm{m}(\tilde{f} * (\varphi * Q)) = \mathrm{m}(f * Q) \quad \text{as } k \to \infty\ , \end{aligned} \tag{1.7}$$

where $\tilde{f} := \varphi \circ f \circ \varphi^{-1}$. Since (f_n) is a Cauchy sequence, we see, by (1.4), that

$$\sup_{m\geq n} d(g_m, g_n) = \sup_{m\geq n} d(f_m, f_n) \to 0 \quad \text{as } n\to\infty\ . \tag{1.8}$$

By (1.7), for all $n\in\mathbb{N}$ and $Q\in \mathrm{HS}(\Omega)$ we have

$$\left|\frac{1}{1+\mathrm{m}(g_{n_k} * Q)} - \frac{1}{1+\mathrm{m}(g_n * Q)}\right| \to \left|\frac{1}{1+\mathrm{m}(f * Q)} - \frac{1}{1+\mathrm{m}(g_n * Q)}\right|$$

as $k\to\infty$. Applying now (1.8) and (1.4) we see that

$$d(f_n, f) = d(g_n, f) \leq \sup_{m\geq n} d(g_m, g_n) \to 0 \quad \text{as } n\to\infty\ ,$$

which proves the completeness of d on $\mathrm{Q}(\Gamma)$. Thus (IV) holds. The property (VI) follows easily from (4.1), and this ends the proof. □

Theorem 1.2. *The functional d^* satisfies the properties* (I), (II) *and* (V) *with $\rho := d^*$ and $\lambda := 1/2$.*

Proof. From (1.3) we easily conclude that the functional d^* satisfies (I), and hence d^* is a pseudo-metric on $\mathrm{Hom}^+(\Gamma)$. Fix $f, g\in \mathrm{Hom}^+(\Gamma)$. If $f\circ g^{-1}\in \mathrm{Q}(\Gamma;1)$, then by (1.2), (1.3) and (1.5)

$$d^*(f,g) \leq d(f,g) = d(f,f) = 0\ . \tag{1.9}$$

Conversely, assume that $d^*(f,g) = 0$. Then

$$\mathrm{m}(f * Q) = \mathrm{m}(g * Q)\ , \quad Q\in \mathrm{HS}(\Omega)\ .$$

Lemma 4.2 now shows that $f\circ g^{-1}\in \mathrm{Q}(\Gamma;1)$. This combined with (1.9) yields (II). The property (V) follows directly from Lemma 4.5. □

Corollary 1.3. *The functional*

$$\tilde{d}(f,g) := \max\{d(f,g)\, , 2d^*(f,g)\}\ , \quad f, g\in \mathrm{Hom}^+(\Gamma)\ ,$$

satisfies the properties (I)-(V) *with $\rho := \tilde{d}$ and $\lambda := 1$.*

Proof. The corollary follows directly from Theorems 1.1 and 1.2, Lemma 4.5, (4.2) and the inequalities

$$d^*(f,g) \leq d(f,g) \leq \tilde{d}(f,g)\ , \quad f,g\in \mathrm{Hom}^+(\Gamma)\ . \quad \square$$

For $f, g \in \mathrm{Hom}^+(\Gamma)$ define

$$d_1(f,g) := \sup\left\{ h_d\left(\frac{1}{1+\mathrm{m}(f*Q)}, \frac{1}{1+\mathrm{m}(g*Q)}\right) : Q \in \mathrm{HS}(\Omega)\right\} ,$$

where

$$h_d(z,w) := \frac{1}{2}\log\frac{1+\left|\frac{z-w}{1-\overline{w}z}\right|}{1-\left|\frac{z-w}{1-\overline{w}z}\right|} , \quad z, w \in \mathbb{D} ,$$

is the hyperbolic distance of z and w in $\mathbb{D}$, and

$$d_2(f,g) := \sup\left\{ \left|\log\frac{1+\mathrm{m}(f*Q)}{1+\mathrm{m}(g*Q)}\right| : Q \in \mathrm{HS}(\Omega)\right\} . \tag{1.10}$$

Theorem 1.4. *For each $k = 1, 2$ the functional d_k satisfies the properties* (I), (II), (IV) *and* (V) *with $\rho := d_k$ and $\lambda := +\infty$. Moreover, for any sequence $f_n \in \mathrm{Q}(\Gamma)$, $n \in \mathbb{N}$ and any $f \in \mathrm{Q}(\Gamma)$,*

$$(\ \tau(f_n, f) \to 0 \quad \textit{as } n \to \infty\ ) \implies (\ d_k(f_n, f) \to 0 \quad \textit{as } n \to \infty\ ) . \tag{1.11}$$

Proof. Assume first $k = 2$. From (1.10) we easily conclude that the functional d_2 satisfies (I). Fix $f, g \in \mathrm{Hom}^+(\Gamma)$. If $h := f \circ g^{-1} \in \mathrm{Q}(\Gamma; 1)$, then by (1.4) and (1.10) we have

$$d_2(f,g) = d_2(f, h \circ g) = d_2(f,f) = 0 . \tag{1.12}$$

Conversely, if $d_2(f,g) = 0$, then $\mathrm{m}(f*Q) = \mathrm{m}(g*Q)$ for all $Q \in \mathrm{HS}(\Omega)$. Lemma 4.2 now shows that $f \circ g^{-1} \in \mathrm{Q}(\Gamma; 1)$. This combined with (1.12) yields (II). From (1.10) and the identity

$$\mathrm{m}(\Omega(z_1,z_2,z_3,z_4))\mathrm{m}(\Omega(z_2,z_3,z_4,z_1)) = 1$$

for all quadrilaterals $\Omega(z_1,z_2,z_3,z_4)$, we see that for all $M \geq 1$,

$$d_2(f,\mathrm{id}) \leq M \iff (2e^M - 1)^{-1} \leq \mathrm{m}(f*Q) \leq 2e^M - 1 ,\ Q \in \mathrm{HS}(\Omega) , \tag{1.13}$$

and consequently (V) holds with $\lambda := +\infty$.

Assume now $f_n \in \mathrm{Q}(\Gamma)$, $n \in \mathbb{N}$ is a Cauchy sequence in $(Q(\Gamma), d_2)$. Then

$$d_2(f_n, \mathrm{id}) \leq M , \quad n \in \mathbb{N} ,$$

for some $M \geq 0$. Combining this with (1.13) we obtain

$$\delta(f_n) \leq 2e^M - 1 , \quad n \in \mathbb{N} .$$

Hence, as in the proof of Theorem 1.1, we can easily deduce (IV). The implication (1.11) follows easily from Lemma 4.6 and Theorem 1.1.

In case $k = 1$ the proof runs in much the same way as in the previous case. The only difference is in a slightly more complicated form of the right hand side of the equivalence (1.13) with d_2 replaced by d_1 and in the proof of the implication (1.11). □

Corollary 1.5. *For each $k = 1, 2$ the functional*

$$\tilde{d}_k(f,g) := d(f,g) + d_k(f,g) \ , \quad f,g \in \mathrm{Hom}^+(\Gamma) \ ,$$

satisfies the properties (I)-(V) *with $\rho := \tilde{d}_k$ and $\lambda := +\infty$.*

Proof. The corollary follows directly from Theorems 1.1 and 1.4 and the inequalities

$$\max\{d_k(f,g)\,,\, d(f,g)\} \le \tilde{d}_k(f,g) \ , \quad f,g \in \mathrm{Hom}^+(\Gamma)\,,\, k = 1,2 \ . \quad \square$$

For $f, g \in \mathrm{Hom}^+(\Gamma)$ we write $f \sim g$ iff $f \circ g^{-1} \in \mathrm{Q}(\Gamma;1)$. It is clear that $\sim$ is an equivalence relation on $\mathrm{Hom}^+(\Gamma)$. Moreover, any pseudo-metric ρ on $\mathrm{Hom}^+(\Gamma)$ taking values in $[0;+\infty)$ and satisfying (II) induces a metric $\rho_{/\sim}$ on the quotient space $\mathrm{Hom}^+(\Gamma)/\mathrm{Q}(\Gamma;1)$ given by

$$\rho_{/\sim}([f/\sim],[g/\sim]) := \rho(f,g) \ , \quad f,g \in \mathrm{Hom}^+(\Gamma) \ .$$

where $[f/\sim]$ denotes the equivalence class of f with respect to $\sim$. Applying now Theorems 1.1 and 1.4, as well as Corollaries 1.3 and 1.5 we obtain

Corollary 1.6. *For each $\rho = d, d_1, d_2, \tilde{d}, \tilde{d}_1, \tilde{d}_2$,* $(\mathrm{Q}(\Gamma)/\mathrm{Q}(\Gamma;1), \rho/\sim)$ *is a complete metric space.*

2. Applications of the pseudo-metric *d*. Let $\Omega \subset \hat{\mathbb{C}}$ be a Jordan domain bounded by a Jordan curve Γ. Given a quadrilateral $Q := \Omega(z_1, z_2, z_3, z_4)$ we define the *conjugate* quadrilateral $Q^* := \Omega(z_4, z_1, z_2, z_3)$.

Lemma 2.1. *For all $f, g \in \mathrm{Hom}(\Gamma)$ the equality*

(2.1)
$$\begin{aligned} d(f,g) &= \sup\left\{\left|\frac{1}{1+\mathrm{m}(f*Q)} - \frac{1}{1+\mathrm{m}(g*Q)}\right| : Q \in \mathrm{HR}(\Omega)\,,\, \mathrm{m}(Q) \ge 1\right\} \\ &= \sup\left\{\left|\frac{1}{1+\mathrm{m}(f*Q)} - \frac{1}{1+\mathrm{m}(g*Q)}\right| : Q \in \mathrm{HR}(\Omega)\,,\, \mathrm{m}(Q) \le 1\right\} \end{aligned}$$

holds. In particular,

(2.2)
$$\begin{aligned} d(f,g) = \sup\{&|[f(z_1),f(z_2),f(z_3),f(z_4)] - [g(z_1),g(z_2),g(z_3),g(z_4)]| : \\ &\Omega(z_1,z_2,z_3,z_4) \in \mathrm{HR}(\Omega)\,,\, [z_1,z_2,z_3,z_4] \ge 1/2\} \\ = \sup\{&|[f(z_1),f(z_2),f(z_3),f(z_4)] - [g(z_1),g(z_2),g(z_3),g(z_4)]| : \\ &\Omega(z_1,z_2,z_3,z_4) \in \mathrm{HR}(\Omega)\,,\, [z_1,z_2,z_3,z_4] \le 1/2\} \ , \end{aligned}$$

provided $\Omega = \mathbb{C}_+$ *or* $\Omega = \mathbb{D}$*.*

Proof. From [15, Definition 1.3] it follows that for every $Q \in \mathrm{HR}(\Omega)$, $\mathrm{m}(Q^*) = 1/\mathrm{m}(Q)$. Since $(f * Q)^* = f * Q^*$ and $(g * Q)^* = g * Q^*$, we see that

$$\frac{1}{1+\mathrm{m}(f*Q)} - \frac{1}{1+\mathrm{m}(g*Q)} = \frac{1}{1+\mathrm{m}(g*Q^*)} - \frac{1}{1+\mathrm{m}(f*Q^*)}\ , \ Q \in \mathrm{HR}(\Omega)\ .$$

Then (2.1) follows from the definition of the pseudo-metric d. The equality (2.2) is a direct consequence of (2.1) and the equality

$$\mathrm{m}(Q) = \frac{1}{[z_1, z_2, z_3, z_4]} - 1\ ,$$

provided $Q \in \mathrm{HR}(\mathbb{D})$ or $Q \in \mathrm{HR}(\mathbb{C}_+)$; cf. [15, Lemma 3.1]. □

For every $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$, i.e. a complex-valued and locally integrable function f on $\mathbb{R}$, set

$$f_I := \frac{1}{|I|_1} \int_I f(t)dt$$

for the average of f over a closed and bounded interval $I \subset \mathbb{R}$ with a positive length $|I|_1 > 0$. The functional

$$\|f\|_* := \sup\left\{ \frac{1}{|I|_1} \int_I |f(t) - f_I|dt : I \subset \mathbb{R} \text{ is a closed interval and } 0 < |I|_1 < +\infty \right\}$$

is a pseudo–norm on the space $\mathrm{BMO}(\mathbb{R}) := \{f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}) : \|f\|_* < +\infty\}$ and for every $f \in \mathrm{BMO}(\mathbb{R})$, $\|f\|_* = 0$ iff f is a constant function almost everywhere on $\mathbb{R}$. We recall that a function $f \in \mathrm{BMO}(\mathbb{R})$ is said to be of *bounded mean oscillation* on $\mathbb{R}$. For a survey of the properties of the space $\mathrm{BMO}(\mathbb{R})$ we refer the reader to [6, Chapter VI].

Theorem 2.2. *Suppose that H is an absolutely continuous homeomorphism of $\hat{\mathbb{R}}$ onto itself such that $h := \log H' \in \mathrm{BMO}(\mathbb{R})$. If*

$$\|h\|_* \le c/2\ , \tag{2.3}$$

then

$$d(H, \mathrm{id}) \le (2Cc^{-1}\|h\|_* + 1)^4 e^{6\|h\|_*} - 1 \to 0 \quad \text{as } \|h\|_* \to 0\ , \tag{2.4}$$

where c and C are the constants from the John-Nirenberg theorem; cf. [6, p. 230].

Proof. Given a closed and bounded interval $I \subset \mathbb{R}$ with a positive length $|I|_1 > 0$ we conclude from (2.3) and [14, Lemma 1.2] that

$$|I|_1 e^{h_I} (2Cc^{-1}\|h\|_* + 1)^{-1} \leq \int_I e^{h(t)} dt \leq |I|_1 e^{h_I} (2Cc^{-1}\|h\|_* + 1) \ .$$

Hence

$$|I|_1 e^{h_I} (2Cc^{-1}\|h\|_* + 1)^{-1} \leq H(I) \leq |I|_1 e^{h_I} (2Cc^{-1}\|h\|_* + 1) \ . \tag{2.5}$$

Fix $z_1, z_2, z_3, z_4 \in \mathbb{R}$ satisfying $z_1 < z_2 < z_3 < z_4$, and set $I_1 := [z_1; z_3]$, $I_2 := [z_2; z_4]$, $I_3 := [z_2; z_3]$ and $I_4 := [z_1; z_4]$. Note that the absolute continuity of H implies $H(\infty) = \infty$. Since

$$\begin{aligned}&[H(z_1), H(z_2), H(z_3), H(z_4)] \\ &\qquad = \frac{H(z_4) - H(z_1)}{H(z_3) - H(z_1)} \cdot \frac{H(z_3) - H(z_2)}{H(z_4) - H(z_2)} = \frac{|H(I_4)|_1}{|H(I_1)|_1} \cdot \frac{|H(I_3)|_1}{|H(I_2)|_1}\end{aligned}$$

and

$$0 < [z_1, z_2, z_3, z_4] = \frac{|I_4|_1}{|I_1|_1} \cdot \frac{|I_3|_1}{|I_2|_1} < 1 \ ,$$

we conclude from (2.5) that

$$\begin{aligned}&|[H(z_1), H(z_2), H(z_3), H(z_4)] - [z_1, z_2, z_3, z_4]| \\ &\quad = \left| \frac{|H(I_4)|_1}{|H(I_1)|_1} \cdot \frac{|H(I_3)|_1}{|H(I_2)|_1} - \frac{|I_4|_1}{|I_1|_1} \cdot \frac{|I_3|_1}{|I_2|_1} \right| \\ &\quad \leq \left((2Cc^{-1}\|h\|_* + 1)^4 e^{|h_{I_4} + h_{I_3} - h_{I_1} - h_{I_2}|} - 1 \right) \frac{|I_4|_1}{|I_1|_1} \cdot \frac{|I_3|_1}{|I_2|_1} \\ &\quad \leq (2Cc^{-1}\|h\|_* + 1)^4 e^{|h_{I_4} + h_{I_3} - h_{I_1} - h_{I_2}|} - 1 \ .\end{aligned} \tag{2.6}$$

Since

$$|I_4|_1 = |I_1|_1 + |I_2|_1 - |I_3|_1 \ ,$$

we have

$$\begin{aligned}0 < [z_1, z_2, z_3, z_4] &= \frac{|I_4|_1}{|I_1|_1} \cdot \frac{|I_3|_1}{|I_2|_1} = \frac{|I_4|_1}{|I_1|_1} + \frac{|I_4|_1}{|I_2|_1} - \frac{|I_4|_1}{|I_1|_1} \cdot \frac{|I_4|_1}{|I_2|_1} \\ &= 1 - \left(\frac{|I_4|_1}{|I_1|_1} - 1 \right) \left(\frac{|I_4|_1}{|I_2|_1} - 1 \right) \ ,\end{aligned}$$

and hence

$$\frac{|I_4|_1}{|I_1|_1} < 2 \quad \text{or} \quad \frac{|I_4|_1}{|I_2|_1} < 2 \ . \tag{2.7}$$

By Lemma 2.1 we may assume that

$$[z_1, z_2, z_3, z_4] \geq 1/2 \ , \tag{2.8}$$

which implies

$$\frac{|I_2|_1}{|I_3|_1} \leq 2\frac{|I_4|_1}{|I_1|_1} \quad \text{and} \quad \frac{|I_1|_1}{|I_3|_1} \leq 2\frac{|I_4|_1}{|I_2|_1} \ .$$

Combining this with (2.7) we obtain

$$\frac{|I_2|_1}{|I_3|_1} \leq 2\frac{|I_4|_1}{|I_1|_1} < 4 \quad \text{or} \quad \frac{|I_1|_1}{|I_3|_1} \leq 2\frac{|I_4|_1}{|I_2|_1} < 4 \ . \tag{2.9}$$

Since $I_3 \subset I_1 \subset I_4$ and $I_3 \subset I_2 \subset I_4$, we deduce from (2.9) that

$$\begin{aligned} |h_{I_4} &+ h_{I_3} - h_{I_1} - h_{I_2}| \\ &\leq \min\{|h_{I_4} - h_{I_1}| + |h_{I_3} - h_{I_2}| \, , |h_{I_4} - h_{I_2}| + |h_{I_3} - h_{I_1}|\} \\ &\leq 2\|h\|_* + 4\|h\|_* = 6\|h\|_* \ . \end{aligned} \tag{2.10}$$

The last inequality follows from $|h_I - h_J| \leq 2\|h\|_*$ provided $I, J \subset \mathbb{R}$ are intervals satisfying $I \subset J$ and $0 < |J|_1 \leq 2|I|_1 < +\infty$; cf. [6, p. 223]. Combining (2.10) with (2.6) we obtain

$$\begin{aligned} |[H(z_1), H(z_2), H(z_3), H(z_4)] - [z_1, z_2, z_3, z_4]| \\ \leq (2Cc^{-1}\|h\|_* + 1)^4 e^{6\|h\|_*} - 1 \ , \end{aligned} \tag{2.11}$$

provided (2.8) holds. Assume now $z_1, z_2, z_3 \in \mathbb{R}$ satisfy $z_1 < z_2 < z_3$ and $z_4 = \infty$. Then

$$[H(z_1), H(z_2), H(z_3), H(z_4)] = \frac{H(z_3) - H(z_2)}{H(z_3) - H(z_1)} = \frac{|H(I_3)|_1}{|H(I_1)|_1} \ ,$$

as well as

$$[z_1, z_2, z_3, z_4] = \frac{|I_3|_1}{|I_1|_1} < 1 \ .$$

Following the proof of (2.11) we obtain

$$\begin{aligned} |[H(z_1), H(z_2), &H(z_3), H(z_4)] - [z_1, z_2, z_3, z_4]| \\ &= \left| \frac{|H(I_3)|_1}{|H(I_1)|_1} - \frac{|I_3|_1}{|I_1|_1} \right| \\ &\leq \left((2Cc^{-1}\|h\|_* + 1)^2 e^{|h_{I_3} - h_{I_1}|} - 1 \right) \frac{|I_3|_1}{|I_1|_1} \\ &\leq (2Cc^{-1}\|h\|_* + 1)^2 e^{2\|h\|_*} - 1 \ , \end{aligned} \tag{2.12}$$

provided (2.8) holds. If now $z_1 = \infty$ and $z_2, z_3, z_4 \in \mathbb{R}$ satisfy $z_2 < z_3 < z_4$, then in a similar way we obtain (2.12) with I_1 replaced by I_2, provided (2.8) holds. The last two cases where $z_2 = \infty$ or $z_3 = \infty$ follow from the two former ones and the identity

$$[w_1, w_2, w_3, w_4] = [w_3, w_4, w_1, w_2] \ ,$$

which holds for every quadruple of distinct points $w_1, w_2, w_3, w_4 \in \hat{\mathbb{C}}$. Combining (2.11) with (2.12) and applying Lemma 2.1 we obtain (2.4). □

Corollary 2.3. *Suppose that $f \in \mathrm{Q}(\hat{\mathbb{R}})$ and $h_n \in \mathrm{Q}(\hat{\mathbb{R}})$, $n \in \mathbb{N}$, is a sequence of absolutely continuous functions on $\mathbb{R}$ such that $\log h_n' \in \mathrm{BMO}(\mathbb{R})$, $n \in \mathbb{N}$. If*

$$\| \log h_n' \|_* \to 0 \quad \text{as } n \to \infty \ , \tag{2.13}$$

then

$$\tau(h_n \circ f, f) \to 0 \quad \text{as } n \to \infty \ . \tag{2.14}$$

Proof. By Lemma 4.1,

$$d(h_n \circ f, f) = d(h_n, \mathrm{id}) \ , \quad n \in \mathbb{N} \ ,$$

and consequently, by Theorem 2.2 and (2.13),

$$d(h_n \circ f, f) \to 0 \quad \text{as } n \to \infty \ .$$

Thus (2.14) follows from Theorem 1.1, which ends the proof. □

Corollary 2.4. *Given $f \in \mathrm{Q}(\hat{\mathbb{R}})$ assume that f and f^{-1} are absolutely continuous on $\mathbb{R}$ and that the inequality*

$$\frac{|f(E)|_1}{|f(I)|_1} \le \alpha \left(\frac{|E|_1}{|I|_1} \right)^{\beta} \tag{2.15}$$

holds for every interval $I \subset \mathbb{R}$, $0 < |I|_1 < \infty$, and every Borel set $E \subset I$, where α and β are some positive constants. If $f_n \in \mathrm{Q}(\hat{\mathbb{R}})$, $n \in \mathbb{N}$, is a sequence of absolutely continuous functions on $\mathbb{R}$ such that

$$\| \log f_n' - \log f' \|_* \to 0 \quad \text{as } n \to \infty \ , \tag{2.16}$$

then $\tau(f_n, f) \to 0$ as $n \to \infty$.

Proof. By the assumption, each function $f_n \circ f^{-1}$, $n \in \mathbb{N}$, is absolutely continuous on $\mathbb{R}$ and the equality

$$\log(f_n \circ f^{-1})' = \log(f_n' \circ f^{-1}) - \log(f' \circ f^{-1}) = (\log f_n' - \log f') \circ f^{-1} \tag{2.17}$$

holds almost everywhere on $\mathbb{R}$. The inequality (2.15) says that the Borel measure $E \mapsto |f(E)|_1$ on $\mathbb{R}$ belongs to the so-called Muckenhoupt class A_∞; cf. [6, p.264] for the definition of the class A_∞. From the Jones result [9] and the Banach invertible operator theorem it follows that the mapping

$$h \mapsto h \circ f^{-1}$$

is a linear homeomorphism of the space $\mathrm{BMO}(\mathbb{R})$ onto itself. Combining now (2.16) with (2.17) we obtain

$$\| \log(f_n \circ f^{-1})' \|_* \to 0 \quad \text{as } n \to \infty \ .$$

Then Corollary 2.3 implies

$$\tau(f_n, f) = \tau((f_n \circ f^{-1}) \circ f, f) \to 0 \quad \text{as } n \to \infty \ ,$$

which ends the proof. □

Remark 2.5. It is easy to show that, if $f \in \mathrm{Hom}^+(\hat{\mathbb{R}})$ satisfies for all $x, y \in \mathbb{R}$ the double inequality

$$\frac{1}{L}|x - y| \leq |f(x) - f(y)| \leq L|x - y|$$

with some constant $L > 0$, i.e., f is a L-bilipschitz homeomorphism of $\mathbb{R}$ onto itself, then f satisfies the inequality (2.15) with $\alpha := L^2$ and $\beta := 1$. In the proof of [14, Lemma 1.4] a more sophisticated result was shown. It says that $f \in \mathrm{Hom}^+(\hat{\mathbb{R}})$ satisfies the inequality (2.15) with $\alpha := \exp(2\|h\|_\infty)(\sqrt{C} + 1)(C + 1)$ and $\beta := 1/2$, provided f is absolutely continuous on $\mathbb{R}$,

$$\log f' \in \mathrm{BMO}(\mathbb{R}) \quad , \quad h \in \mathrm{L}^\infty(\mathbb{R}) \quad \text{and} \quad \| \log f' - h \|_* \leq c/4 \ ,$$

where c and C are the constants from the John-Nirenberg theorem; cf. [6, p. 230].

Using the stronger pseudo-norm $\| \cdot \|_\infty$ instead of $\| \cdot \|_*$ we may omit the absolute continuity of f^{-1} and the assumption (2.15) in Corollary 2.4. We now prove

Theorem 2.6. *Suppose that* $f_n \in \mathrm{Q}(\hat{\mathbb{R}})$, $n = 0, 1, 2, \ldots$, *is a sequence of absolutely continuous functions on* $\mathbb{R}$ *such that*

$$\lambda_n := \| \log f_n' - \log f' \|_\infty \to 0 \quad \text{as } n \to \infty \ , \tag{2.18}$$

where $f := f_0$. *Then* $\tau(f_n, f) \to 0$ *as* $n \to \infty$.

Proof. Setting $h_n := \log f_n' - \log f'$, $n = 1, 2, \ldots$, we see by (2.18) that the inequalities

$$e^{-\lambda_n} f' \leq e^{h_n} f' = f_n' \leq e^{\lambda_n} f' \ , \quad n = 1, 2, \ldots \ , \tag{2.19}$$

hold almost everywhere on $\mathbb{R}$. Given a closed interval $I \subset \mathbb{R}$ we have

$$|f_n(I)|_1 = \int_I f_n'(t)dt\ , \quad n = 0, 1, 2, \dots\ .$$

Hence by (2.19),

$$e^{-\lambda_n}|f(I)|_1 \le |f_n(I)|_1 \le e^{\lambda_n}|f(I)|_1\ , \quad n = 1, 2, \dots\ . \tag{2.20}$$

Fix $z_1, z_2, z_3, z_4 \in \mathbb{R}$ satisfying $z_1 < z_2 < z_3 < z_4$, and set $I_1 := [z_1; z_3]$, $I_2 := [z_2; z_4]$, $I_3 := [z_2; z_3]$ and $I_4 := [z_1; z_4]$. Since for every $n = 0, 1, 2 \dots$,

$$\begin{aligned}[f_n(z_1), f_n(z_2), f_n(z_3), f_n(z_4)] &= \frac{f_n(z_4) - f_n(z_1)}{f_n(z_3) - f_n(z_1)} \cdot \frac{f_n(z_3) - f_n(z_2)}{f_n(z_4) - f_n(z_2)} \\ &= \frac{|f_n(I_4)|_1}{|f_n(I_1)|_1} \cdot \frac{|f_n(I_3)|_1}{|f_n(I_2)|_1}\ ,\end{aligned}$$

we conclude from (2.20) that

$$\begin{aligned} e^{-4\lambda_n}[f(z_1), f(z_2), f(z_3), f(z_4)] &\le [f_n(z_1), f_n(z_2), f_n(z_3), f_n(z_4)] \\ &\le e^{4\lambda_n}[f(z_1), f(z_2), f(z_3), f(z_4)]\ , \quad n = 1, 2, \dots\ . \end{aligned} \tag{2.21}$$

Since $0 < [f(z_1), f(z_2), f(z_3), f(z_4)] < 1$, (2.21) yields

$$\begin{aligned} &|[f_n(z_1), f_n(z_2), f_n(z_3), f_n(z_4)] - [f(z_1), f(z_2), f(z_3), f(z_4)]| \\ &\quad \le (e^{4\lambda_n} - 1)[f(z_1), f(z_2), f(z_3), f(z_4)] \le e^{4\lambda_n} - 1\ ,\ n = 1, 2, \dots\ . \end{aligned} \tag{2.22}$$

Suppose now that one of the points z_1, z_2, z_3, z_4 is equal to ∞. For simplicity we may restrict ourselves to the case where $z_4 = \infty$ and $z_1, z_2, z_3 \in \mathbb{R}$ satisfy $z_1 < z_2 < z_3$. Then

$$[f_n(z_1), f_n(z_2), f_n(z_3), f_n(z_4)] = \frac{f_n(z_3) - f_n(z_2)}{f_n(z_3) - f_n(z_1)} = \frac{|f_n(I_3)|_1}{|f_n(I_1)|_1}\ ,\ n = 1, 2, \dots\ ,$$

and a reasoning similar to that in (2.22) leads to

$$\begin{aligned} &|[f_n(z_1), f_n(z_2), f_n(z_3), f_n(z_4)] - [f(z_1), f(z_2), f(z_3), f(z_4)]| \\ &\qquad \le e^{2\lambda_n} - 1\ , \quad n = 1, 2, \dots\ . \end{aligned} \tag{2.23}$$

Combining (2.22) with (2.23) we obtain for every $Q \in \mathrm{HR}(\mathbb{C}_+)$,

$$\left| \frac{1}{1 + \mathrm{m}(f_n * Q)} - \frac{1}{1 + \mathrm{m}(f * Q)} \right| \le e^{4\lambda_n} - 1\ , \quad n = 1, 2, \dots\ . \tag{2.24}$$

By the definition of the pseudo-metric d we conclude from (2.24) and (2.18) that

$$d(f_n, f) \le e^{4\lambda_n} - 1 \to 0 \quad \text{as } n \to \infty \ .$$

Theorem 1.1 now shows that $\tau(f_n, f) \to 0$ as $n \to \infty$, which ends the proof. $\square$

Remark 2.7. All the results presented above have their counterparts in the case $\Omega := \mathbb{D}$ and $\Gamma := \mathbb{T}$. However, we omit the details.

3. The pseudo-metric $\hat{d}$. Let $S := \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ and let ρ_S be the Poincaré metric on S. For $f, g \in \mathrm{Hom}^+(\Gamma)$ we define

$$\hat{d}(f, g) := \sup\{\rho_S(-\mathrm{m}(f * Q)\,,\, -\mathrm{m}(g * Q)) : Q \in \mathrm{HR}(\Omega)\} \ . \tag{3.1}$$

To show that $\hat{d}$ satisfies all the properties (I)-(VI) we need the following lemma related to Hamilton's result [7, Lemmma 2]. For $K \ge 1$ denote by $\mathrm{QC}'(\hat{\mathbb{C}}; K)$ the class of all $F \in \mathrm{QC}(\hat{\mathbb{C}}; K)$ such that $F(t) = t$ for $t = 0, 1, \infty$.

Lemma 3.1. *If $K \ge 1$ and if $F \in \mathrm{QC}'(\hat{\mathbb{C}}; K)$, then*

$$\rho_S(F(z), z) \le \frac{1}{2} \log K \ , \quad z \in S \ . \tag{3.2}$$

Proof. Given $z \in S$ let $w := F(z)$ and $\pi : \mathbb{D} \to S$ be a holomorphic universal covering satisfying $\pi(0) = z$. By the definition of ρ_S there exists some $\lambda \in \mathbb{D}$ such that

$$\pi(\lambda) = w \quad \text{and} \quad \rho_S(w, z) = \inf\{\rho_h(0, t) : t \in \pi^{-1}(w)\} = \rho_h(0, \lambda) \ , \tag{3.3}$$

where ρ_h is the hyperbolic metric on $\mathbb{D}$. For every function $\mu \in \mathrm{L}^\infty(\hat{\mathbb{C}})$ with $\|\mu\|_\infty < 1$, let B^μ denote the uniquely determined homeomorphic solution $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the Beltrami equation

$$\bar{\partial}\varphi = \mu \partial \varphi$$

which keeps the points 0, 1 and ∞ fixed; cf. [12, p. 194]. From the Bers-Royden lemma, cf. [3] it follows that every point of $T(\hat{\mathbb{C}} \setminus \{0, 1, \infty, z\})$ is of the form $[B^\mu]$ where $\mu \in \mathrm{L}^\infty(\hat{\mathbb{C}})$, $\|\mu\|_\infty < 1$ and that there exists a holomorphic universal covering $p : T(\hat{\mathbb{C}} \setminus \{0, 1, \infty, z\}) \to S$ which sends every $[B^\mu] \in T(\hat{\mathbb{C}} \setminus \{0, 1, \infty, z\})$ into $B^\mu(z)$. Here $T(\hat{\mathbb{C}} \setminus \{0, 1, \infty, z\})$ stands for the Teichmüller space of $\hat{\mathbb{C}} \setminus \{0, 1, \infty, z\}$ and $[B^\mu]$ stands for the equivalence class of B^μ. Thus there exists a biholomorphic mapping $\Phi : \mathbb{D} \to \hat{T}(\mathbb{C} \setminus$

$\{0,1,\infty,z\}$) such that $\Phi(0)=[\mathrm{id}]$ and $p\circ\Phi=\pi$. Since in $\mathbb{D}$ the Kobayashi distance between 0 and a given $t\in\mathbb{D}$ is equal to $\rho_h(0,t)$, it follows that

(3.4) the Kobayashi distance between [id] and $\Phi(t)$ is equal to $\rho_h(0,t)$.

By Theorem 3[5, Chapter 7], the Kobayashi and Teichmüller metrics coincide. Combining this with (3.4) we see that for every $t\in\mathbb{D}$,

$$\frac{1}{2}\inf\{\log \mathrm{K}(B^\mu):[B^\mu]=\Phi(t)\}=\rho_h(0,t)=\frac{1}{2}\log\frac{1+|t|}{1-|t|}\ . \tag{3.5}$$

Given $\mu\in\mathrm{L}^\infty(\hat{\mathbb{C}})$ with $\|\mu\|_\infty<1$ it is easy to check that $B^\mu(z)=w$ iff there exists $t\in\mathbb{D}$ such that $\pi(t)=w$ and $\Phi(t)=[B^\mu]$. Thus by (3.3) and (3.5) we obtain

$$\begin{aligned}\rho_S(w,z)&=\inf\{\rho_h(0,t):\pi(t)=w\}\\&=\frac{1}{2}\inf\{\inf\{\log \mathrm{K}(B^\mu):[B^\mu]=\Phi(t)\}:\pi(t)=w\}\\&=\frac{1}{2}\inf\{\log \mathrm{K}(B^\mu):B^\mu(z)=w\}\ .\end{aligned}$$

Hence

$$\rho_S(w,z)\le\frac{1}{2}\log \mathrm{K}(F)\le\frac{1}{2}\log K\ ,$$

which proves (3.2). □

Theorem 3.2. *The functional* $\rho:=\hat{d}$ *satisfies all the properties* (I), (II), (III), (IV), (V) *with* $\lambda:=+\infty$ *and* (VI). *Moreover, for all* $f,g\in\mathrm{Q}(\Gamma)$,

$$\hat{d}(f,g)\le\frac{1}{2}\log \mathrm{K}(f\circ g^{-1})=\tau(f,g)\ . \tag{3.6}$$

Proof. The property (I) follows directly from the definition (3.1).

From (3.1) we also see that for all $f,g\in\mathrm{Hom}^+(\Gamma)$,

$$\hat{d}(f,g)=0\iff \mathrm{m}(f*Q)=\mathrm{m}(g*Q)\ ,\quad Q\in\mathrm{HR}(\Omega)\ .$$

Hence, as in the proof of Theorem 1.2, we deduce the property (II).

To prove the property (III) we first show the inequality (3.6). Fix $f,g\in\mathrm{Hom}^+(\Gamma)$ and $Q:=\Omega(z_1,z_2,z_3,z_4)\in\mathrm{HR}(\Omega)$. By the Riemann and Taylor–Osgood–Carathéodory theorems there exist homeomorphic mappings φ_1 and φ_2 of $\overline{\mathbb{C}_+}$ onto $\overline{\Omega}$ and conformal on $\mathbb{C}_+$ such that

$$\begin{aligned}\varphi_1(0)&=f\circ g^{-1}(z_2) & &\varphi_2(0)=z_2\\ \varphi_1(1)&=f\circ g^{-1}(z_3) &\text{and}\qquad &\varphi_2(1)=z_3\\ \varphi_1(\infty)&=f\circ g^{-1}(z_4) & &\varphi_2(\infty)=z_4\ .\end{aligned}$$

Setting $z := \varphi_2^{-1}(z_1)$ and $w := \varphi_1^{-1} \circ f \circ g^{-1}(z_1)$ we conclude from the conformal invariance of the second module and from [15, Lemma 3.1] that

$$\mathrm{m}(Q) = \mathrm{m}(\varphi_2^{-1} * Q) = \mathrm{m}(\mathbb{C}_+(z, 0, 1, \infty)) = \frac{1}{[z, 0, 1, \infty]} - 1 = -z \tag{3.7}$$

and similarly,

$$\mathrm{m}((f \circ g^{-1}) * Q) = \mathrm{m}((\varphi_1^{-1} \circ f \circ g^{-1}) * Q) = \mathrm{m}(\mathbb{C}_+(w, 0, 1, \infty)) = -w\ . \tag{3.8}$$

Since $\varphi_1^{-1} \circ f \circ g^{-1} \circ \varphi_2 \in \mathrm{Q}(\hat{\mathbb{R}}; K)$ with $K := \mathrm{K}(f \circ g^{-1})$, there exists $F \in \mathrm{QC}(\hat{\mathbb{C}}; K)$ such that

$$F(t) = \varphi_1^{-1} \circ f \circ g^{-1} \circ \varphi_2(t)\ , \quad t \in \hat{\mathbb{R}}\ . \tag{3.9}$$

Hence $F(t) = t$ for $t = 0, 1, \infty$, and so $F \in \mathrm{QC}'(\hat{\mathbb{C}}; K)$. Since by (3.9), $F(z) = w$, we conclude from (3.7), (3.8) and Lemma 3.1 that

$$\begin{aligned} \rho_S(-\mathrm{m}(f * (g^{-1} * Q)), -\mathrm{m}(g * (g^{-1} * Q)) &= \rho_S(-\mathrm{m}((f \circ g^{-1}) * Q), -\mathrm{m}(Q)) \\ &= \rho_S(w, z) = \rho_S(F(z), z) \le \frac{1}{2} \log K\ . \end{aligned}$$

Then (3.6) follows from (3.1) and the equality $\{g^{-1} * Q : Q \in \mathrm{HR}(\Omega)\} = \mathrm{HR}(\Omega)$. Let $f \in \mathrm{Q}(\Gamma)$ and $f_n \in \mathrm{Q}(\Gamma)$, $n \in \mathbb{N}$, be arbitrarily fixed. If $\tau(f_n, f) \to 0$ as $n \to \infty$, then by (3.6),

$$\hat{d}(f_n, f) \le \tau(f_n, f) \to 0 \quad \text{as } n \to \infty\ . \tag{3.10}$$

Conversely, assume that $\hat{d}(f_n, f) \to 0$ as $n \to \infty$. Then

$$\begin{aligned} &\sup\{\rho_S(-\mathrm{m}((f_n \circ f^{-1}) * Q)\,,\, -1) : Q \in \mathrm{HS}(\Omega)\} \\ &\quad \le \sup\{\rho_S(-\mathrm{m}((f_n \circ f^{-1}) * Q)\,,\, -m(Q)) : Q \in \mathrm{HR}(\Omega)\} \\ &\quad = \sup\{\rho_S(-\mathrm{m}(f_n * Q)\,,\, -m(f * Q)) : Q \in \mathrm{HR}(\Omega)\} \\ &\quad = \hat{d}(f_n, f) \to 0 \quad \text{as } n \to \infty\ , \end{aligned} \tag{3.11}$$

and consequently,

$$\delta(f_n \circ f^{-1}) = \sup\{\mathrm{m}((f_n \circ f^{-1}) * Q) : Q \in \mathrm{HS}(\Omega)\} \to 1 \quad \text{as } n \to \infty\ . \tag{3.12}$$

Lemma 4.4 now implies that $\tau(f_n, f) \to 0$ as $n \to \infty$, which combined with (3.10) yields the property (III).

Suppose now that $\hat{d}(f_n, f_m) \to 0$ as $n, m \to \infty$. Replacing f by f_m in the inequalities and equalities in (3.11) and (3.12) we have

$$\delta(f_n \circ f_m^{-1}) \to 1 \quad \text{as } n, m \to \infty \ ,$$

and consequently by (0.2),

$$\mathrm{K}(f_n \circ f_m^{-1}) \to 1 \quad \text{as } n, m \to \infty \ .$$

Applying now Lemma 4.1 we see that $d(f_n, f_m) \to 0$ as $n, m \to \infty$. By Theorem 1.1 there exists $f \in \mathrm{Q}(\Gamma)$ such that $d(f_n, f) \to 0$ as $n \to \infty$. Applying Theorem 1.1 once again we have $\tau(f_n, f) \to 0$ as $n \to \infty$. By the property (III) we obtain $\hat{d}(f_n, f) \to 0$ as $n \to \infty$, which proves the property (IV).

If $f \in \mathrm{Q}(\Gamma)$ then by (3.6),

$$\hat{d}(f, \mathrm{id}) \le \frac{1}{2} \log \mathrm{K}(f) < +\infty \ . \tag{3.13}$$

Conversely, assume that $f \in \mathrm{Hom}^+(\Gamma)$ and $\hat{d}(f, \mathrm{id}) < +\infty$. Then

$$\sup\{\rho_S(-\mathrm{m}(f * Q)\,,\, -1) : Q \in \mathrm{HS}(\Omega)\}$$
$$\le \sup\{\rho_S(-\mathrm{m}((f * Q)\,,\, -m(Q)) : Q \in \mathrm{HR}(\Omega)\} = \hat{d}(f, \mathrm{id}) < +\infty \ ,$$

and consequently there exists $M \ge 1$ such that

$$1/M \le \mathrm{m}(f * Q) \le M \ , \quad Q \in \mathrm{HS}(\Omega) \ .$$

By [15, Thm. 2.2], $f \in \mathrm{Q}(\Gamma)$. Combining this with (3.13) we derive the property (V) with $\lambda := +\infty$.

The property (VI) is an immediate consequence of (3.1) and the equality $\{h * Q : Q \in \mathrm{HR}(\Omega)\} = \mathrm{HR}(\Omega)$ for $h \in \mathrm{Hom}^+(\Gamma)$. □

4. Supplementary results. Throughout this section we collect a number of technical lemmas that complete considerations in the previous section.

Lemma 4.1. *For all* $f, g \in \mathrm{Hom}^+(\Gamma)$,

$$d(f, g) = d(f \circ g^{-1}, \mathrm{id}) \ . \tag{4.1}$$

Moreover, if $K \ge 1$ *and* $f \circ g^{-1} \in \mathrm{Q}(\Gamma; K)$, *then*

$$d(f, g) \le M(K) := 2\Phi^2_{\sqrt{K}}(1/\sqrt{2}) - 1 \ . \tag{4.2}$$

Proof. Since $g*Q\in \mathrm{HR}(\Omega)$ iff $Q\in \mathrm{HR}(\Omega)$, we see by (1.2)

$$\begin{aligned} d(f,g) &= \sup\left\{\left|\frac{1}{1+\mathrm{m}((f\circ g^{-1})(g*Q))}-\frac{1}{1+\mathrm{m}(g*Q)}\right| : Q\in \mathrm{HR}(\Omega)\right\} \\ &= d(f\circ g^{-1},\mathrm{id})\ , \end{aligned}$$

which yields (4.1). Assume that $K\ge 1$ and $h:=f\circ g^{-1}\in \mathrm{Q}(\Gamma;K)$ and that $Q\in \mathrm{HR}(\Omega)$. As in the proof of [15, Thm. 2.2] we can show that

$$\Phi_{1/K}\left(\frac{1}{\sqrt{1+\mathrm{m}(Q)}}\right)\le \frac{1}{\sqrt{1+\mathrm{m}(h*Q)}}\le \Phi_K\left(\frac{1}{\sqrt{1+\mathrm{m}(Q)}}\right)\ .$$

Therefore

$$\begin{aligned} \Phi_{1/K}\left(\frac{1}{\sqrt{1+\mathrm{m}(Q)}}\right)^2-\frac{1}{1+\mathrm{m}(Q)} &\le \frac{1}{1+\mathrm{m}(h*Q)}-\frac{1}{1+\mathrm{m}(Q)} \\ &\le \Phi_K\left(\frac{1}{\sqrt{1+\mathrm{m}(Q)}}\right)^2-\frac{1}{1+\mathrm{m}(Q)} \end{aligned}$$

and applying the identity ([2, Thm. 3.3])

$$\Phi_K(r)^2+\Phi_{1/K}(\sqrt{1-r^2}\,)^2=1\ ,\quad 0\le r\le 1\ ,$$

we obtain by (1.2)

$$\begin{aligned} d(f,g) &\le \max\left\{\max_{0\le t\le 1}(\Phi_K(\sqrt{t}\,)^2-t)\,,\ \max_{0\le t\le 1}(t-\Phi_{1/K}(\sqrt{t}\,)^2)\right\} \\ &= \max_{0\le t\le 1}(\Phi_K(\sqrt{t}\,)^2-t)\ . \end{aligned}$$

Combining this with [13, Thm. 3.1] we obtain (4.2), which completes the proof. $\square$

Lemma 4.2. *If* $f,g\in \mathrm{Hom}^+(\Gamma)$ *and if*

$$\mathrm{m}(f*Q)=\mathrm{m}(g*Q)\ ,\quad Q\in \mathrm{HS}(\Omega)\ , \tag{4.3}$$

then $f\circ g^{-1}\in \mathrm{Q}(\Gamma;1)$.

Proof. By the Riemann and Taylor–Osgood–Carathéodory theorems there exist homeomorphic mappings φ, φ_1 and φ_2 of $\overline{\mathbb{C}_+}$ onto $\overline{\Omega}$ and conformal on $\mathbb{C}_+$ such that $f\circ\varphi(t)=\varphi_1(t)$ and $g\circ\varphi(t)=\varphi_2(t)$ for $t=0,1,\infty$. Then the mappings $\tilde{f} := \varphi_1^{-1} \circ f \circ \varphi$ and $\tilde{g} := \varphi_2^{-1} \circ g \circ \varphi$ belong to $\mathrm{Hom}^+(\hat{\mathbb{R}})$ and satisfy $\tilde{f}(t) = \tilde{g}(t) = t$ for $t = 0, 1, \infty$. By (4.3) and the conformal invariance of the second module $\mathrm{m}(Q)$,

$$\mathrm{m}(\tilde{f} * Q) = \mathrm{m}(\tilde{g} * Q) \ , \quad Q \in \mathrm{HS}(\mathbb{C}_+) \ . \tag{4.4}$$

From [15, Example 1.4] it follows that

$$\mathrm{m}(Q) = \frac{x_2 - x_1}{x_3 - x_2} \ , \quad x_1, x_2, x_3 \in \mathbb{R} \, , \, x_1 < x_2 < x_3 \ , \tag{4.5}$$

where $Q := \mathbb{C}_+(x_1, x_2, x_3, \infty)$. Combining (4.4) and (4.5) we see that

$$\frac{\tilde{f}(x) - \tilde{f}(x-t)}{\tilde{f}(x+t) - \tilde{f}(x)} = \frac{\tilde{g}(x) - \tilde{g}(x-t)}{\tilde{g}(x+t) - \tilde{g}(x)} \ , \quad x \in \mathbb{R} \, , \, t > 0 \ . \tag{4.6}$$

Since $\tilde{f}(t) = \tilde{g}(t) = t$ for $t = 0, 1, \infty$, we conclude from (4.6) that

$$\tilde{f}\left(\frac{k}{2^n}\right) = \tilde{g}\left(\frac{k}{2^n}\right) \ , \quad n = 0, 1, 2, \dots, \ , \ k = \dots, -1, 0, 1, \dots \ .$$

By continuity, $\tilde{f}(t) = \tilde{g}(t)$ for all $t \in \mathbb{R}$. Hence

$$\varphi_1^{-1} \circ f \circ \varphi = \varphi_2^{-1} \circ g \circ \varphi$$

and finally

$$f \circ g^{-1} = \varphi_1 \circ \varphi_2^{-1} \in \mathrm{Q}(\Gamma; 1) \ ,$$

which proves the lemma. □

Lemma 4.3. *Suppose that $f_n \in \mathrm{Hom}^+(\Gamma)$, $n \in \mathbb{N}$ is a sequence satisfying*

$$\delta(f_n) \le M \ , \quad n \in \mathbb{N} \ , \tag{4.7}$$

with some real constant $M \ge 1$. Then there exist $f \in \mathrm{Q}(\Gamma)$ and sequences $g_n \in \mathrm{Q}(\Gamma)$, $n \in \mathbb{N}$ and $n_k \in \mathbb{N}$, $k \in \mathbb{N}$ such that $\delta(f) \le M$,

$$g_n \circ f_n^{-1} \in \mathrm{Q}(\Gamma; 1) \ , \quad n \in \mathbb{N} \tag{4.8}$$

and

$$g_{n_k}(z) \to f(z) \quad \text{as } k \to \infty \ , \quad z \in \Gamma \ . \tag{4.9}$$

Proof. By the Riemann and Taylor–Osgood–Carathéodory theorems there exist homeomorphic mappings φ and φ_n, $n \in \mathbb{N}$ of $\overline{\mathbb{C}_+}$ onto $\overline{\Omega}$ and conformal on $\mathbb{C}_+$ such that $f_n \circ \varphi(t) = \varphi_n(t)$ for $n \in \mathbb{N}$ and $t = 0, 1, \infty$. Then $\tilde{f}_n := \varphi_n^{-1} \circ f_n \circ \varphi \in \mathrm{Hom}^+(\hat{\mathbb{R}})$ and $\tilde{f}_n(t) = t$ for $n \in \mathbb{N}$ and $t = 0, 1, \infty$. By (4.7) and the conformal invariance of the second module $\mathrm{m}(Q)$,

$$\delta(\tilde{f}_n) \leq M \ , \quad n \in \mathbb{N} \ ,$$

and hence, by [15, Example 1.4 and Thm. 2.2], we obtain

$$\tilde{f}_n \in \mathrm{QS}(\mathbb{R}; M) \ , \quad n \in \mathbb{N} \ , \tag{4.10}$$

where $\mathrm{QS}(\mathbb{R}; M)$ denotes the class of all sense-preserving homeomorphic self-mappings of $\hat{\mathbb{R}}$ that keep the point ∞ fixed and are M-quasisymmetric in the sense of Beurling and Ahlfors; cf. [4], [11, p. 31] or [12, p. 88]. The class $\{h \in \mathrm{QS}(\mathbb{R}; M) : h(0) = 0 \, , \, h(1) = 1\}$ is compact in the locally uniform convergence topology; cf. [11, p. 32] or [1, p. 66, Lemma 1]. Combining this with (4.10) we see that

$$\tilde{f}_{n_k}(z) \to \tilde{f}(z) \quad \text{as } k \to \infty \ , \quad z \in \hat{\mathbb{R}} \ , \tag{4.11}$$

for some $\tilde{f} \in \mathrm{QS}(\mathbb{R}; M)$ and a sequence $n_k \in \mathbb{N}$, $k \in \mathbb{N}$. Setting $f := \varphi \circ \tilde{f} \circ \varphi^{-1}$ and $g_n := \varphi \circ \tilde{f}_n \circ \varphi^{-1}$ for $n \in \mathbb{N}$, we conclude from (4.11) that (4.9) holds. Furthermore,

$$g_n \circ f_n^{-1} = \varphi \circ \varphi_n^{-1} \in \mathrm{Q}(\Gamma; 1) \ , \quad n \in \mathbb{N} \ ,$$

which yields (4.8). Given $Q = \mathbb{C}_+(z_1, z_2, z_3, z_4) \in \mathrm{HS}(\mathbb{C}_+)$ we conclude from (4.11) and (1.1) that

$$\begin{aligned} \mathrm{m}(\tilde{f}_n * Q) &= \frac{1}{[\tilde{f}_{n_k}(z_1), \tilde{f}_{n_k}(z_2), \tilde{f}_{n_k}(z_3), \tilde{f}_{n_k}(z_4)]} - 1 \\ &\to \frac{1}{[\tilde{f}(z_1), \tilde{f}(z_2), \tilde{f}(z_3), \tilde{f}(z_4)]} - 1 = \mathrm{m}(\tilde{f} * Q) \quad \text{as } k \to \infty \ . \end{aligned} \tag{4.12}$$

Applying the conformal invariance of the second module $\mathrm{m}(Q)$ we deduce from (4.7) that

$$1/M \leq \mathrm{m}(\tilde{f}_n * Q) \leq M \ , \quad n \in \mathbb{N} \, , \, Q \in \mathrm{HS}(\mathbb{C}_+) \ ,$$

and hence, by (4.12), that

$$1/M \leq \mathrm{m}(\tilde{f} * Q) \leq M \ , \quad Q \in \mathrm{HS}(\mathbb{C}_+) \ .$$

The last inequality yields $\delta(f) = \delta(\tilde{f}) \leq M$, which completes the proof. □

Lemma 4.4. *For every* $f \in \mathrm{Q}(\Gamma)$ *and every sequence* $f_n \in \mathrm{Q}(\Gamma)$, $n \in \mathbb{N}$,

$$(4.13)\quad \left(\ \delta(f_n \circ f^{-1}) \to 1 \quad \text{as } n \to \infty\ \right) \iff \left(\ \tau(f_n, f) \to 0 \quad \text{as } n \to \infty\ \right) .$$

Proof. If $\delta(f_n \circ f^{-1}) \to 1$ as $n \to \infty$, then by [15, Remark 2.4] we have

$$(4.14)\quad 1 \le K(f_n \circ f^{-1}) \le \min\{\delta(f_n \circ f^{-1})^{3/2}\,,\, 2\delta(f_n \circ f^{-1}) - 1\} \to 1 \quad \text{as } n \to \infty .$$

Conversely, if $\tau(f_n, f) \to 0$ as $n \to \infty$, then by [15, Remark 2.4] we have

$$(4.15)\qquad 1 \le \delta(f_n \circ f^{-1}) \le \lambda(K(f_n \circ f^{-1})) \to 1 \quad \text{as } n \to \infty .$$

Combining (4.14) with (4.15) we obtain (4.13). □

Lemma 4.5. *For every* $f \in \mathrm{Hom}^+(\Gamma)$,

$$d^*(f, \mathrm{id}) = \frac{1}{2}\frac{\delta(f) - 1}{\delta(f) + 1} .$$

In particular, $f \in \mathrm{Q}(\Gamma)$ *iff* $d^*(f, \mathrm{id}) < 1/2$.

Proof. The lemma follows from the equivalence

$$\left|\frac{1}{1+u} - \frac{1}{2}\right| \le v \iff \frac{1-2v}{1+2v} \le u \le \frac{1+2v}{1-2v}\ , \quad u > 0\,,\ 0 \le v < \frac{1}{2}\ ,$$

and the definitions of δ and d^*. □

Lemma 4.6. *Let* $M_1, M_2 \ge 1$ *and let* $f \in \mathrm{Q}(\Gamma; M_1)$ *and* $g \in \mathrm{Q}(\Gamma; M_2)$. *Then*

$$d_2(f, g) \le (1 + \lambda(M_1))(1 + \lambda(M_2))d^*(f, g) .$$

Proof. The lemma follows from (0.3), (1.3), (1.10) and from the inequality

$$\left|\log\frac{1+u}{1+v}\right| \le |u - v| = (1+u)(1+v)\left|\frac{1}{1+u} - \frac{1}{1+v}\right|\ , \quad u, v > 0\ . \quad \square$$

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