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On the construction of a stable sequence with given density

Abstract. The notion of a stable sequence of events generalizes the notion of mixing sequence and was introduced by A. Rényi. A sequence of random elements X_n is said to be stable if for every $B \in \mathcal{A}$ with $P(B) > 0$ there exists a probability measure μ_B on $(S, \mathcal{B})$ such that $\lim_{n\to\infty} P([X_n \in A] \mid B) = \mu_B(A)$ for every $A \in \mathcal{A}$ with $\mu_B(\delta A) = 0$. Given a density function, the aim of this note is to give a martingale construction of a stable sequence of random elements having the given density function. The problem was solved in the special case $\Omega =< 0, 1 >$ by the second named author and S.Gutkowska.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. By (S, ρ) we denote a metric space and $\mathcal{B}$ stands for the $\sigma-$field generated by open sets of S.

Let $\mathcal{X}$ be the set of all random elements (r.e.):

$$\mathcal{X} = \{X : \Omega \to S : \ X^{-1}(A) \in \mathcal{A}, A \in \mathcal{B}\}$$

Definition 1. An infinite sequence of events $A_1, A_2, \ldots, A_n, \ldots$ $(A_i \in \mathcal{A}, i \geq 1)$ will be called a *stable sequence* if the limit

$$\lim_{n\to\infty} P(A_n B) = Q(B)$$

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exists for every $B \in \mathcal{A}$.

Thus Q is a bounded measure on $\mathcal{A}$ which is absolutely continuous with respect to the measure P and consequently

$$Q(B) = \int_B \alpha \, dP$$

for every $B \in \mathcal{A}$, where $\alpha = \alpha(\omega)$ is a measurable function on Ω such that $0 \le \alpha(\omega) \le 1$ almost surely (a.s.).

In the case when the local density is constant, the sequence $\{A_n, n \ge 1\}$ will be called a *mixing sequence* of events with density α.

In the special case when $\Omega =< 0, 1 >$ a construction of a stable sequence with given continuous density function α is described, cf. [7]. In this paper we give a construction in a more general situation.

It is well known [6] that any sample space Ω can be represented as

$$\Omega = B \cup \bigcup_{k=1}^{\infty} B_k, B_m \cap B_n = \emptyset \text{ for } m \neq n, B \cap B_n = \emptyset, n = 1, 2, \ldots$$

where each B_k is an atom or an empty set and B has the property that for any given $A \in \mathcal{A}$ such that $A \subset B$ and any ε, $0 < \varepsilon < P(A)$, there exists $C \in \mathcal{A}$, $C \subset A$, such that $P(C) = \varepsilon$. Random elements are constant a.s. on atoms.

Theorem 1. *Assume that $(\Omega, \mathcal{A}, P)$ is an atomless probability space. Then for every measurable real function α ($0 \le \alpha \le 1$ a.s.) there exists a stable sequence of events $\{A_n, n \ge 1\}$ such that*

$$\lim_{n \to \infty} P(A_n B) = \int_B \alpha \, dP = Q(B).$$

Proof. Let $\mathcal{A}' \subset \mathcal{A}$ be the σ-field generated by the sets $\alpha^{-1}(B(x_i, r_j))$, where x_i and r_j are rational numbers ($0 \le x_i \le 1$, $r_j > 0$) and

$$B(x_i, r_j) = \{x : |x - x_i| < r_j\}.$$

We can assume that $\mathcal{A}'$ is generated by $B_1, B_2, \ldots, B_n, \ldots$ with $B_i \in \mathcal{A}$, $i \ge 1$. We denote by $\mathcal{C}_n = \sigma(B_1, B_2, \ldots, B_n)$ the σ-field generated by the set $B_1, B_2, \ldots, B_n$. $\mathcal{C}_n$ is generated by the measurable partition $\{C_n^1, C_n^2, \ldots C_n^{k_n}\}$.

By the martingale convergence theorem, we have

$$\alpha(\omega) = \lim_{n \to \infty} E^{\mathcal{C}_n} \alpha(\omega) \text{a.s.},$$

where $E^{\mathcal{C}_n}$ denotes the conditional expectation with respect to the σ-field $\mathcal{C}_n$.

Since $(\Omega, \mathcal{A}', P)$ is atomless, for every n and $1 \le i \le k_n$ there exists in $\mathcal{A}'$ a set $A_n^i \subset C_n^i$ such that

$$P(A_n^i) = \int_{C_n^i} \alpha(\omega)\, dP.$$

We put $A_n = \bigcup_{i=1}^{k_n} A_n^i$. For $\omega \in C_n^i$, we have

$$E^{\mathcal{C}_n}(I_{A_n})(\omega) = \frac{P(A_n \cap C_n^i)}{P(C_n^i)} = \frac{P(A_n^i)}{P(C_n^i)} = \frac{\int_{C_n^i} \alpha(\omega)\, dP}{P(C_n^i)} = E^{\mathcal{C}_n}\alpha(\omega).$$

If $B \in \mathcal{C}_n$ for some $n \ge 1$ then we have

$$\begin{aligned}\lim_{n\to\infty} E(I_{A_n} I_B) &= \lim_{n\to\infty} E(E^{\mathcal{C}_n}(I_{A_n} I_B)) = \lim_{n\to\infty} EI_B E^{\mathcal{C}_n}(I_{A_n}) \\ &= \lim_{n\to\infty} EI_B E^{\mathcal{C}_n}\alpha = EI_B\alpha.\end{aligned}$$

Let now $\mathcal{K} = \{B \in \mathcal{A}' : \lim_{n\to\infty} E(I_{A_n} I_B) = EI_B\alpha\}$. The set $\mathcal{K}$ contains $\emptyset$ and $\bigcup_{n=1}^{\infty} \mathcal{C}_n \subset \mathcal{K}$. We prove that $\mathcal{K}$ is a σ-field. It is easy to see that if $B \in \mathcal{K}$ then $B^c \in \mathcal{K}$. Let now $B_n \in \mathcal{K}$, $n \ge 1$, be an increasing sequence and $B = \bigcup_{n=1}^{\infty} B_n$. For any $\varepsilon > 0$ there exists n_0 such that $P(B) \le P(B_{n_0}) + \varepsilon$. Then we have $\liminf_{n\to\infty} E(I_{A_n} I_B) \ge \lim_{n\to\infty} E(I_{A_n} I_{B_{n_0}}) = E\alpha I_{B_{n_0}} \ge E\alpha I_B - \varepsilon$ and $\limsup_{n\to\infty} E(I_{A_n} I_{B)} \le \lim_{n\to\infty} E(I_{A_n} I_{B_{n_0}}) + \varepsilon = E\alpha I_{B_{n_0}} + \varepsilon \le E\alpha I_B + \varepsilon$ which implies

$$\lim_{n\to\infty} E(I_{A_n} I_{B)} = E\alpha I_B$$

and this proves that $\mathcal{K}$ is a σ-field and $\mathcal{K}$ contains $\mathcal{A}'$.

Next, we show that equality $\lim_{n\to\infty} E(I_{A_n} I_B) = E(\alpha I_B)$ remains true for each $B \in \mathcal{A}$. If $g : \Omega \to < 0, 1 >$ is some $\mathcal{A}'$-measurable function, we can find for each $\varepsilon > 0$ a step function $f : \Omega \to < 0, 1 >$ which is $\mathcal{A}'$-measurable and such that $|f - g| < \varepsilon$ on a set Ω' with $P(\Omega') > 1 - \varepsilon$. Then as $f = \sum_{s=1}^{m} \lambda_s I_{D_s}$ where $D_s \in \mathcal{A}'$ and $\lambda_s \in \mathcal{R}$ for $s = 1, 2, l \ldots, m$, we have

$$\begin{aligned}\lim_{n\to\infty} \int f I_{A_n}\, dP &= \lim_{n\to\infty} \int (\sum_{s=1}^{m} \lambda_s I_{D_s}) I_{A_n}\, dP \\ &= \lim_{n\to\infty} \sum_{s=1}^{m} \lambda_s \int I_{D_s} I_{A_n}\, dP \\ &= \lim_{n\to\infty} \sum_{s=1}^{m} \lambda_s \int I_{D_s} \alpha\, dP \\ &= \int f\alpha\, dP\end{aligned}$$

Thus

$$\begin{aligned}\liminf_{n\to\infty} E(gI_{A_n}) &\ge \liminf_{n\to\infty} E(gI_{A_n} I_{\Omega'}) \\ &\ge \lim_{n\to\infty} E(fI_{A_n} I_{\Omega'}) - \varepsilon \\ &= E(f\alpha I_{\Omega'}) - \varepsilon \\ &\ge E(g\alpha I_{\Omega'}) - 2\varepsilon \\ &\ge E(g\alpha) - 3\varepsilon\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\to\infty} E(gI_{A_n}) &\le \limsup_{n\to\infty} E(gI_{A_n} I_{\Omega'}) + \varepsilon \\ &\le \lim_{n\to\infty} E(fI_{A_n} I_{\Omega'}) + 2\varepsilon \\ &= E(f\alpha I_{\Omega'}) + 2\varepsilon \\ &\le E(g\alpha I_{\Omega'}) + 3\varepsilon \\ &\le E(g\alpha) + 4\varepsilon.\end{aligned}$$

Since ε is arbitrary, we have

$$\lim_{n\to\infty} E(gI_{A_n}) = E(g\alpha) \tag{1}$$

for each $\mathcal{A}'$-measurable function g such that $0 \le g \le 1$.

Now, let $B \in \mathcal{A}$. We have

$$\lim_{n\to\infty} E(I_{A_n} I_B) = \lim_{n\to\infty} E(E^{\mathcal{A}'}(I_{A_n} I_B)) = \lim_{n\to\infty} E(I_{A_n} E^{\mathcal{A}'} I_B)$$

because $A_n \in \mathcal{A}'$, $n \ge 1$, and by (1) we have

$$\begin{aligned}\lim_{n\to\infty} E(I_{A_n} I_B) &= \lim_{n\to\infty} E(I_{A_n} E^{\mathcal{A}'} I_B) = E(\alpha E^{\mathcal{A}'} I_B) = E(E^{\mathcal{A}'} \alpha I_B) \\ &= E(\alpha I_B),\end{aligned}$$

which completes the proof. $\square$

By this construction we see that if α', α are measurable real functions such that $0 \le \alpha' \le \alpha \le 1$, then there exist stable sequences $\{A'_n, n \ge 1\}$ and $\{A_n, n \ge 1\}$ with density α' and α, respectively, such that $A'_n \subset A_n$, $n \ge 1$. It is obvious that the sequence $\{A_n \setminus A'_n, n \ge 1\}$ is stable with density $\alpha - \alpha'$. If α', α are nonnegative measurable real functions such that $0 \le \alpha' + \alpha \le 1$, then there exist stable sequences $\{A'_n, n \ge 1\}$ and $\{A_n, n \ge 1\}$ with density α' and α respectively, such that $A_n \cap A'_n = \emptyset$, $n \ge 1$.

Definition 2. A sequence $\{X_n, n \ge 1\}$ of r.e. is said to be *stable* if for every $A \in \mathcal{A}_+ = \{A \in \mathcal{A} : P(A) > 0\}$ there exists a probability measure μ_A, defined on $(S, \mathcal{B})$, such that

$$\lim_{n\to\infty} P([X_n \in B] \mid A) = \mu_A(B) \tag{2}$$

for every $B \in \mathcal{C}_{\mu_A} = \{B \in \mathcal{B} : \mu_A(\partial B) = 0\}$ where ∂B denotes the boundary of B.

If $\mu_A(B) = \mu(B)$ for every $A \in \mathcal{A}_+$ and $B \in \mathcal{B}$ then the sequence $\{X_n, n \ge 1\}$ of r.e. is said to be *μ-mixing*.

Let $Q_B(A) = \mu_A(B)P(A)$. Obviously Q_B is an absolutely continuous measure with respect to P. By the Radon-Nikodym Theorem there exists a nonnegative function $\alpha_B : \Omega \to R^+$, such that

$$Q_B(A) = \int_A \alpha_B \, dP.$$

The function α_B is called the *density* of the stable sequence $\{X_n, n \ge 1\}$.

The set $\mathcal{P}_{\mathcal{A}}(S) = \{\mu_A : A \in \mathcal{A}_+\}$ of all probability measures defined by (2) satisfies the following condition:

$$P\Big(\bigcup_{i=1}^{n} A_i\Big)\mu_{\bigcup_{i=1}^{n} A_i}(B) = \sum_{i=1}^{n} \mu_{A_i}(B)P(A_i) \tag{3}$$
$$\text{for every } A_i \in \mathcal{A}_+,\ i = 1, 2, \ldots, n,\ n \ge 1,\ A_i \cap A_j = \emptyset,\ i \ne j.$$

Moreover, it is known [10] that a sequence $\{X_n, n \ge 1\}$ of r.e. converges in probability to a r.e. X iff $\{X_n, n \ge 1\}$ is a stable sequence and $\mathcal{P}_{\mathcal{A}}(S)$ satisfies the following condition:

$$\begin{aligned} &\text{If } \mu_A(B) > 0 \text{ then there exists a set } A' \in \mathcal{A}_+, A' \subset A \\ &\text{such that } \mu_{A'}(B) = 1. \end{aligned} \tag{4}$$

Theorem 2. *Assume that* $(\Omega, \mathcal{A}, P)$ *is an atomless probability space. If the set* $P_{\mathcal{B}}(S) = \{\mu_A : A \in \mathcal{A}_+\}$ *of probability measures on* $(S, \mathcal{B})$ *satisfies Condition (3) then there exists a stable sequence* $\{X_n, n \geq 1\}$ *such that*

$$\lim_{n\to\infty} P([X_n \in B], A) = \mu_A(B)P(A),\ B \in \mathcal{B},\ A \in \mathcal{A}_+ \quad .$$

Remark. It is easy to check that Condition (3) expresses the fact that the set function $\widetilde{\mu}(A \times B) = \mu_A(B)P(A)$ can be extended to a probability measure on the σ–algebra $\mathcal{A} \otimes \mathcal{B}$, whereas Condition (3) means that the measure $\widetilde{\mu}$ is supported by the graph of a r.e.

Proof of Theorem 2. Let $Q_B(A) = \mu_A(B)P(A)$, $B \in \mathcal{B}, A \in \mathcal{A}_+$ and $Q_B(A) = 0$ for $P(A) = 0$. Obviously Q_B is an absolutely continuous measure with respect to P and there exists a measurable function α_B such that

$$Q_B(A) = \int_A \alpha_B \, dP,\ 0 \leq \alpha_B \leq 1 \text{ a.e..}$$

Now, there exists a variant $\lambda(B, \cdot)$ of $\alpha(B, \cdot)$ such that with probability 1 $\lambda(\cdot\ , \omega)$ is a probability measure on $(S, \mathcal{B})$ $(P\{\omega : \lambda(B, \omega) \neq \alpha(B, \omega)\} = 0$ for every $B \in \mathcal{B}$ [9].

Let us choose a sequence of Borel subsets $S_{i_1,i_2,\ldots,i_k} \in \mathcal{C}_{\mu_\Omega}$ satisfying the following conditions [8]:

(a) $S_{i_1,i_2,\ldots,i_k} \cap S_{i'_1,i'_2,\ldots,i'_k} = \emptyset$ if $i_s \neq i'_s$ for some $1 \leq s \leq k$,

(b) $\bigcup_{i_k=1}^{\infty} S_{i_1,i_2,\ldots,i_{k-1},i_k} = S_{i_1,i_2,\ldots,i_{k-1}},\ \bigcup_{i_1=1}^{\infty} S_{i_1} = S$,

(c) $d(S_{i_1,i_2,\ldots,i_k}) < \frac{1}{2^k}$, where $d(B)$ denotes the diameter of the set $B \subset S$.

By Theorem 1, for every $S_{i_1,i_2,\ldots,i_k}$ there exists a stable sequence $\{A^n_{i_1,i_2,\ldots,i_k},\ n \geq 1\}$ with density $\alpha(S_{i_1,i_2,\ldots,i_k}, \cdot)$ such that

(a') $A^n_{i_1,i_2,\ldots,i_k} \cap A^n_{i'_1,i'_2,\ldots,i'_k} = \emptyset$ if $i_s \neq i'_s$ for some $1 \leq s \leq k$
and

(b') $A^n_{i_1,i_2,\ldots,i_{k+1}} \subset A^n_{i_1,i_2,\ldots,i_k}$, $n \geq 1$, $k \geq 1$ and $\bigcup_{i_{k+1}=1}^{\infty} A^n_{i_1,i_2,\ldots,i_k,i_{k+1}} = A^n_{i_1,i_2,\ldots,i_k},\ \bigcup_{i_1=1}^{\infty} A^n_{i_1} = \Omega,\ n \geq 1.$

If $z_{i_1,i_2,\ldots,i_k} \in S_{i_1,i_2,\ldots,i_k}$ we can define

$$X^k_n(\omega) = z_{i_1,i_2,\ldots,i_k} \text{ for } \omega \in A^n_{i_1,i_2,\ldots,i_k},\ n \geq 1.$$

Then for every ω the sequence $\{X^k_n, k \geq 1\}$ satisfies the Cauchy condition and therefore converges to some r.e. X_n.

Moreover, for every k, the sequence $\{X_n^k, n \geq 1\}$ is stable.

Let $A \in \mathcal{A}$ and $\varepsilon > 0$. We can choose $\delta > 0$ such that

$$\int_A \alpha(S^{2\delta}_{i_1,i_2,\dots,i_l}, \cdot)\, dP \leq \int_A \alpha(S_{i_1,i_2,\dots,i_l}, \cdot)\, dP + \varepsilon,$$

where $B^\delta = \{x : \inf_{y \in B} \rho(x,y) < \delta\}$.

Hence, if we set

$$S'(\delta) = \bigcup_{\{i_1,i_2,\dots,i_s : s > \log_2 \frac{1}{\delta},\, S_{i_1,i_2,\dots,i_s} \cap S^\delta_{i_1,i_2,\dots,i_l} \neq \emptyset\}} S_{i_1,i_2,\dots,i_s},$$

we have

$$\begin{aligned} P([X_n \in S_{i_1,i_2,\dots,i_l}] \cap A) &\leq P([X_n^k \in S^\delta_{i_1,i_2,\dots,i_l}] \cap A) \\ &\leq P([X_n^k \in S'(\delta)] \cap A) \\ &\underset{n\to\infty}{\longrightarrow} \int_A \alpha(S'(\delta), \cdot)\, dP \\ &\leq \int_A \alpha(S^{2\delta}_{i_1,i_2,\dots,i_l}, \cdot)\, dP \\ &\leq \int_B \alpha(S_{i_1,i_2,\dots,i_l}, \cdot)\, dP + \varepsilon. \end{aligned}$$

Similarly,

$$\lim_{n\to\infty} P([X_n \in S_{i_1,i_2,\dots,i_l}] \cap A) \geq \int_A \alpha(S_{i_1,i_2,\dots,i_l}, \cdot)\, dP - \varepsilon,$$

which proves that

$$\lim_{n\to\infty} P([X_n \in S_{i_1,i_2,\dots,i_l}] \cap A) = \int_A \alpha(S_{i_1,i_2,\dots,i_l}, \cdot)\, dP\,.$$

This completes the proof, since the sets $S_{i_1,i_2,\dots,i_l}$ form a convergence-determining class. $\square$

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