ANNALES
UNIVERSITATIS MARIAE CURIE – SKŁODOWSKA
LUBLIN – POLONIA

VOL. LVI, 4 SECTIO A 2002

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On the stochastic convergence of conditional expectations of some random sequences

Abstract. Let $(\Omega, \mathfrak{F}, P)$ be a non-atomic probability space and (X_n) be a sequence of integrable random variables. We shall indicate several conditions under which the following conclusion holds: for any random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields such that $\mathbb{E}\,(X_n|\mathfrak{A}_n) \to Y$ in probability, for $n \to \infty$.

1. Introduction. Let $(\Omega, \mathfrak{F}, P)$ be a non-atomic probability space and (X_n) be a sequence of integrable random variables. The aim of this paper is to find possibly weakest assumptions on (X_n) under which the following conclusion holds:

(α) *for any random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields such that*

$$\lim_{n\to\infty} \mathbb{E}\,(X_n|\mathfrak{A}_n) = Y \text{ in probability.}$$

It is easily seen that condition (α) forces:

$$\lim_{n\to\infty} \mathbb{E}X_n^+ = \lim_{n\to\infty} \mathbb{E}X_n^- = \infty. \tag{0}$$

However, as shown by Ex. 4.1 in [3], (0) is not sufficient for (α). Results presented in this paper generalize the following ones obtained in [2]:

2000 *Mathematics Subject Classification.* 60A10.
Key words and phrases. Conditional expectation, stochastic convergence.

Theorem 1. *Let (X_n) be a sequence of random variables satisfying the following conditions*

$$\lim_{n\to\infty} X_n = 0 \quad \textit{in probability}$$

and

$$\lim_{n\to\infty} \mathbb{E}X_n^+ = \lim_{n\to\infty} \mathbb{E}X_n^- = \infty.$$

Then for any random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields such that

$$\lim_{n\to\infty} \mathbb{E}\left(X_n|\mathfrak{A}_n\right) = Y \quad \textit{in probability.}$$

Theorem 2. *Let (X_n) be a sequence of random variables satisfying the following conditions*

$$\lim_{n\to\infty} X_n = 0 \ \textit{in probability}$$

and

$$\lim_{n\to\infty} \mathbb{E}X_n^+ = \infty.$$

Then for any nonnegative random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ -fields such that

$$\lim_{n\to\infty} \mathbb{E}\left(X_n|\mathfrak{A}_n\right) = Y \quad \textit{in probability.}$$

Similar theorems for almost sure convergence can be found in [3].

2. Main results. The following lemma has been proved in [3]:

Lemma 3. *Let X be an integrable random variable. For any simple random variable Y of the form*

$$Y = \sum_{i=1}^{k} \alpha_i \mathbf{1}_{A_i} + \beta \mathbf{1}_B, \quad \emptyset \subsetneq B \subsetneq \Omega$$

satisfying

$$\sum_{i=1}^{k} |\alpha_i|\, P(A_i) + \max_{i=1,\dots,k} |\alpha_i|\, P(B)$$
$$\leqslant \min\left\{\mathbb{E}X^+\mathbf{1}_B - \mathbb{E}X^-\mathbf{1}_{B^c}, \mathbb{E}X^-\mathbf{1}_B - \mathbb{E}X^+\mathbf{1}_{B^c}\right\}$$

there exists a σ-field $\mathfrak{A}$ such that

$$\mathbb{E}\left(X|\mathfrak{A}\right)(\omega) = Y(\omega) \quad \textit{a.s. for} \quad \omega \in B^c.$$

Now let us prove the following:

Theorem 4. *Let (X_n) be a sequence of integrable random variables such that for some sequence of events (B_n) we have*

$$\lim_{n\to\infty} P\left(B_n^c\right) = 1 \tag{1}$$

and

$$\mathbb{E}X_n^+ \mathbf{1}_{B_n} - \mathbb{E}X_n^- \mathbf{1}_{B_n^c} \to \infty \quad \text{for } n \to \infty, \tag{2}$$

$$\mathbb{E}X_n^- \mathbf{1}_{B_n} - \mathbb{E}X_n^+ \mathbf{1}_{B_n^c} \to \infty \quad \text{for } n \to \infty. \tag{3}$$

Then for any random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields such that

$$\lim_{n\to\infty} \mathbb{E}\left(X_n|\mathfrak{A}_n\right) = Y \quad \text{in probability.}$$

Proof. For sequences (X_n) and (B_n) satisfying (1), (2) and (3) we have

$$\min\left\{\mathbb{E}X_n^+ \mathbf{1}_{B_n} - \mathbb{E}X_n^- \mathbf{1}_{B_n^c}, \mathbb{E}X_n^- \mathbf{1}_{B_n} - \mathbb{E}X_n^+ \mathbf{1}_{B_n^c}\right\} \to \infty \quad \text{for } n \to \infty.$$

Now let (Y_n) be a sequence of simple random variables of the form

$$Y_n = \sum_{i=1}^{k(n)} \alpha_i\left(n\right) \mathbf{1}_{A_i(n)} + \beta_n \mathbf{1}_{B_n}$$

such that

$$\lim_{n\to\infty} Y_n = Y \quad \text{a.s.}$$

and

$$\max_{i=1,\dots,k(n)} \left|\alpha_i\left(n\right)\right| \leqslant \min\left\{\mathbb{E}X_n^+ \mathbf{1}_{B_n} - \mathbb{E}X_n^- \mathbf{1}_{B_n^c}, \mathbb{E}X_n^- \mathbf{1}_{B_n} - \mathbb{E}X_n^+ \mathbf{1}_{B_n^c}\right\}.$$

Lemma 3 implies now the existence of a sequence $(\mathfrak{A}_n)$ of σ-fields such that

$$\mathbb{E}\left(X_n|\mathfrak{A}_n\right)(\omega) = Y_n\left(\omega\right) \quad a.s. \text{ for } \omega \in B_n^c.$$

Since $\lim_{n\to\infty} P\left(B_n^c\right) = 1$, we finally get

$$\lim_{n\to\infty} \mathbb{E}\left(X_n|\mathfrak{A}_n\right) = Y \quad \text{in probability},$$

which ends the proof of the theorem. □

Theorem 5. *Let (p_n) be a sequence of probability distributions for which there exist sequences (a_n) and (b_n) of nonnegative real numbers satisfying*

$$\lim_{n\to\infty} p_n\left((-\infty,\,-b_n)\cup(a_n,\,\infty)\right)=0$$

and

$$\int_{(a_n,\,\infty)} x dp_n(x) + \int_{[-b_n,\,0]} x dp_n(x) \to \infty \quad \text{for } n\to\infty,$$

$$\int_{(-\infty,\,-b_n)} x dp_n(x) + \int_{[0,\,a_n]} x dp_n(x) \to -\infty \quad \text{for } n\to\infty.$$

Then for any sequence (X_n) of integrable random variables such that $p_{X_n} = p_n$ and any random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields satisfying

$$\mathbb{E}\left(X_n|\mathfrak{A}_n\right)\to Y \quad \text{in probability.}$$

Proof. Under the assumptions of the theorem we put

$$B_n = X_n^{-1}\left[(-\infty,\,-b_n)\cup(a_n,\,\infty)\right].$$

Now the conclusion follows from Theorem 4. □

Theorem 6. *Let (X_n) be a sequence of integrable random variables such that the sequence of distributions (p_{X_n}) is tight and*

$$\lim_{n\to\infty}\mathbb{E}X_n^+ = \lim_{n\to\infty}\mathbb{E}X_n^- = \infty. \tag{4}$$

Then for any random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields satisfying

$$\lim_{n\to\infty}\mathbb{E}\left(X_n|\mathfrak{A}_n\right) = Y \quad \text{in probability.}$$

Proof. The fact that the sequence (p_{X_n}) is tight means that for any $\varepsilon>0$ there exists $a>0$ such that

$$P\left(|X_n|<a\right)>1-\varepsilon \quad \text{for } n\geqslant 1.$$

Let (a_j) be a sequence of real numbers such that

$$P\left(|X_n|<a_k\right)>1-2^{-k} \quad \text{for } n\geqslant 1.$$

We put

$$B_{n,\,k}=\left\{|X_n|\geqslant a_k\right\}.$$

From (4) we easily get

$$\mathbb{E}X_n^+\mathbf{1}_{B_{n,\,k}} - \mathbb{E}X_n^-\mathbf{1}_{B^c_{n,\,k}} \to \infty \quad \text{for } n \to \infty$$

and

$$\mathbb{E}X_n^-\mathbf{1}_{B_{n,\,k}} - \mathbb{E}X_n^+\mathbf{1}_{B^c_{n,\,k}} \to \infty \quad \text{for } n \to \infty.$$

Now let (n_k) be an increasing sequence of integers such that

$$\mathbb{E}X_n^+\mathbf{1}_{B_{n,\,k}} - \mathbb{E}X_n^-\mathbf{1}_{B^c_{n,\,k}} \geqslant k \quad \text{for } n \geqslant n_k \tag{5}$$

and

$$\mathbb{E}X_n^-\mathbf{1}_{B_{n,\,k}} - \mathbb{E}X_n^+\mathbf{1}_{B^c_{n,\,k}} \geqslant k \quad \text{for } n \geqslant n_n. \tag{6}$$

Let us put

$$B_n = B_{n,\,k} \quad \text{for } n_k \leqslant n < n_{k+1}.$$

From (5) and (6) it follows that

$$\mathbb{E}X_n^+\mathbf{1}_{B_n} - \mathbb{E}X_n^-\mathbf{1}_{B^c_n} \to \infty \quad \text{for } n \to \infty$$

and

$$\mathbb{E}X_n^-\mathbf{1}_{B_n} - \mathbb{E}X_n^+\mathbf{1}_{B^c_n} \to \infty \quad \text{for } n \to \infty.$$

We also easily observe that

$$\lim_{n\to\infty} P\,(B_n^c) = 1.$$

The conclusion of the theorem is a direct consequence of Theorem 2.7. □

The following lemma has been proved in [3]:

Lemma 7. *Let (p_n) be a sequence of probability distributions on the real line weakly convergent to a probability distribution p satisfying*

$$\int\limits_o^\infty tp\,(dt) = -\int\limits_{-\infty}^0 tp\,(dt) = \infty.$$

Then

$$\lim_{n\to\infty}\int\limits_0^\infty tp_n\,(dt) = -\lim_{n\to\infty}\int\limits_{-\infty}^0 tp_n\,(dt) = \infty.$$

The next proposition provides quite a large class of sequences for which condition (α) holds.

Proposition 8. *Let (X_n) be a sequence of integrable random variables weakly convergent to a random variable X such that*

$$\mathbb{E}X^+ = \mathbb{E}X^- = \infty.$$

Then for any random variable Y there exists a sequence $(\mathfrak{A}_n)$ of σ-fields satisfying

$$\lim_{n\to\infty} \mathbb{E}\,(X_n|\mathfrak{A}_n) = Y \text{ in probability.}$$

Proof. It is well known (see for instance [1]) that if (X_n) is a weakly convergent sequence of random variables then the sequence of probability distributions (p_{X_n}) is tight. The above lemma implies that

$$\lim_{n\to\infty} \mathbb{E}X_n^+ = \lim_{n\to\infty} \mathbb{E}X_n^- = \infty.$$

The conclusion follows now from Theorem 6. □

References

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received November 21, 2001