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On the size of the ideal boundary of a finite Riemann surface

Abstract. The ideal boundary of a non-compact Riemann surface R_0 becomes visible if R_0 is embedded into some compact surface R which naturally should have the same genus g as R_0. All these compactifications of R_0 can be compared in a certain quotient space of $\mathbb{C}^g$. With respect to the canonical metric in this space the diameters of all models of the ideal boundary of R_0 are known to be bounded (cf. [4]) by a number depending only on R_0.

In this paper we prove that the diameter of each component has either a positive lower bound, depending only of R_0, or this component appears to be a single point in any compactification R.

Introduction. There are several definitions of the ideal boundary of Riemann surfaces (cf. [2]). In this article we consider a finitely connected, non-compact Riemann surface R_0 of finite genus g. If $\iota : R_0 \to R$ is a conformal embedding of R_0 into some compact surface R of genus g, then we call the boundary $\partial\iota(R_0) \subset R$ the *ideal boundary* of R_0 with respect to the compactification (R, ι) of R_0. We will ask for properties of this ideal boundary which are independent of (R, ι) and such characteristics of R_0. As in [4] we use a suitable Jacobian manifold, a quotient space of $\mathbb{C}^g$, in

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which each embedding $\iota(R_0) \subset R$ can again be embedded. On the Jacobian manifold we have a natural metric, induced by the euclidean metric on $\mathbb{C}^g$. With respect to this metric we may compare the diameter of the ideal boundaries which we obtain for all the different embeddings in any surfaces R as described above. In [4] is proved that there is some uniform bound for all these diameters.

The ideal boundary, realized as a portion of a compact surface R, consists of components. Because R_0 is provided as a finitely connected surface we have only finitely many components of the ideal boundary. It is easy to verify that there is a one-to-one correspondence of these components if we consider two or more different embeddings $\iota_1 : R_0 \to R_1, \iota_2 : R_0 \to R_2$. In this sense we understand *the components of the ideal boundary* of R_0. The purpose of this article is to show that for each such component we have (besides the supremum obtained in [4]) also a non trivial infimum for the diameter of the corresponding subset of the Jacobian manifold, which is valid for all such compactifications R of R_0. If the infimum is 0, then the component in view is always (i.e. on each such R) a singleton.

1. Notations and Definitions. Let, as before, R_0 denote some finitely connected non-compact Riemann surface of finite genus $g > 0$. Then we can fix g pairs of piecewise smooth curves a_j^0, b_j^0 such $\chi_0 = \{a_j^0, b_j^0\}_{j=1}^g$ represents a canonical homology basis modulo dividing cycles on R_0 (cf. [1]). Now we consider some compact Riemann surface R of genus g together with some conformal embedding $\iota : R_0 \to R$ and define

$$\iota(a_j^0) =: a_j \text{ and } \iota(b_j^0) =: b_j \; (1 \leq j \leq g)$$

It can be easily seen that the g pairs of curves $\chi = \{a_j, b_j\}_{j=1}^g$ represent a canonical homology basis for R.

We say that the triple $\mathcal{R} = (R, \chi, \iota)$ gives a conformal compactification of the (marked) Riemann surface (R_0, χ_0).

Remark: For each j, $1 \leq j \leq g$ there is one and only one closed holomorphic differential $\phi^{(j)}$ on R with

$$\int_{a_k} \phi^{(j)} = \delta_{jk}, \quad \int_{b_k} \phi^{(j)} =: \tau_{jk} \quad (j, k = 1, 2, \cdots, g), \tag{1}$$

where δ_{jk} denotes the Kronecker symbol(cf. [3] III.2.8).

We write $\tau_k(R, \chi)$ resp. ϵ_k for the kth column of the matrix (τ_{jk}) resp. (δ_{jk}).

Let Π stand for the linear span with integer coefficients of the $2g$ vectors

$$\tau_1, \tau_2, \cdots, \tau_g, \; \epsilon_1, \epsilon_2, \cdots, \epsilon_g$$

and we call

$$\text{Jac }(R,\chi) := \mathbb{C}^g/\Pi$$

the Jacobian manifold of the marked Riemann surface (R,χ). We have the canonical projection $\pi : \mathbb{C}^g \to \text{Jac }(R,\chi)$.

Now we fix some point p^0 on R and take for each $p \in R$ a piecewise smooth curve γ_p on R with initial point p^0 and endpoint p. This defines a map $\tilde{\Phi}_{\mathcal{R}} : R \to \mathbb{C}^g$ via

$$\tilde{\Phi}_{\mathcal{R}}(p) = \left(\int_{\gamma_p} \phi^{(1)}, \int_{\gamma_p} \phi^{(2)}, \cdots, \int_{\gamma_p} \phi^{(g)} \right).$$

Note that the image $\tilde{\Phi}_{\mathcal{R}}(p)$ depends on p *and* on the contour γ_p. However, the composition map $\Phi_{\mathcal{R}} := \pi \circ \tilde{\Phi}_{\mathcal{R}} : R \to \text{Jac }(R,\chi)$ turns out to be independent of the special choice of γ_p.
Relating to the conformal compactification $\mathcal{R} = (R,\chi,\iota)$ of (R_0,χ_0) we define the ideal boundary of R_0 as the topological boundary of the set $\iota(R_0) \subset R$, i.e.

$$\partial_{\mathcal{R}} R_0 := \overline{\iota(R_0)} \setminus \iota(R_0).$$

The set $R \setminus \iota(R_0)$ consists, by the assumption on R_0 and the compactness of R, of finitely many components $B_R^1, \ldots, B_R^n$. Now we consider another conformal compactification S instead of R, which gives the components $B_S^1, \ldots, B_S^n$. Then, by means of pairwise disjoint, simple closed curves on R_0 whose images under ι_R resp. ι_S separate the components B_R^j on R as well as B_S^j on S, we get a one-to-one correspondence of the sets B_R^j and B_S^j for $j = 1, \ldots, n$. In this sense we can speak of the n components $B^1, \ldots, B^n$ (with respect to some fixed denumeration) of the ideal boundary $\partial_{\mathcal{R}} R_0$ independently of R. Moreover, let

$$\Delta_{\mathcal{R}} R_0 := \Phi_{\mathcal{R}}(\partial_{\mathcal{R}} R_0) \text{ as well as } \Delta_{\mathcal{R}}^j R_0 := \Phi_{\mathcal{R}}(\partial B_R^j) \quad (j = 1, \ldots, n).$$

We denote by $d_{\mathcal{R}}(M)$ the diameter of a subset M of $\text{Jac}(R,\chi)$ with respect to the canonically induced metric of $\mathbb{C}^g$.

2. Universal bounds.

Theorem 1. *Let* (R_0,χ_0) *denote a non compact, finitely connected, marked Riemann surface of finite genus* $g > 0$ *with the ideal boundary components* $B^1, \ldots, B^n$ *(defined as above). Then there exist numbers* c_j, C_j $(j = 1, \ldots, n)$ *such that*

$$c_j \le d_{\mathcal{R}}(\Delta_{\mathcal{R}}^j R_0) \le C_j \qquad (j = 1, \ldots, n)$$

for all conformal compactifications $\mathcal{R} = (R,\chi,\iota)$ *of* (R_0,χ_0). *Each lower bound* c_j *can be taken strictly positive except for the case where* $B_R^j \subset R$ *is a singleton for some (and thus for all) conformal compactification of* (R_0,χ_0).

In the proof we will need the following

Lemma. *Let Ω denote a doubly connected domain in the complex plane, bounded by the piecewise smooth Jordan curves Γ_1, Γ_2. For each $m \in \mathbb{N}$ let some complex-valued function f_m, continuous on $\overline{\Omega}$ and holomorphic on Ω be given. We assume that the sequence f_m is uniformly bounded on Ω and tends to some constant c uniformly on Γ_2.*
Let f denote the limit function of some locally convergent subsequence of f_m on Ω. Then $f \equiv c$ on Ω or Γ_2 consists of a single point.

Proof. We assume that the cycle $\Gamma := \Gamma_1 - \Gamma_2$ represents a positively oriented parametrization of $\partial\Omega$, where the boundary of the unbounded component C_1 of $\mathbb{C} \setminus \Omega = C_1 \cup C_2$ is given by Γ_1. By Cauchy's formula we have for $m \in \mathbb{N}, z \in \Omega$

$$\begin{aligned} f_m(z) &= \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f_m(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f_m(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{f_m(\zeta)}{\zeta - z}\, d\zeta \\ &=: g_m^1(z) - g_m^2(z). \end{aligned}$$

Each function g_m^1 admits an analytic continuation on $I(\Gamma_1) := \Omega \cup C_2$. Because Γ_1 has winding number 1 with respect to the points on Γ_2 and $f_m \to c$ uniformly on Γ_2 we have $g_m^1 \to c$ as $m \to \infty$ on this this contour. The functions g_m^1 are uniformly bounded on $I(\Gamma_1)$. By Montel's theorem we may assume that the sequence g_m^1 is locally uniformly convergent on $I(\Gamma_1)$. The limit function g is obviously an analytic continuation of $f = \lim f_m$ on $I(\Gamma_1)$. But we have just proved $g \equiv c$ on Γ_2. So, if Γ_2 is a continuum, we conclude $g \equiv c$ on $I(\Gamma_1)$, and thus $f \equiv c$ on Ω. $\square$

Now we are ready to give the proof of Theorem 1.
According to [4, Satz 2] there exists some C with $d_{\mathcal{R}}(\Delta_{\mathcal{R}} R_0) \leq C$ simultaneously for all conformal compactifications $\mathcal{R} = (R, \chi, \iota)$ of (R_0, χ_0).
Since $\Delta^j_{\mathcal{R}} R_0 \subset \Delta_{\mathcal{R}} R_0\, (j = 1, \ldots, n)$, we get the existence of the upper bounds C_j already by the mentioned result in [4].
Now we fix some $j \in \{1, \ldots, n\}$ and assume that there is no strictly positive lower bound c_j. This means, there exists some sequence of conformal compactifications $\mathcal{R}_m = (R_m, \chi_m, \iota_m)$ of (R_0, χ_0) in the described sense with the property

$$d_{\mathcal{R}_m}(\Delta^j_{\mathcal{R}_m} R_0)) \to 0 \text{ as } m \to \infty. \tag{2}$$

On the Riemann surface $R^j_m := R_m \setminus B^j_{R_m}$ we can find some domain Λ^0_m with the following properties:

(i) Λ^0_m has genus g,

(ii) $B^{\mu}_{R_m} \subset \Lambda^0_m$ for $\mu = 1, \ldots, j-1, j+1, \ldots, n$,

(iii) $\partial\Lambda^0_m$ can be parametrized as a Jordan curve ω^0_m on R^j_m.

In $R_m^j \setminus \overline{\Lambda_m^0}$ we fix another Jordan curve ω_m^1, homotopic to ω_m^0 on R_m^j. By A_m we denote the domain bounded by these curves and let $\Lambda_m^1 := \overline{\Lambda_m^0} \cup A_m$. As proved (with slight modifications) in [4], p.42, the following estimate is valid:

$$d_{R_m}(\Phi_{\mathcal{R}_m}(R_m \setminus \Lambda_m^1) \leq B, \tag{3}$$

where B depends only on A_m and the periods $\tau_{\nu\nu}$. Note that we can give the conformal annulus A_m via ι_m by the curves $C_0 := \iota^{-1}(\omega_m^0)$ and $C_1 := \iota^{-1}(\omega_m^1)$ on R_0 as well as on R_m. Thus B is determined by considerations purely on the Riemann surface R_0 and we may assume that the boundary curves C_0, C_1 are the same for all $m \in \mathbb{N}$.
Note that (3) can also be expressed as:

(4) The variation of $\tilde{\Phi}_{\mathcal{R}_m} \circ \iota_m (m \in \mathbb{N})$ on $M_m := R_m \setminus \Lambda_m^1$ is uniformly bounded.

The set M_m is, for each $m \in \mathbb{N}$, a simply connected domain. We may assume that for all m the starting point p_m^0 of the contours in the definition of $\tilde{\Phi}_{\mathcal{R}_m}$ belongs to M_m and also that for each $p \in M_m$ the contour γ_p is a curve in M_m. Moreover, we take $p_m^0 = \iota_m(p_0)$ where p_0 is some fixed point on R_0. By the monodromy theorem the value $\tilde{\Phi}_{\mathcal{R}_m}(p)$ for $p \in M_m$ comes out to be independent of the special choice of the contours γ_p.
The set $H := \iota_m^{-1}(M_m \cap \iota_m(R_0))$ is a planar domain on R_0 and does not depend on m.

Let $G \subset \mathbb{C}$ be a domain bounded by Jordan curves which admits a conformal map θ of G onto H. It follows from our construction that the boundary of G consists of *two* components. One of them, which we denote by Γ_1, corresponds under θ to the Jordan curve C_1 on R_0, the other one, Γ_2, to the ideal boundary component B^j of R_0.
The functions $f_m := \tilde{\Phi}_{\mathcal{R}_m} \circ \iota_m \circ \theta$ map G holomorphically in $\mathbb{C}^g$ and have a continuous extension on Γ_1 and Γ_2. From (2) we know that the sequence f_m tends on Γ_2 uniformly to some constant. The functions f_m are uniformly bounded on G, as follows from (4) and the normalization

$$f_m(\theta^{-1}(p_0)) = \tilde{\Phi}_{\mathcal{R}_m}(\iota_m(p_0)) = \tilde{\Phi}_{\mathcal{R}_m}(p_m^0) = 0.$$

We apply Montel's theorem to the coordinate functions of f_m and may assume that the sequence f_m itself is locally convergent on G. By our Lemma we see that the limit function f is constant, or Γ_2 consists of a single point.
But the first case cannot happen: the canonical lifting of the function f_m on $H \subset R_0$ is given by $F_m := \tilde{\Phi}_{\mathcal{R}_m} \circ \iota_m$ and has an unrestricted analytic

continuation on R_0 along every curve on R_0 starting in H. This defines an analytic element $\tilde{F_m}$ on R_0. On the universal covering surface Σ_0 of R_0 this element $\tilde{F_m}$ appears as a holomorphic function $F_m^* : \Sigma_0 \to \mathbb{C}^g$. Let this be done for all $m \in \mathbb{N}$. By (4) and the definition of the functions $\tilde{\Phi}_{\mathcal{R}_m}$ we see that the functions F_m^* are uniformly bounded on every compact subset of Σ_0. This shows that the sequence F_m^* tends, locally uniformly on Σ_0, to a constant as $m \to \infty$ if the sequence f_m does the same on G. But this contradicts (cf.(1))

$$\int_{a_k} \phi^{(k)} = 1 \qquad (k = 1, \ldots, g).$$

Thus Γ_2 is a constant curve. By elementary considerations we see that in this case $B_R^j \subset R$ must be a singleton for *all* conformal compactifications of R_0 in the described sense. □

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