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On the boundary behaviour of functions of several complex variables

ABSTRACT. In this paper we study the boundary behaviour of holomorphic functions defined in either the unit ball, or in the unit polydisk.

I. Functions in the unit ball. Let $\mathbb{C}^n$ denote the n-dimensional complex space of all ordered n-tuples $z = (z_1, z_2, \dots, z_n)$ of complex numbers with the inner product $\langle z, w\rangle = z_1\bar{w}_1 + \ldots + z_n\bar{w}_n$. For $z \in \mathbb{C}^n$ let $z = (z_1, z')$, where $z' = (z_2, \dots, z_n) \in \mathbb{C}^{n-1}$. The unit ball $\mathbf{B}^n$ of $\mathbb{C}^n$ is the set of all $z \in \mathbb{C}^n$ with $\|z\| = (\langle z, z\rangle)^{\frac{1}{2}} < 1$. For $\varepsilon > 0$ let $\mathbf{B}^n_\varepsilon = \varepsilon\mathbf{B}^n$ and let $\mathbf{B}_\varepsilon$ denote $\mathbf{B}^1_\varepsilon$. Let $\mathbf{S}$ be the unit sphere. To every fixed $a \in \mathbf{B}^n$ corresponds an automorphism φ_a of $\mathbf{B}^n$ that interchanges a and $\mathbb{O} = (0, \dots, 0)$. Let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[a] = \{\lambda a : \lambda \in \mathbb{C}\}$, i.e.

$$\mathrm{P}_a\, z = \begin{cases} \frac{\langle z,a\rangle}{\langle a,a\rangle} a, & a \neq \mathbb{O} \\ 0, & a = \mathbb{O}, \end{cases}$$

and let $\mathrm{Q}_a = \mathrm{I} - \mathrm{P}_a$ be the projection onto the orthogonal complement of $[a]$. For $s_a = (1 - \|a\|^2)^{\frac{1}{2}}$ write

$$\varphi_a(z) = \frac{a - \mathrm{P}_a\, z - s_a\, \mathrm{Q}_a\, z}{1 - \langle z, a\rangle}.$$

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Now, let us fix $a = (r, 0, \dots, 0) \in \mathbf{B}^n$ and ε, $0 < \varepsilon < 1$. Then the image of the ball $\mathbf{B}^n_\varepsilon$ under φ_a is an ellipsoid

$$\text{(1.1)} \qquad \frac{|z_1 - c|^2}{\varepsilon^2\rho^2} + \frac{t^2}{\varepsilon^2\rho} < 1,$$

where $c = a(1-\varepsilon^2)/(1-\varepsilon^2 r^2)$, $\rho = (1-r^2)/(1-\varepsilon^2 r^2)$, $t = \|z'\|^2$.

For $\alpha > 0$ and $\zeta \in \mathbf{S}$ let a Korányi-Stein wedge Ω^ζ_α (see [Ru]) be the set of all $z \in \mathbf{B}^n$ such that

$$|1 - \langle z, \zeta\rangle| < \frac{\alpha}{2}(1 - \|z\|^2).$$

For $\alpha \le 1$, $\Omega^\zeta_\alpha = \emptyset$, and for $\alpha \to \infty$ the regions Ω^ζ_α fill up $\mathbf{B}^n$ for every fixed $\zeta \in \mathbf{S}$. In the paper [GS1] the authors obtained results on the boundary behaviour of functions holomorphic in the unit disk. If $\zeta = e_1 := (1, 0, \dots, 0) \in \mathbb{C}^n$ then the Korányi-Stein wedge is given by the inequality

$$\text{(1.2)} \qquad |1 - z_1| < \frac{\alpha}{2}(1 - |z_1|^2 - \|z'\|^2).$$

Then set $\Omega_\alpha = \Omega^{e_1}_\alpha$. Put $\Phi_\varepsilon = \cup_{r\in(0,1)}\varphi_a(\mathbf{B}^n_\varepsilon)$. We shall need the following result.

Lemma 1.1. *Let $\alpha > 1$ and $0 < \varepsilon < 1$.*

1° *If $(\frac{1+\varepsilon}{1-\varepsilon})^2 < \alpha$, then $\Phi_\varepsilon \subset \Omega_\alpha$ in a sufficiently small neighbourhood of e_1.*

2° *If $\min\{1+\varepsilon^2, \sqrt{1 + \frac{4\varepsilon^2}{(1+\varepsilon^2)^2}})\} > \alpha$, then $\Omega_\alpha \subset \Phi_\varepsilon$ in a sufficiently small neighbourhood of e_1.*

Proof.

1° Let us fix $\|z'\|^2 = t$. Note that the inequalities (1.1) and (1.2) can be written in the following form

$$\text{(1.1')} \qquad |z_1 - c|^2 < \varepsilon^2\rho^2 - \rho t^2$$

and

$$\text{(1.2')} \qquad |1 - z_1| < \frac{\alpha}{2}(1 - |z_1|^2 - t),$$

respectively. Denote by $\Phi_\varepsilon(t)$ and $\Omega_\alpha(t)$ the sets of $z_1 \in \mathbb{C}$ such that (1.1') and (1.2') hold, respectively. We show that the region $\Omega_\alpha(t)$ is convex in the direction of the imaginary axis. Let $z_1 = x + iy$, $y^2 = \tau$. Then (1.2') can be written in the form

$$(1.3)\qquad (1-x)^2 - \frac{\alpha^2}{4}(1-t-x^2)^2 < \frac{\alpha^2}{4}[\tau^2 - 2\tau(1-t-x^2) - \tau\frac{4}{\alpha^2}].$$

One can show that the right–hand side expression in (1.3) decreases with respect to τ. Thus, if (1.3) holds for some τ_0, then the same is true for $0 < \tau \leq \tau_0$. This means that $\Omega_\alpha(t)$ is convex in the direction of the imaginary axis.

Note that for the rest of the proof it suffices to prove that for every sufficiently small t the region $\Omega_\alpha(t)$ contains all the disks (1.1') in a small neighbourhood of $z_1 = 1$. From (1.1) it follows that in (1.1') we have $t \leq \varepsilon^2\rho$. Since $c \to 1$ and $\rho \to 0$ for $r \to 1^-$, we show that for r close to 1 the disks (1.1') are contained in $\Omega_\alpha(t)$.

Since there is λ such that $t = \varepsilon^2\rho\lambda$, we have $\rho = (1-r)\frac{2}{1-\varepsilon^2} + o(1-r)$, $1 - c = (1-r)\frac{1+\varepsilon^2}{1-\varepsilon^2} + o(1-r)$, $t = \frac{2\varepsilon^2\lambda}{1-\varepsilon^2}(1-r) + o(1-r)$, $\lambda \in [0,1]$, for $r \to 1^-$. Since $\Omega_\alpha(t)$ is a simply connected region (because of its convexity in the direction of the imaginary axis), it suffices to show that the boundaries of the disks (1.1') lie in $\overline{\Omega_\alpha(t)}$. We show that

$$(1.4)\qquad \begin{aligned} \overline{\Omega_\alpha(t)} \ni z_1 &= c + e^{i\theta}\sqrt{\varepsilon^2\rho^2 - \rho t} \\ &= 1 - \frac{1+\varepsilon^2}{1-\varepsilon^2}(1-r) + e^{i\theta}\frac{2\varepsilon\sqrt{1-\lambda}}{1-\varepsilon^2}(1-r) + o(1-r), \end{aligned}$$

for $\theta \in [0, 2\pi]$. Let us insert (1.4) into (1.2'). Then

$$\left|\frac{1-\varepsilon^2}{1-\varepsilon^2}(1-r) - e^{i\theta}\frac{2\varepsilon\sqrt{1-\lambda}}{1-\varepsilon^2}(1-r) + o(1-r)\right|$$

$$\leq \frac{\alpha}{2}\left[1 - \left(1 - \frac{1+\varepsilon^2}{1-\varepsilon^2}(1-r) + \cos\theta\frac{2\varepsilon\sqrt{1-\lambda}}{1-\varepsilon^2}(1-r)\right)^2 - \frac{2\varepsilon^2\lambda}{1-\varepsilon^2}(1-r)\right],$$

or equivalently

$$(1-r)\sqrt{\left(\frac{1+\varepsilon^2}{1-\varepsilon^2}\right)^2 - 2\frac{2\varepsilon\sqrt{1-\lambda}(1+\varepsilon^2)}{(1-\varepsilon^2)^2}\cos\theta + \frac{4\varepsilon^2(1-\lambda)}{(1-\varepsilon^2)^2}} + o(1-r)$$

$$\leq \frac{\alpha}{2}\left[2\frac{1+\varepsilon^2}{1-\varepsilon^2}(1-r) - \frac{4\varepsilon\sqrt{1-\lambda}}{1-\varepsilon^2}(1-r)\cos\theta - \frac{2\varepsilon^2\lambda}{1-\varepsilon^2}(1-r)\right].$$

The last inequality is a consequence of the following one:

$$\begin{aligned}(1.5)\qquad &\sqrt{(1+\varepsilon^2)^2+4\varepsilon(1+\varepsilon^2)\sqrt{1-\lambda}+4\varepsilon^2(1-\lambda)}+o(1)\\ &\le \frac{\alpha}{2}\left[2(1+\varepsilon^2)-4\varepsilon\sqrt{1-\lambda}\cos\theta-2\varepsilon^2\lambda\right].\end{aligned}$$

It is sufficient to show that (1.5) is true with $\cos\theta = 1$:

$$(1.6)\qquad \sqrt{(1+\varepsilon^2)^2+4\varepsilon(1+\varepsilon^2)\sqrt{1-\lambda}+4\varepsilon^2(1-\lambda)} \le \alpha(1-\varepsilon\sqrt{1-\lambda})^2.$$

The left–hand side expression in (1.6) increases and the right–hand side decreases with respect to $v = \sqrt{1-\lambda}$. Therefore it suffices to prove (1.6) for $\lambda = 0$. Then we have

$$\sqrt{(1+\varepsilon^2)^2+4\varepsilon(1+\varepsilon^2)+4\varepsilon^2} = (1+\varepsilon)^2 \le \alpha(1-\varepsilon)^2,$$

which is equivalent to $(\frac{1+\varepsilon}{1-\varepsilon})^2 \le \alpha$. For such an ε we have $\Omega_\alpha \subset \Phi_\varepsilon$ in a sufficiently small neighbourhood of e_1.

2° Let us fix $\|z'\|^2 = t$ and $x = \mathrm{Re} z_1$. We show that

$$Y_1 := \{y : z = x+iy \in \Omega_\alpha(t)\} \subset Y_2 := \{y : z = x+iy \in \Phi_\varepsilon(t)\}.$$

Let $M_\varepsilon := \{(x,t) \in \mathbb{R}^2 : \exists y\ \exists z'\ \|z'\|^2 = t,\ \ (x+iy, z') \in \Phi_\varepsilon\}$ and $N_\alpha := \{(x,t) \in \mathbb{R}^2 : \exists y \ge 0\ \exists z'\ \|z'\|^2 = t,\ \ (x+iy,z') \in \Omega_\alpha\}$. Since $x \to 1$ in an arbitrary way, we may assume that $x = c = 1-(1-r)\frac{1+\varepsilon^2}{1-\varepsilon^2}+o(1-r)$, $(r \to 1^-)$ is the centre of the disc (1,1'). Note that we have to prove that

$$(1.7)\qquad N_\alpha \subset M_\varepsilon$$

in a neighbourhood of $(1,0) \in \mathbb{R}^2$. Let $M_\varepsilon(x) := \{t : (x,t) \in M_\varepsilon\}$ and $N_\alpha(x) = \{t : (x,t) \in N_\alpha\}$. We shall show that $N_\alpha(x) \subset M_\varepsilon(x)$ for x close to 1. The right–hand side expression in (1.3) decreases with respect to τ. Thus the supremum of t_x from $N_\alpha(x)$ fulfills the following equation: $(1-x)^2 - \frac{\alpha^2}{4}(1-t_x-x^2)^2 = 0$, or equivalently $t_x = 1-x^2-\frac{2}{\alpha}(1-x) = (1-r)[2\frac{1+\varepsilon^2}{1-\varepsilon^2} - \frac{2}{\alpha}\frac{1+\varepsilon^2}{1-\varepsilon^2}] + o(1-r)$, for $r \to 1$ (that is for $x = 1-(1-r)\frac{1+\varepsilon^2}{1-\varepsilon^2} + o(1-r) \to 1$). Note that the supremum of t from $M_\varepsilon(x)$ is greater or equal to $t'_x = \varepsilon^2\rho = (1-r)\frac{2\varepsilon^2}{1-\varepsilon^2} + o(1-r)$. (Note that from (1.1') and (1.2') it follows that the sets M_ε and N_α are convex in the direction of t-axis.) The inclusion $N_\alpha(x) \subset M_\varepsilon(x)$ will be shown if $t_x \le t'_x$ for x sufficiently small

$(r \to 1)$, that is if $\frac{2\varepsilon^2}{1-\varepsilon^2} \geq 2\frac{1+\varepsilon^2}{1-\varepsilon^2}(1-\frac{1}{\alpha})$ or equivalently $\alpha \leq 1+\varepsilon^2$. Thus (1.7) holds. Now, we will show that $Y_1 \subset Y_2$ for $r \to 1$ ($x = c = c(r) \to 1$, $t = t(r) \to 0$ and $\rho = \rho(r) \to 0$). From (1.1') we have

$$\sup Y_2 \geq \sqrt{\varepsilon^2\rho^2 - \rho t} = [(1-r)^2\frac{4\varepsilon^2(1-\lambda)}{(1-\varepsilon^2)^2} + o((1-r^2))]^{\frac{1}{2}}.$$

We have to show that

$$\forall y \in Y_1 : \ \tau = (\sup Y_1)^2 \leq (1-r)^2\frac{4\varepsilon^2(1-\lambda)}{(1-\varepsilon^2)^2} + o((1-r^2)). \tag{1.8}$$

From (1.2') we see that τ is a solution of the equation

$$\sqrt{(1-c)^2+\tau} = \frac{\alpha}{2}(1-c^2-\tau-t), \tag{1.9}$$

for fixed $x = c = c(r)$ close to 1. Evidently $\tau = \tau(r) = (1-r)K + (1-r)^2L + o((1-r)^2)$ for $r \to 1$, where K, L are constants. And now we express (1.9) in r-terms.

$$\sqrt{(1-r)^2\left(\frac{1+\varepsilon^2}{1-\varepsilon^2}\right)^2 + (1-r)K + (1-r)^2L + o((1-r)^2)}$$

$$= \frac{\alpha}{2}\left[2(1-r)\frac{1+\varepsilon^2}{1-\varepsilon^2} - (1-r)K - (1-r^2)L - 2\frac{\varepsilon^2\lambda}{1-\varepsilon^2}(1-r)\right] + o(1-r).$$

From the above it follows that $K = 0$ and

$$L = (\alpha^2-1)\left(\frac{1+\varepsilon^2}{1-\varepsilon^2}\right)^2 - 2\alpha^2\frac{\lambda\varepsilon^2(1+\varepsilon^2)}{(1-\varepsilon^2)^2} + \frac{\alpha^2\lambda^2\varepsilon^4}{(1-\varepsilon^2)^2}.$$

For $r \to 1$ the inequality (1.8) is equivalent to the following one:

$$(\alpha^2-1)(1+\varepsilon^2)^2 - 4\varepsilon^2 \leq -\lambda^2\alpha^2\varepsilon^4 + \lambda(2\alpha^2\varepsilon^2(1+\varepsilon^2) - 4\varepsilon^2).$$

Minimum with respect to $\lambda \in [0,1]$ of the right–hand side in the last inequality is attained for $\lambda = 0$ or $\lambda = 1$. Thus let us consider two cases:

(i) $\lambda = 0$. Then $(\alpha^2-1)(1+\varepsilon^2)^2 - 4\varepsilon^2 \leq 0$, or equivalently

$$\alpha^2 \leq \frac{4\varepsilon^2}{(1+\varepsilon^2)^2} + 1. \tag{1.10}$$

(ii) $\lambda = 1$. Then $\alpha \leq 1+\varepsilon^2$.

Now note that $1+\varepsilon^2$ is less than the right-hand side of (1.10). □

Theorem 1.2. *Let f be a function holomorphic in $\mathbf{B}^n$, $c_2, \dots, c_n$ be real integers, $c_1 \in \mathbb{C}$ and let Ω_α be a Korányi-Stein wedge at e_1. If*

$$\lim_{\Omega_\alpha \ni z \to e_1} \left[f(z)(1-z_1)^{c_1} \prod_{k=2}^{n} z_k^{c_k} \right] = A \neq \infty,$$

then there exists $\alpha_1 < \alpha$ such that

$$\lim_{\Omega_{\alpha_1} \ni z \to e_1} \frac{\partial f(z)}{\partial z_1}(1-z_1)^{c_1+1} \prod_{k=2}^{n} z_k^{c_k} = Ac_1$$

and

$$\lim_{\Omega_{\alpha_1} \ni z \to e_1} \frac{\partial f(z)}{\partial z_l}(1-z_1)^{c_1} \prod_{k=2}^{n} z_k^{c_k} z_l = -Ac_l, \quad l = 2, \dots, n.$$

Proof. Let us consider the function

$$h(z) = f(\varphi_a(z))(1-\varphi_a^{(1)}(z))^{c_1} \prod_{k=2}^{n} (\varphi_a^{(k)}(z))^{c_k}.$$

The automorphism φ_a, with $a = (r, 0, \dots, 0)$ and r close to 1, maps every ball $\mathbf{B}^n_{\varepsilon(\alpha)-\delta}$, with δ sufficiently small, into a Korányi-Stein wedge $\Omega_\alpha = \Omega_\alpha^{e_1}$. Therefore, if there exists $\lim_{\Omega_\alpha \ni z \to e_1} h(z) = A \in \mathbb{C}$, then $f(\varphi_a(z))(1 - \varphi_a^{(1)}(z))^{c_1} \prod_{k=2}^{n} (\varphi_a^{(k)}(z))^{c_k}$ tends uniformly in $\mathbf{B}^n_\varepsilon$ to A for $r \to 1$. Note that for the above a we have $\varphi_a(z) = (\varphi_a^{(1)}(z), \dots, \varphi_a^{(n)}(z))$, with $\varphi_a^{(1)}(z) = \frac{r-z_1}{1-rz_1}$, and $\varphi_a^{(k)}(z) = \frac{-\sqrt{1-r^2}z_k}{1-rz_1}$, $k = 2, \dots, n$.

Then

$$\frac{\partial h}{\partial z_1}(z) = \left[\frac{\partial f}{\partial \varphi^{(1)}}(\varphi_a(z)) \frac{-1+r^2}{(1-rz_1)^2}(1-\varphi_a^{(1)}(z))^{c_1+1} \prod_{k=2}^{n} (\varphi_a^{(k)}(z))^{c_k} \right.$$

$$\left. - c_1 f(\varphi_a(z))(1-\varphi_a^{(1)}(z))^{c_1} \prod_{k=2}^{n} (\varphi_a^{(k)}(z))^{c_k} \frac{-1+r^2}{(1-rz_1)^2} \right] \frac{1}{1-\varphi_a^{(1)}(z)}$$

$$+ \sum_{j=2}^{n} \left[\frac{\partial f}{\partial \varphi^{(j)}}(\varphi_a(z)) \left(\frac{-r\sqrt{1-r^2}z_j}{(1-rz_1)^2} \right) (1-\varphi_a^{(1)}(z))^{c_1} \prod_{k=2}^{n} (\varphi_a^{(k)}(z))^{c_k} \right.$$

$$\left. + f(\varphi_a(z))(1-\varphi_a^{(1)}(z))^{c_1} \prod_{k=2}^{n} (\varphi_a^{(k)}(z))^{c_k} \frac{c_j}{\varphi_a^{(j)}(z)} \frac{-r\sqrt{1-r^2}z_j}{(1-rz_1)^2} \right]$$

and this uniformly tends to 0, as $r \to 1$ in $\mathbf{B}^n_\varepsilon$.

Now, let us observe that

$$\frac{r^2-1}{(1-rz_1)^2}\frac{1}{1-\varphi_a^{(1)}(z)} = -\frac{1+r}{(1+z_1)(1-rz_1)}$$

and that the last term is bounded for r close to 1. Moreover, each term under the sign of sum $\sum_{j=2}^n$ has the following form

(1.11)

$$\left[\frac{\partial f}{\partial w_j}(w)(1-w_1)^{c_1}\prod_{k=2}^n w_k^{c_k} w_j + f(w)(1-w_1)^{c_1}\prod_{k=2}^n w_k^{c_k} c_j\right]\frac{r}{1-rz_1},$$

where the expression $\frac{r}{1-rz_1}$ is bounded for r close to 1. Therefore, using Lemma 1.1 one can see that for ε sufficiently small (1.11) tends to 0 as $w \to e_1$ in Φ_ε.

Moreover, from the definition of h we get

$$\frac{\partial h}{\partial z_l}(z) = \frac{\partial f}{\partial z_l}(\varphi_a(z))\frac{-\sqrt{1-r^2}}{1-rz_1}(1-\varphi_a^{(1)}(z))^{c_1}\prod_{k=2}^n(\varphi_a^{(k)}(z))^{c_k}$$
$$+f(\varphi_a(z))(1-\varphi_a^{(1)}(z))^{c_1}\prod_{k=2}^n(\varphi_a^{(k)}(z))^{c_k}c_l\frac{1}{z_l} \to 0$$

uniformly, as $r \to 1$ in $\mathbf{B}^n_\varepsilon$. Then

$$\frac{\partial f}{\partial z_l}(\varphi_a(z))\frac{\sqrt{1-r^2}z_l}{1-rz_1}(1-\varphi_a^{(1)}(z))^{c_1}\prod_{k=1}^n(\varphi_a^{(k)}(z))^{c_k} \to c_l A,$$

uniformly, as $r \to 1$ in $\mathbf{B}^n_\varepsilon$. Thus

$$\lim_{\Phi_\varepsilon \ni w \to e_1}\left[\frac{\partial f}{\partial z_l}(w)(1-w_1)^{c_1}\prod_{k=1}^n w_k^{c_k} w_l\right] = -c_l A.$$

The proof is complete. □

Corollary 1.3. *Let f be a function holomorphic in $\mathbf{B}^n$. If $\lim_{\Omega_\alpha \ni z \to e_1} f(z) = A \neq \infty$, then there exists $\alpha_1 < \alpha$ such that in Ω_{α_1} we have $\frac{\partial f(z)}{\partial z_1} = o(\frac{1}{|1-z_1|})$ and $\frac{\partial f(z)}{\partial z_l} = o(\frac{1}{|z_l|})$ for $z \to e_1$ and every $l = 2, \ldots, n$.*

In the next theorem we give results concerning the behaviour of $\frac{\partial f}{\partial z_j}$, which is essentially different from that presented in Theorem 1.2.

Theorem 1.4. *Let f be a function holomorphic in $\mathbf{B}^n$, $c \in \mathbb{C}$ and let Ω_α be a Korányi-Stein wedge at e_1. Assume that there exists the limit*

$$\lim_{\Omega_{\alpha_1} \ni z \to e_1} f(z)(1 - z_1^2 - \ldots - z_n^2)^c = A \in \mathbb{C}.$$

Then

(i) for every $l = 2, \ldots, n$ the expression $\frac{\partial f(z)}{\partial z_l}(1 - z_1^2 - \ldots - z_n^2)^{c+\frac{1}{2}}$ is bounded in Ω_{α_1} for $z \to e_1$, but the limit $\lim_{\Omega_{\alpha_1} \ni z \to e_1} \dfrac{\partial f(z)}{\partial z_l}(1 - z_1^2 - \ldots - z_n^2)^{c+\frac{1}{2}}$ does not exist with $c \neq 0$.

(ii) there exists $\alpha_1 < \alpha$ such that

$$\lim_{\Omega_{\alpha_1} \ni z \to e_1} \frac{\partial f(z)}{\partial z_1}(1 - z_1^2 - \ldots - z_n^2)^{c+1} = 2cA.$$

Proof. Let us consider an automorphism

$$\varphi_a(z) = \left(\frac{r - z_1}{1 - r z_1}, -\frac{\sqrt{1 - r^2} z_2}{1 - r z_1}, \ldots, -\frac{\sqrt{1 - r^2} z_n}{1 - r z_1} \right),$$

with $a = (r, 0, \ldots, 0)$. Then $\varphi(\mathbf{B}_\varepsilon^n) \subset \Phi_\varepsilon \subset \Omega_\alpha$, $(\frac{1+\varepsilon}{1-\varepsilon})^2 < \alpha$. Write

$$h(z) = f(\varphi_a(z))(1 - (\varphi_a^{(1)}(z))^2 - \ldots - (\varphi_a^{(n)}(z))^2)^c,$$

and $w_j = \varphi_a^{(j)}(z)$. From the assumption we have $\lim_{\mathbf{B}_\varepsilon^n \ni z \to e_1} h(z) = A$.

First we prove (i).

For every $j = 2, \ldots, n$ we get (after some calculations)

$$\frac{\partial h(z)}{\partial z_j} = \frac{\partial f}{\partial w_j}(w)[1 - (\varphi_a^{(1)}(z))^2 - \ldots - (\varphi_a^{(n)}(z))^2)]^{c+1} \frac{1 - r z_1}{\sqrt{1 - r^2}(1 - z_1^2 - \ldots - z_n^2)}$$
$$- f(w)c[1 - (\varphi_a^{(1)}(z))^2 - \ldots - (\varphi_a^{(n)}(z))^2]^c \frac{2 z_j}{1 - z_1^2 - \ldots - z_n^2},$$

which tends to 0 uniformly for $z \in \mathbf{B}_\varepsilon^n$ and $r \to 1$. From the above we see that

$$\frac{\partial h(z)}{\partial z_j} = -\frac{\partial f}{\partial w_j}(w)(1 - w_1^2 - \ldots - w_n^2)^{c+\frac{1}{2}} \frac{(1 - w_1^2 - \ldots - w_n^2)^{\frac{1}{2}}(1 - r z_1)}{\sqrt{1 - r^2}(1 - z_1^2 - \ldots - z_n^2)}$$
$$- f(w)c[1 - w_1^2 - \ldots - w_n^2]^c \frac{2 z_j}{1 - z_1^2 - \ldots - z_n^2},$$

tends to 0 uniformly for $z \in \mathbf{B}^n_\varepsilon$ and $r \to 1$. Since $\sqrt{\frac{1-w^2}{1-r^2}} = \frac{\sqrt{1-z_1^2-\ldots-z_n^2}}{1-rz_1}$ and $\sqrt{1-z_1^2-\ldots-z_n^2}$ are bounded in $\mathbf{B}^n_\varepsilon$,

$$\frac{\partial f}{\partial w_j}(w)(1-w_1^2-\ldots-w_n^2)^{c+\frac{1}{2}} + f(w)c[1-w_1^2-\ldots-w_n^2]^c \frac{2z_j}{1-z_1^2-\ldots-z_n^2}$$

tends to 0 uniformly for $z \in \mathbf{B}^n_\varepsilon$ and $r \to 1$. Therefore

$$\frac{\partial f(z)}{\partial z_l}(1-z_1^2-\ldots-z_n^2)^{c+\frac{1}{2}}$$

is bounded in Ω_{α_1} for $z \to e_1$ and $j = 2, \ldots, n$.

We will show that the expression

$$\frac{\partial f(z)}{\partial z_l}(1-z_1^2-\ldots-z_n^2)^{c+\frac{1}{2}}$$

with $c \neq 0$, has no limit for $\Omega_{\alpha_1} \ni z \to e_1$. In the case $n = 2$ let us consider the function

$$f(z) = \frac{1}{1-z_1^2-z_2^2}.$$

Note that $\lim_{\Omega_{\alpha_1} \ni z \to e_1} f(z)(1-z_1^2-z_2^2) = 1$, with $c = 1$ and $A = 1$. Then

$$\lim_{\Omega_{\alpha_1} \ni z \to e_1} \frac{\partial f(z)}{\partial z_2}(1-z_1^2-z_2^2)^{1+\frac{1}{2}} = 2 \lim_{\Omega_{\alpha_1} \ni z \to e_1} \frac{z_2}{\sqrt{1-z_1^2-z_2^2}}.$$

We will prove that the last limit does not exist. By the definition of the Korányi-Stein wedge in $\mathbb{C}^2$ we have

$$|1-z_1| < \frac{\alpha}{2}(1-|z_1|^2-|z_2|^2).$$

Then for $z_1 = 1-r$ we get $|z_2|^2 \leq r(2(1-\frac{1}{\alpha})-r)$. Note that for r sufficiently small we may take $z_2^2 = r(1-\frac{1}{\alpha})t$, where $t \in [0,1]$. Then

$$\sqrt{1-z_1^2-z_2^2} = \sqrt{2r - r^2\left(1+\left(1-\frac{1}{\alpha}\right)^2 t^2\right)}$$

and therefore

$$\lim_{\Omega_{\alpha_1} \ni z \to e_1} \frac{z_2}{\sqrt{1-z_1^2-z_2^2}} = \sqrt{\frac{1-\frac{1}{\alpha}}{2}t}.$$

The last expression depends on t, so that $\lim_{\Omega_{\alpha_1}\ni z\to e_1} \frac{z_2}{\sqrt{1-z_1^2-z_2^2}}$ does not exist. For $n>2$ one may consider the function

$$f(z)=\frac{1}{1-z_1^2-\ldots-z_n^2}.$$

Now we prove (ii).

Put $w=\varphi_a(z)$. Then

$$\begin{aligned}\frac{\partial h(z)}{\partial z_1}=&\left[\frac{\partial f}{\partial w_1}(w)\frac{r^2}{(1-rz_1)^2}(1-w_1^2-\ldots-w_n^2)^c\right.\\&\left.+cf(w)(1-w_1^2-\ldots-w_n^2)^{c-1}\left(-2w_1\frac{r^2-1}{(1-rz_1)^2}\right)\right]_1\\&-\sum_{k=2}^n\left[\left(\frac{\partial f}{\partial w_k}(w)(1-w_1^2-\ldots-w_n^2)^c\right.\right.\\&\left.\left.-2cf(w)(1-w_1^2-\ldots-w_n^2)^{c-1}w_k\right)\frac{r\sqrt{1-r^2}z_k}{(1-rz_1)^2}\right]_k\end{aligned}$$

tends to 0 uniformly for $z\in\mathbf{B}^n_\varepsilon$ and $r\to 1^-$.
Since $(1-w_1^2-\ldots-w_n^2)^{\frac12}=\sqrt{1-r^2}\frac{\sqrt{1-z_1^2-\ldots-z_n^2}}{1-rz_1}$, we get

$$\begin{aligned}[\ldots]_k=&\left(\frac{\partial f}{\partial w_k}(w)(1-w_1^2-\ldots-w_n^2)^{c+\frac12}\frac{1-rz_1}{\sqrt{1-r^2}\sqrt{1-z^2}}\right.\\&\left.+2cf(w)(1-w_1^2-\ldots-w_n^2)^c\frac{(1-rz_1)z_k}{\sqrt{1-r^2}(1-z^2)}\right)\frac{r\sqrt{1-r^2}z_k}{(1-rz_1)^2}.\end{aligned}$$

From the first part of the proof we have

$$\frac{\partial f}{\partial w_k}(w)(1-w_1^2-\ldots-w_n^2)^{c+\frac12}=-cf(w)(1-w_1^2-\ldots-w_n^2)^c\frac{2z_k}{\sqrt{1-z^2}}+o(1).$$

Therefore

$$\begin{aligned}[\ldots]_k=&-cf(w)(1-w_1^2-\ldots-w_n^2)^c\frac{rz_k}{\sqrt{1-z^2}}\frac{1}{1-rz_1}+o(1)\\&+cf(w)(1-w_1^2-\ldots-w_n^2)^c\frac{rz_k}{\sqrt{1-z^2}}\frac{1}{1-rz_1}=o(1)\end{aligned}$$

tends to 0 uniformly for $z \in \mathbf{B}^n_\varepsilon$ and $r \to 1^-$. Moreover

$$[\dots]_1 = \frac{\partial f}{\partial w_1}(w)(1 - w_1^2 - \dots - w_n^2)^{c+1}\frac{-1}{1-z^2} + cf(w)(1 - w_1^2 - \dots - w_n^2)^c \frac{2}{1-z^2}\frac{r-z_1}{1-rz_1}.$$

Thus from the above considerations we get

$$\lim_{\Omega_{\alpha_1} \ni z \to e_1} \frac{\partial f(z)}{\partial z_1}(1 - z_1^2 - \dots - z_n^2)^{c+1} = 2cA. \quad \square$$

Rudin obtained the following result ([Ru], Lemma 6.4.6).

Theorem R. *If f is a function holomorphic in $\mathbf{B}^n$, $c \geq 0$, and*

$$|f(z)| \leq (1 - \|z\|)^{-c} \quad \text{for} \quad z \in \mathbf{B}^n,$$

then for $l = 2, \dots, n$, $0 < r < 1$,

$$\left|\frac{\partial f(re_1)}{\partial z_l}\right| \leq A_c(1-r)^{-c-\frac{1}{2}}.$$

Note that Theorem R is interesting in the case when $|f(re_1)| \to \infty$ as $r \to 1^-$. The following corollary describing the behaviour of functions, in the case of existence of a finite limit $\lim_{\Omega_{\alpha_1} \ni z \to e_1} f(z)$, may be concluded from the proof of Theorem 1.4.

Corollary 1.5. *If there exists a finite limit $\lim_{\Omega_{\alpha_1} \ni z \to e_1} f(z)$ then for every $l = 2, \dots, n$*

$$\lim_{\Omega_{\alpha_1} \ni z \to e_1} \frac{\partial f(z)}{\partial z_l}(1 - z_1^2 - \dots - z_n^2)^{\frac{1}{2}} = 0.$$

From the proof of Theorem 1.4 also the next corollary follows.

Corollary 1.6. *If there exists $\lim_{\Omega_{\alpha_1} \ni z \to e_1} f(z)(1 - z_1^2 - \dots - z_n^2)^c = A$, then for every $l = 2, \dots, n$*

$$\lim_{\Omega_{\alpha_1} \ni z \to e_1} \frac{\partial f(z_1, 0, \dots, 0)}{\partial z_l}(1 - z_1^2)^{c+\frac{1}{2}} = 0.$$

Corollary 1.7. *Let f be a function holomorphic in $\mathbf{B}^n$. If there exists a finite limit* $\lim\limits_{\Omega_\alpha \ni z \to e_1} f(z)$*, then*

$$\frac{\partial f(z)}{\partial z_1} = o\left(\frac{1}{1 - z_1^2 - \ldots - z_n^2}\right),$$

for $z \to e_1$ in Ω_{α_1}.

II. Functions in the unit polydisk. Let $\mathbf{\Delta}$ be the unit disk in the plane. For $e^{i\theta} = (e^{i\theta_1}, \ldots, e^{i\theta_n})$ and $\eta = (\eta_1, \ldots, \eta_n)$ let us consider Stolz domains at $e^{i\theta_k}$, i.e. the domains

$$W_{\eta_k}(e^{i\theta_k}) = \{z_k \in \mathbf{\Delta} : |\arg(1 - z_k e^{-i\theta_k})| < \eta_k\},$$

where $\eta_k \in (0, \pi/2]$, $\rho > 0$, $k = 1, \ldots, n$. Let

$$W_\eta(e^{i\theta}) = W_{\eta_1}(e^{i\theta_1}) \times \ldots \times W_{\eta_n}(e^{i\theta_n})$$

be the Stolz domain.

In this part of the paper we solve some problems concerning the behaviour of functions holomorphic in the polydisk near "the vertex" of a Stolz domain.

Theorem 2.1. *Let $A \in \mathbb{C}$, $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ and let*

$$\lim_{W_\eta \ni z \to e^{i\theta}} f(z) \prod_{k=1}^{n} (1 - z_k e^{-i\theta_k})^{c_n} = A. \tag{2.1}$$

Then

1° *for every $\varepsilon_k \in (0, \eta_k)$, $k = 1, \ldots, n$ and each $l = 1, \ldots, n$*

$$\lim_{W_{\eta-\varepsilon} \ni z \to e^{i\theta}} \frac{\partial f}{\partial z_l} \prod_{k=1}^{n} (1 - z_k e^{-i\theta_k})^{c_n} = A c_l e^{-i\theta_l};$$

2° *if $A \neq 0$ and if the limit (2.1) exists for every W_η with the vertex at $\mathbf{1}$, then*

$$\lim_{a \to \mathbf{1}} \frac{f(\varphi_a(z))}{f(a)} = \prod_{k=1}^{n} \left(\frac{1 + z_k}{1 - z_k}\right)^{c_k},$$

where $\varphi_a(z) = (\frac{z_1 + a_1}{1 + a_1 z_1} e^{i\theta_1}, \ldots, \frac{z_n + a_n}{1 + a_n z_n} e^{i\theta_n})$ is an automorphism of $\mathbf{\Delta}^n$.

Proof.

1° For $\delta > 0$ sufficiently small and $l = 1, \dots, n$, put $K_{\eta_l}(\delta) = \{z_l : |z_l| \leq r_{\eta_k} - \delta\}$ and $K_\eta(\delta) = K_{\eta_1}(\delta) \times \dots \times K_{\eta_n}(\delta)$. Then for $a = (a_1, \dots, a_n) \in (0,1)^n$ we have

$$f(w_1, \dots, w_n) \prod_{k=1}^{n} (1 - w_k)^{c_k} \to A$$

uniformly in $K_\eta(\delta)$, as $a \to \mathbf{1}$ where $w_k = \frac{z_k + a_k}{1 + a_k z_k}$. Now for fixed l we get

$$\frac{\partial f}{\partial z_l}(w) \prod_{k=1}^{n} (1 - w_k)^{c_k} e^{i\theta_l} \frac{1 - a_l^2}{(1 + a_l z_l)^2} - f(w) \prod_{k=1, k \neq l}^{n} (1 - w_k)^{c_k} \frac{1 - a_l^2}{(1 + a_l z_l)^2} c_l (1 - w_l)^{c_l - 1} \to 0$$

uniformly in $K_\eta(\delta)$, as $a \to \mathbf{1}$. Note that $\frac{1 - a_l^2}{(1 + a_l z_l)^2} = (1 - w_l) \frac{1 + a_l}{(1 + a_l z_l)(1 - z_l)}$ and

$$\left[\frac{\partial f}{\partial z_l}(w) \prod_{k=1}^{n} (1 - w_k)^{c_k} e^{i\theta_l} (1 - w_l) - f(w) \prod_{k=1}^{n} (1 - w_k)^{c_k} c_l \right] \frac{1 + a_l}{(1 + a_l z_l)(1 - z_l)} \to 0$$

uniformly in $K_\eta(\delta)$, as $a \to \mathbf{1}$. Therefore

(2.2)
$$\frac{\partial f}{\partial z_l}(w) \prod_{k=1}^{n} (1 - w_k e^{-i\theta_k})^{c_k} e^{i\theta_l} (1 - w_l e^{-i\theta_l}) - f(w) \prod_{k=1}^{n} (1 - w_k e^{-i\theta_k})^{c_k} c_l \to 0$$

as $w = (w_1, \dots, w_n) \to \mathbf{1}$ in a domain Ω_α which is the image of $K_\eta(\delta)$ under the map $(\frac{z_1 + a_1}{1 + a_1 z_1} e^{i\theta_1}, \dots, \frac{z_n + a_n}{1 + a_n z_n} e^{i\theta_n})$. In the same way as in the case $n = 1$ ([GS1]) one can show that $W_{\eta - \varepsilon} \subset \Omega_\alpha$ for every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ with sufficiently small $\|\varepsilon\|$. Thus from (2.2) we obtain

$$\lim_{W_{\eta - \varepsilon} \ni w \to \mathbf{1}} \frac{\partial f}{\partial z_l}(w) \prod_{k=1}^{n} (1 - w_k e^{-i\theta_k})^{c_k} e^{i\theta_l} (1 - w_l e^{-i\theta_l}) = A c_l.$$

2° Note that for $g(z) = \frac{(1 + z_1)^{c_1 - 2} \dots (1 + z_n)^{c_n - 2}}{(1 - z_1)^{c_1} \dots (1 - z_n)^{c_n}}$ there exists the limit

$$\lim_{W_\eta \ni z \to \mathbf{1}} \frac{f(z)}{g(z)} = A \cdot 2^q, \tag{2.3}$$

where $q = 2n = \sum_{k=1}^{n} c_k$. In particular, $\lim\limits_{W_\eta \ni a \to \mathbf{1}} \frac{f(a)}{g(a)} = A \cdot 2^q$. Similarly as in the proof of 1^o one can rewrite (2.3) in the following form

$$\lim_{W_\eta \ni a \to \mathbf{1}} \frac{f(\varphi_a(z))g(a)}{g(\varphi_a(z))f(a)} = 1, \tag{2.4}$$

where the convergence is uniform in $\mathbf{\Delta}^n$. Since $g(z) = \frac{g(\varphi_a(z))}{g(a)\prod_{k=1}^{n}(1+a_k z_k)^2}$, from (2.4) we get

$$\lim_{W_\eta \ni a \to \mathbf{1}} \frac{f(\varphi_a(z))}{f(z)} = g(z)\prod_{k=1}^{n}(1+z_k)^2 = \prod_{k=1}^{n}\left(\frac{1+z_k}{1-z_k}\right)^{c_k}. \quad \square$$

Corollary 2.2. *Let $c = \mathbb{O}$. If the limit $\lim\limits_{W_\eta \ni z \to e^{i\theta}} f(z)$ is finite then for every $l = 1, \dots, n$ and every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ $\frac{\partial f}{\partial z_l}(z) = o(\frac{1}{1-\|z\|})$ for $z \to \mathbf{1}$ in $W_{\eta-\varepsilon}$. Moreover $\frac{\partial^m f}{\partial^{k_1} z_1 \dots \partial^{k_n} z_n}(z) = o((\frac{1}{1-\|z\|})^m)$, where $m = k_1 + \dots + k_n$, and on the right–hand side of the last equality it is not possible to put a number less than m.*

The proof of Theorem 2.1 implies a modification of Hardy-Littlewood theorem ([Du], [Ru]; also cf. [GS2]).

Theorem 2.3. *Let f be a function holomorphic in W_η with the vertex at $\mathbf{1}$, where η is sufficiently small and suppose that for fixed $c \in \mathbb{C}^n$ the limit $\lim\limits_{W_\eta \ni z \to \mathbf{1}} f(z) \prod_{k=1}^{n}(1-z_k)^{c_k} = A \in \mathbb{C}$ does exist. Then for every $l = 1, \dots, n$*

$$\lim_{\mathbf{r} \to \mathbf{1}} \frac{\partial f}{\partial z_l}(\mathbf{r}) \prod_{k=1}^{n}(1-r_k)^{c_k}(1-r_l) = Ac_l,$$

where $\mathbf{r} = (r_1, \dots, r_n) \in (0,1)^n$.

Theorem 2.4. *Let $c = (c_1, \dots, c_n) \in \mathbb{C}^n$, $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$. If*

$$\lim_{W_\eta \ni z \to \mathbf{1}} f(z) \prod_{k=1}^{n}\left((1-z_k)^{c_k}\left(\log\frac{1}{1-z_k}\right)^{\mu_k}\right) = A \in \mathbb{C},$$

$$\left(\textit{or} \quad \lim_{W_\eta \ni z \to \mathbf{1}} f(z) \prod_{k=1}^{n} \exp\frac{c_k}{1-z_k} = A \in \mathbb{C}\right),$$

then for every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $0 < \varepsilon_k < \eta_k$ and every $l = 1, \dots, n$

$$\lim_{W_{\eta-\varepsilon} \ni z \to \mathbf{1}} \frac{\partial f}{\partial z_l}(w) \prod_{k=1}^{n}\left((1-w_k)^{c_k}\left(\log\frac{1}{1-w_k}\right)^{\mu_k}\right)(1-w_k) = Ac_l.$$

$$\left(\lim_{W_{\eta-\varepsilon}\ni z\to\mathbf{1}}\frac{\partial f}{\partial z_l}(w)\prod_{k=1}^{n}\exp\frac{c_k}{1-w_k}=-Ac_l\right).$$

The proof of this theorem is similar to that of Theorem 2.1.

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