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Quantum backreaction of massless scalar field on the Schwarzschild black hole

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ABSTRACT

The approximate regularized stress-energy tensor of the quantized conformally coupled scalar field in Hartle-Hawking state in Schwarzschild spacetime is constructed. It is shown that outside the event horizon its maximal deviation from the exact value is less than 0.7%. The linearized semiclassical Einstein field equations are solved and quantum-corrected geometry of the black hole exterior and anisotropic interior is studied.

1. INTRODUCTION

The mean value of the renormalized stress-energy tensor of the quantized fields in a suitable state in the region *outside* the event horizon of the Schwarzschild spacetime is well documented. Indeed, we have detailed knowledge of the $\langle T_{\nu}^{\mu}\rangle$ of the scalar and vector fields in the Hartle-Hawking, Unruh, and Boulware state [1–10]. Less known case of the massless spinor fields has been treated in Ref. [11]. Especially useful in this context are the results of the (semi)analytical approximations which reproduce exact stress-energy tensor with a great accuracy [12–20]. Such models allow construction of the analytic solutions of the semiclassical Einstein field equations.

Treating the Page [2] approximation of the renormalized stress-energy tensor of the conformally coupled massless scalar field in the Hartle-Hawking state as a source term of the semiclassical Einstein field equations, York [21] was able to solve their linearized version and to gain insight into the nature of the quantum-corrected Schwarzschild spacetime. Subsequently the back

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reaction on the metric and its consequences to various physical phenomena have been extensively studied by a number of authors [22–28].

Recently, in a very interesting paper, the first-order semiclassical perturbations of the classical spacetime has been analyzed in rather unusual context — inside the event horizon of the Schwarzschild black hole [29]. In that paper Hiscock, Larson, and Anderson undertook an extensive study of the quantum-corrected black hole interiors for both massive and massless case and for scalar, spinor, and vector fields. They analysed the influence of the semiclassical effects on the anisotropy of the Schwarzschild interior and the behavior of the Kretschmann scalar of the perturbed metric. For quantized scalar field they assumed validity of the Page approximation in the region $r < 2M$. Similar calculations in the interior region of the spacetime of the cylindrical black hole has been performed recently by DeBenedictis [30].

The construction of the Page approximation consists of two steps. First, employing relative simplicity of the optical companion of the Schwarzschild metric and the Gaussian approximation of the propagator one constructs the renormalized stress-energy tensor in the conformal space. Subsequently, thus obtained tensor is transformed back to the physical spacetime by means of the appropriate (conformal) transformation. The meaning of the method is clear: the better approximation in the optical space is constructed, the better physical tensor is finally obtained. As the stress-energy tensor in the optical space coincides with

$$\langle T_{\mu}^{\nu} \rangle = \frac{\pi^2}{90\,(8\pi M)^4}\,\mathrm{diag}\,(-3,1,1,1)_{\mu}^{\nu}\,, \tag{1}$$

i. e. the thermal tensor of the massless particles and since the curvature effects are not taken into account, the Page approximation agrees only qualitatively with the numerical results.

A more sophisticated model has been proposed by Frolov and Zel'nikov on the basis of the geometrical considerations. In the Schwarzschild spacetime it depends on one free parameter, that can be unambiguously determined from the knowledge of the horizon values of the exact stress-energy tensor. In fact, due to spherical symmetry and known value of the anomalous trace, the equality

$$\langle T_{\mu}^{\nu}\,(2M) \rangle^{FZ} = \langle T_{\mu}^{\nu}\,(2M) \rangle^{\mathrm{exact}}, \tag{2}$$

yields only one independent condition. With this additional piece of information the Frolov-Zel'nikov model improves the quality of the approximation, and, when applied in the Schwarzschild spacetime it could be expressed in terms of sixth-order polynomials in $x = 2M/r$.

In this paper, using Christensen-Fulling asymptotic conditions, we shall construct the (semi)analytical approximation of the stress-energy tensor of the quantized massless and conformally invariant scalar field in the Schwarzschild spacetime that reproduces numerical calculations outside the event horizon with a great accuracy. We shall incorporate the curvature effects assuming that the tensor satisfies the weak thermal hypothesis within $N = 8$ ansatz in the optical space [15], and, as a result we obtain the most general form of $\langle T_{\mu}^{\nu} \rangle$. Constructed tensor will be employed to study the back reaction effects.

The aforementioned problems, of course, do require some numeric. For example, one can use numerically evaluated stress energy tensor and numerically solve the semiclassical Einstein field equations. On the other hand however, one can construct the (semi)analytical approximation of $\langle T_{\mu}^{\nu} \rangle$ which is covariantly conserved, has a proper trace and asymptotics, and, subsequently, solve analytically the back reaction equations. As the second approach is, till the very end of calculations, purely analytical, and since we do not loose the tensorial character of the equations, we shall employ this very method.

Although the improved $\langle T_{\mu}^{\nu} \rangle$ reproduces the numerical calculations of Anderson, Hiscock, and Samuel [9, 10] and those of Howard [5] with a great accuracy outside the event horizon, one should be very careful in in the attempts to extend this results for $r < 2M$. As there are no numerical analyses inside the Schwarzschild black hole (except the calculations of the field fluctuation, $\langle \varphi^2 \rangle$, carried out some times ago by Candelas and Jensen [31]) one could not *a priori* decide if the obtained approximation is satisfactory there. Moreover, since both Page and improved approximations are divergent as $r \to 0$, the results of the first-order perturbative analyses are limited to the region $r_m \leq r < 2M$, where the exact value of r_m is dictated by the analyses of the applicability of the semiclassical Einstein field equations on the one hand and reasonableness of the approximation on the other.

In our calculations that partially extend results of Ref. [29], we shall assume that the first order calculations are legitimate in the region $(r_m, 2M)$, (the discussion of admissible values of r_m will be given later) and analyse the back reaction problem employing the improved stress-energy tensors of the massless and conformally invariant scalar field. The issues concerning the motivations and the philosophy of the back reaction calculations are well known and will not be repeated here.

In our previous paper on the back reaction of the quantized massless scalar field on the geometry of the exterior region of the Schwarzschild black hole we accepted the strong thermal hypothesis. This paper is its natural continuation and extension.

2. APPROXIMATE RENORMALIZED STRESS-ENERGY TENSOR IN THE SCHWARZSCHILD SPACETIME

The anomalous trace of the stress-energy tensor of the massless and conformally invariant quantized field is given by

$$\langle \overline{T}^{a}_{a} \rangle = aH + bG + c\Box R, \tag{3}$$

where H is a square of the Weyl tensor

$$H = C^{\mu\nu\sigma\tau}C_{\mu\nu\sigma\tau} = R^{\mu\nu\sigma\tau}R_{\mu\nu\sigma\tau} - R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^{2}, \tag{4}$$

and

$$G = {}^{*}R^{\mu\nu\sigma\tau}\,{}^{*}R_{\mu\nu\sigma\tau} = R^{\mu\nu\sigma\tau}R_{\mu\nu\sigma\tau} - 4R^{\mu\nu}R_{\mu\nu} + R^{2}. \tag{5}$$

The numerical coefficients depend on the helicity of the field and when computed with the aid of the ζ-function regularization, in the particular case of the scalar field are: $a = 2/(3840\pi^2)$, $b = -1/3a$,and $c = 2/3a$. Accepting polynomial $N = 8$ ansatz for the angular component $\langle \overline{T}^{\theta}_{\theta} \rangle$ in the optical companion of the Schwarzschild metric

$$ds^2 = -dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} + \frac{r^2}{1 - \frac{2M}{r}}\left(dr^2 + \sin^2\theta\, d\phi^2\right), \tag{6}$$

solving the covariant conservation equations

$$\nabla_b \langle \overline{T}^{b}_{a} \rangle = 0, \tag{7}$$

for the radial component, and finally making use of the Christensen-Fulling asymptotic conditions [32], one can reduce the number of the unknown parameters from 10 to 3. Indeed, after much algebra, one obtains the most general stress-energy tensor in the physical spacetime in the form

$$\begin{aligned}\langle T^{t}_{t} \rangle \;=\; & -3p\left[1 + 2\,x + 3\,x^2 + 4\,x^3 + \frac{x^4\,(438 + 6\,\Theta + 26\,a_4 - a_5)}{102}\right.\\ & + \frac{x^5\,(444 + 14\,\Theta + 72\,a_4 + 43\,a_5)}{102}\\ & \left. + \frac{x^6\,(-1767 + 7\,\Theta - 49\,a_4 - 21\,a_5)}{51}\right],\end{aligned} \tag{8}$$

$$\langle T_r^r \rangle = p \left[1 + 2\,x + 3\,x^2 + \frac{2\,x^3\,(174 - 6\,\Theta + 8\,a_4 + a_5)}{51} + \frac{x^4\,(870 - 30\,\Theta - 62\,a_4 + 5\,a_5)}{102} - \frac{x^5\,(-324 + 10\,\Theta + 32\,a_4 + 21\,a_5)}{34} - \frac{3\,x^6\,(-97 + \Theta - 7\,a_4 - 3\,a_5)}{17} \right], \tag{9}$$

and

$$\langle T_\theta^\theta \rangle = p \left[1 + 2\,x + 3\,x^2 + \frac{x^3\,(132 + 6\,\Theta - 8\,a_4 - a_5)}{51} + \frac{x^4\,(111 + 12\,\Theta + 35\,a_4 - 2\,a_5)}{51} + \frac{2\,x^5\,(15 + 3\,\Theta + 13\,a_4 + 8\,a_5)}{17} + \frac{x^6\,(-213 + 5\,\Theta - 35\,a_4 - 15\,a_5)}{17} \right], \tag{10}$$

where $p = \pi^2/90\,(8\pi M)^4$ and $p\,\Theta$ is the horizon value of the tangential pressure. The remaining parameters are to be determined from additional data.

It could be easily shown that the choice

$$a_4 = a_5 = 0, \qquad \Theta = 12 \tag{11}$$

results in the Page approximation, however, it should be noted that near the event horizon of the Schwarzschild black hole the stress-energy tensor evaluated with (11) substantially differs from its numerical estimates, and, therefore, it is not a good candidate for the source term of the semiclassical Einstein field equations. For example the deviation of $\langle T_\theta^\theta \rangle^{\rm Page}$ at $r = 2.5M$ is approximately 48% and about 17% on the event horizon.

A more sophisticated approximation has been invented by Frolov and Zel'nikov [6]. Assuming that the stress-energy tensor could be constructed from the curvature, the Killing vector, and their covariant derivatives, they proposed $\langle T_\mu^\nu \rangle$, that for the quantized massless field in the Schwarzschild geometry coincides with (8–10) for

$$a_4 = \frac{12 - \Theta}{10} \text{ and } a_5 = 0. \tag{12}$$

To specify the Frolov-Zel'nikov approximation further it is necessary to know $\langle T_\theta^\theta\,(2M) \rangle$.

Making use of the analyses of the behaviour of the Green functions in Hartle-Hawking state in the Schwarzschild spacetime, and the analyses carried out in Ref. [1], one can easily estimate value of the stress-energy tensor at the bifurcation point of the event horizon, and for the (rescaled) tangential pressure, one has

$$\Theta = 10.29. \tag{13}$$

The Frolov-Zel'nikov approximation is, of course, exact at the horizon and at infinity and with Θ given by (13), deviation of its angular component from the exact value is below 37%.

Since the extensive numerical studies of the vacuum polarization effects in the Schwarzschild spacetime have been carried out by Howard [5], and Anderson, Hiscock and Samuel [10], one can determine free parameters of (8–10) using a best fit argument. Indeed, accepting Candelas' value of the tangential pressure at the event horizon to be exact and performing linear least-square fit to the set of numerical data which consists of 63 points, one obtains

$$a_4 = 10.398\,, \qquad a_5 = -74.370\,. \tag{14}$$

The agreement of this simple approximation with the numerical results of Anderson, Hiscock and Samuel is amazing: the maximal deviation of the energy density is below 0.7% whereas the maximal deviation of the tangential and radial pressure is below 0.1% and 0.2% respectively. It seems that outside the event horizon the more complicated models are of little use, simply because the higher terms practically do not contribute to the stress energy tensor at great distances, and, moreover, the numerical calculations are believed to be accurate only to three significant digits. Therefore having established a good fit near the event horizon and in the intermediate region there is no use of more complicated models. However, this does not necessarily means that the approximation remains satisfactory also in the region inside the event horizon.

In Figures 1–3 the radial dependence of the components of the Page approximation and the stress-energy tensor (8–10) with (14) are exhibited, and, moreover, Figure 3 is supplemented with the analogous graph for the Frolov-Zel'nikov approximation. For comparison the numerical results of Anderson are also presented there.

The situation inside the event horizon of the Schwarzschild black hole changes drastically: now the leading terms of the stress–tensor as one approaches the singularity are, of course, the highest order terms in $2M/r$. Since the models (11), (12) with (13), and (14) differ only by a particular choice of parameters, their predictions could be compared even inside the event horizon. This is one of the reasons why we have accepted the weak thermal hypothesis and $N = 8$.

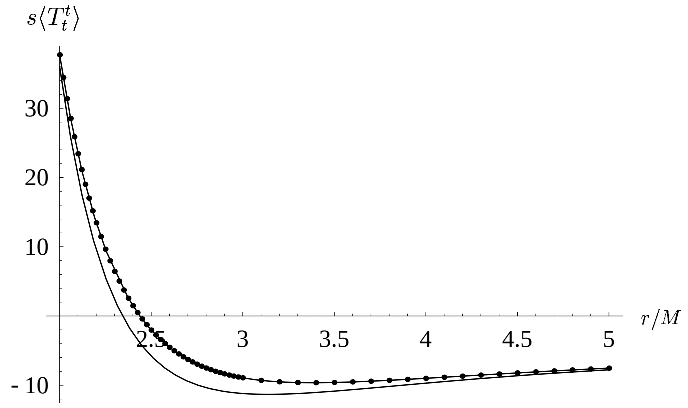

Fig. 1. This graph shows the radial dependence of the rescaled component $\langle T_t^t\rangle$ [$s = 90\pi^2(8M)^4$] as a function of r/M. Top to bottom the functions are for the improved and Page approximation respectively. Dots represent numerical values

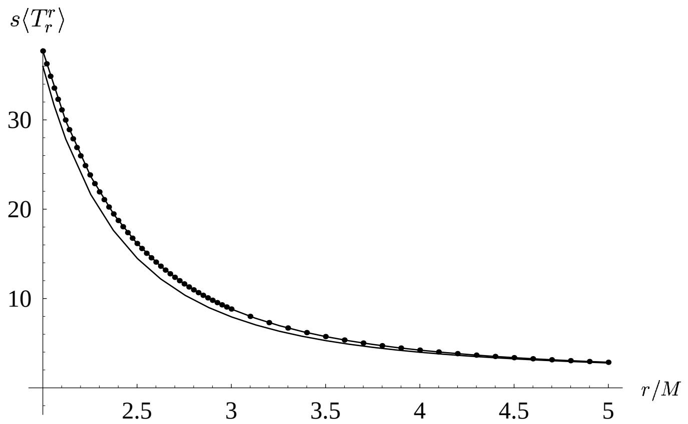

Fig. 2. This graph shows the radial dependence of the rescaled component $\langle T_r^r\rangle$ [$s = 90\pi^2(8M)^4$] as a function of r/M. Top to bottom the functions are for the improved and Page approximation respectively. Dots represent numerical values

3. QUANTUM CORRECTED EXTERIOR OF THE SCHWARZSCHILD BLACK HOLE

The quantum corrected, spherically-symmetric line element is generally of the form

$$ds^2 = -A(r)dt^2 + B^{-1}(r)dr^2 + r^2 d\Omega^2, \tag{15}$$

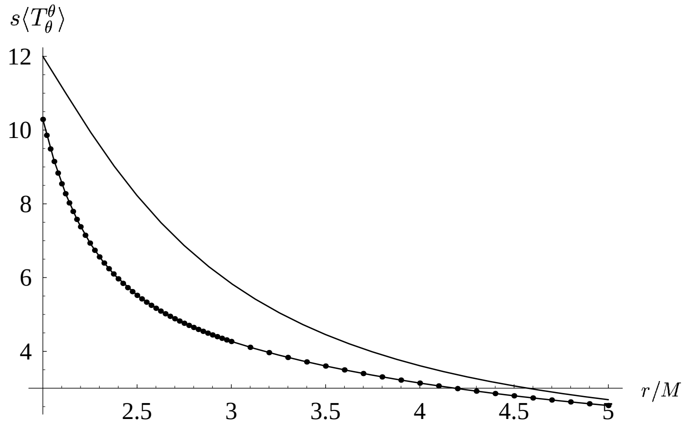

Fig. 3. This graph shows the radial dependence of the rescaled component $\langle T_\theta^\theta\rangle$ [$s = 90\pi^2(8M)^4$] as a function of r/M. Top to bottom the functions are for the Page and improved approximation respectively. Dots represent numerical values

where

$$A(r) = B(r)e^{2\psi(r)} \tag{16}$$

and

$$B(r) = 1 - 2m(r)/r. \tag{17}$$

The semi-classical Einstein field field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle, \tag{18}$$

may be therefore solved perturbatively to first order in $\varepsilon = (M_P/M)^2$, M_P is the Planck mass. Indeed, assuming that to $\mathcal{O}(\varepsilon)$ the functions e^ψ and $m(r)$ have the expansions

$$e^\psi = 1 + \varepsilon\rho(r) \tag{19}$$

and

$$m(r) = M(1 + \varepsilon\mu(r)), \tag{20}$$

one obtains two independent linear equations governing $\mu(r)$ and $\rho(r)$.

$$\frac{\varepsilon M}{4\pi r^2}\frac{d\mu}{dr} = -\langle T_t^t\rangle, \tag{21}$$

and

$$\frac{\varepsilon}{4\pi r}\frac{d\rho}{dr} = \left(1-\frac{2M}{r}\right)^{-1}\left(\langle T_r^r\rangle - \langle T_t^t\rangle\right). \tag{22}$$

Simple quadratures give

$$\begin{aligned} K\mu &= \frac{1}{3}x^{-3} + x^{-2} + 3\,x^{-1} - 4\,\ln x - 5\,x - \frac{13}{51}\,x\,a_4 + \frac{1}{102}\,x\,a_5 + \frac{1}{17}\,x\,s \\ &\quad - 3\,x^2 - \frac{6}{17}\,x^2 a_4 - \frac{43}{204}\,x^2 a_5 + \frac{7}{102}\,x^2 s + 11\,x^3 + \frac{49}{153}\,x^3 a_4 \\ &\quad + \frac{7}{51}\,x^3 a_5 + \frac{7}{153}\,s\,x^3 - \frac{22}{3} - \frac{53}{306}\,s + \frac{44}{153}\,a_4 + \frac{13}{204}\,a_5 + k_0 \\ &= K\mu_0 + k_0, \end{aligned} \tag{23}$$

$$\begin{aligned} K\rho &= 2\,x^{-1} + \frac{1}{3}\,x^{-2} - \frac{82}{17}\,x^3 - \frac{2}{51}\,x^3 a_5 + \frac{2}{153}\,x^3\,(12-s) - \frac{14}{153}\,x^3 a_4 \\ &\quad + \frac{1}{34}\,x^2\,(12-s) - \frac{2}{51}\,x^2 a_4 - \frac{91}{17}\,x^2 - \frac{1}{204}\,x^2 a_5 + \frac{2}{51}\,x\,(12-s) \\ &\quad - \frac{8}{153}\,x\,a_4 - \frac{364}{51}\,x - \frac{1}{153}\,x\,a_5 + 14 + \frac{28}{153}\,a_4 + \frac{31}{612}\,a_5 \\ &\quad + \frac{25}{306}\,s - 4\,\ln x + C_0 \\ &= K\rho_0 + C_0, \end{aligned} \tag{24}$$

where $K = 3840\pi$, and, k_0 and C_0 are integration constants. The equation (20) suggests that k_0 may be included into the mass, renormalizing the black hole bare mass. Since the source term of the semiclassical Einstein equations asymptotically approaches (1), it is necessary to enclose the black hole in a massless spherical box of a definite radius, say, R. In doing so, one ignores the boundary terms which are expected to be small if the cavity radius is bigger than the length of the least damped quasinormal mode. Moreover, the semiclassical corrections of the quantized field to the Schwarzschild geometry outside the spherical wall could be ignored too. Imposing microcanonical boundary conditions (fixed total energy of the system), and assuming the continuity of the line element, one has

$$k_0 = -K\rho\,(R)\,. \tag{25}$$

As expected k_0 is a function of the radius R.

In Figure 4 the radial dependence of the effective mass is presented for Page, Frolov and Zel'nikov, and our approximation. Inspection of the figure indicates that near the event horizon the quantum processes tend to decrease the effective mass, that could be ascribed to the existence of the pocket of negative energy there, and, moreover, both the Page and Frolov-Zel'nikov approximations overestimate the mass function.

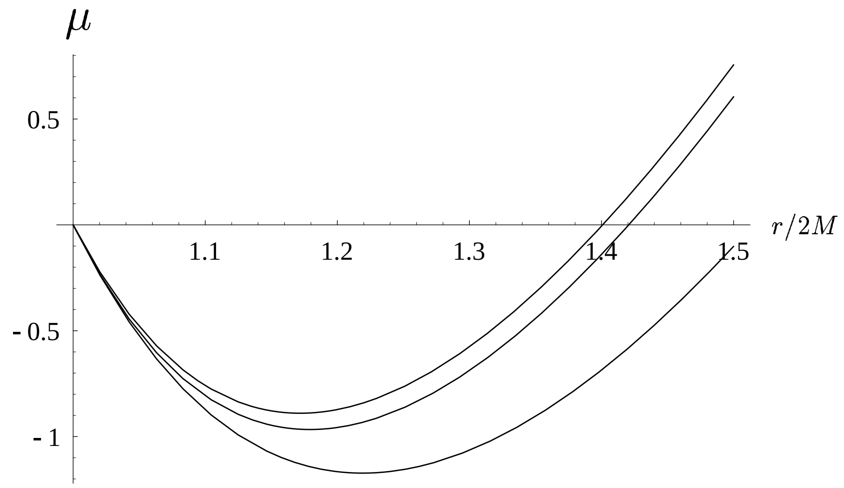

Fig. 4. The effective mass μ as a function of $r/2M$. Top to bottom the curves are for the Page, Frolov-Zel'nikov, and the improved approximation

Assuming that the changes of the metric caused by the quantum effects remain small for $r < R$, i. e.

$$\varepsilon \left| h_{\mu}^{\nu} \right| = \varepsilon_0 < 1, \tag{26}$$

where ε_0 is a dimensionless parameter, the maximal radius of the spherical wall may be easily determined. The fractional corrections to the metric, h_{μ}^{ν}, may be obtained from its definition:

$$g_{\mu\nu} = g_{\mu\sigma}^{S} \left(\delta_{\nu}^{\sigma} + \varepsilon h_{\nu}^{\sigma} \right), \tag{27}$$

where $g_{\mu\nu}^{S}$ is the Schwarzschild unperturbed metric.

Behaviour of geodesics in the quantum corrected spacetime is very interesting and surely deserves a separate study. Here we shall confine ourselves to the case of circular orbits and discuss the modifications of the Kepler law caused by the quantized field assuming somewhat idealized situation. In the inner region, $2M < r < R$, one can easily solve the geodesic equations for

$$\Omega = \frac{d\phi}{dt}. \tag{28}$$

Indeed, computing connection coefficients for the metric (15) and retaining $\mathcal{O}(\varepsilon)$ terms, after simple manipulations, one has

$$\Omega^2 = \frac{M}{r^3} + \frac{\varepsilon}{r^3}\left(r^2\frac{d}{dr}\tilde{\rho} + 2\,\tilde{\rho}M - 2\,Mr\frac{d}{dr}\tilde{\rho} + M\mu_0 - Mr\frac{d}{dr}\mu_0\right), \qquad (29)$$

where by (25) $\tilde{\rho} = \rho(r) - \rho(R)$. Outside the wall the metric is exactly Schwarzschildian, and, therefore, the Kepler law bears simple form:

$$\Omega^2 = \frac{M_{\text{total}}}{r^3}, \qquad (30)$$

where $M_{\text{total}} = M\,(1 + \varepsilon\mu_0\,(R))$.

The equilibrium temperature of static and self gravitating system, $T_{\text{loc}}(r)$, is given by the Tolman formula

$$T_{\text{loc}}(r) \;=\; T|g_{tt}|^{-1/2}. \qquad (31)$$

One expects that for the corrected Schwarzschild black hole, the temperature T is no longer of the form $T = 1/(8\pi M)$. The $O(\varepsilon)$ black hole temperature may be calculated either from the general formula

$$T \;=\; \frac{\kappa}{2\pi}, \qquad (32)$$

where κ is the black hole surface gravity or from the examination of the complexified Schwarzschild metric. Indeed, regularity of the line element as $r \to 2M$ requires τ to be periodic with a period β_H given by

$$\beta_H \;=\; \frac{1}{T} \;=\; 4\pi \lim_{r\to 2M}\,(g_{00}\,g_{11})^{1/2}\left(\frac{dg_{11}(r)}{dr}\right)^{-1}, \qquad (33)$$

and the Euclideanized version of the line element allows to identify the inverse of the local temperature, for fixed r, θ, and ϕ, with the periodically identified proper length of the τ

$$\frac{1}{T_{\text{loc}}(r)} \;=\; \int_0^{\beta_H} \sqrt{g_{00}(r)}d\tau. \qquad (34)$$

The main features of the entropy of the quantized field, S_{rad}, have been extensively investigated in numerous papers; a general expression, the proof of positivity, and monotonic increase with r is firmly established for conformally invariant massless scalar (Page approximation), vector (Jensen and

Ottewill approximation), and to certain extent spinor field. Unfortunately, there are no numerical calculations of the $\langle T_{\nu}^{\mu}\rangle$ of the quantized conformal spinor and therefore validity of the Brown-Ottewill-Page approximation in this context is unknown. Detailed discussion and the method for constructing entropy has been presented in Ref. [5]. Recently, we have extended the analyses of the entropy of the massless, conformally invariant scalar field beyond the Page approximation employing the regularized stress-energy tensor constructed with the aid of the strong thermal hypothesis.

Simple and elegant expression describing S has been recently derived by Zaslavskii [33]

$$S_{\rm rad} = 32\pi^2 M \int_{2M}^{R} dr' r'^2 \left[\langle T_r^r\rangle - \langle T_t^t\rangle - \langle T_{\mu}^{\mu}\rangle \ln\left(\frac{R}{r'}\right)\right], \tag{35}$$

where the arbitrary integration constant has been fixed by the demand of vanishing of ΔS at the event horizon of the black hole (no room for quantum radiation). When applied to regular stress energy tensor it exhibits some general features: the radial derivative of the entropy vanishes at the event horizon, and S is a positive function of r monotonically increasing with radius. It should be noted that this features, as have been observed by Hochberg, Kephart, and York, do not hold if one ignores the back reaction. Indeed, substituting in (35) appropriate components of the stress tensor (8–10), after simple integration, one has

$$S_{\rm rad} = S_{\rm Page} + \Delta S, \tag{36}$$

$$S_{\rm Page} = -\frac{1}{540} + \frac{1}{1080\, w^3} + \frac{1}{360\, w^2} + \frac{1}{120\, w} - \frac{w}{72} - \frac{w^2}{120} + \frac{13\, w^3}{1080} + \frac{\ln w}{90}, \tag{37}$$

$$\begin{aligned}\Delta S \;=\; & \frac{s}{73440} + \frac{a_4}{7344} - w^2\left(\frac{s}{24480} + \frac{a_4}{2448} + \frac{11\, a_5}{48960}\right) \\ & + w\left(\frac{s}{12240} - \frac{a_4}{9180} - \frac{a_5}{73440}\right) - w^3\left(\frac{s}{18360} - \frac{7\, a_4}{18360} - \frac{a_5}{6120}\right) \\ & + \frac{11\, a_5}{146880} + \frac{s\ln s}{6120} - \frac{a_4 \ln s}{4590} - \frac{a_5\, \log s}{36720},\end{aligned} \tag{38}$$

where $w = 2M/R$ and $s = \Theta - 12$. At the event horizon

$$\frac{dS_{rad}}{dr} = 0, \tag{39}$$

and

$$\frac{d^2 S_{\rm rad}}{dr^2} = \frac{1}{45\,M^2} - \frac{7\,s}{48960\,M^2} + \frac{31\,a_4}{73440\,M^2} + \frac{41\,a_5}{293760\,M^2}, \qquad (40)$$

and therefore $S_{\rm rad}$ has a local minimum there. The total entropy

$$S = 4\pi M^2 + S_{\rm rad}, \qquad (41)$$

where the first term is the usual Bekenstein-Hawking expression for the black hole entropy. The total entropy (41) may be easily obtained also from the thermodynamic considerations.

4. GEOMETRY OF THE SCHWARZSCHILD BLACK HOLE INTERIOR

The region of the Schwarzschild solution inside the event horizon has been analysed by Novikov forty years ago [35]. Since the role of the time coordinate is now playing by the radial coordinate and vice versa, it is useful to rewrite the line element describing the interior of the Schwarzschild black hole has in the form

$$ds^2 \,=\, -\left(\frac{2M}{T} - 1\right)^{-1} dT^2 \,+\, \left(\frac{2M}{T} - 1\right) dx^2 \,+\, T^2 d\Omega, \qquad (42)$$

where new coordinates x and T are defined as

$$x \,=\, t, \qquad T \,=\, r. \qquad (43)$$

The manifold may be thought of as a homogeneous, anisotropic cosmology of the Kantowski-Sachs type, that near the singularity takes the form of the Kasner solution.

In (x,T) coordinates, the quantum-corrected metric describing the region inside the event horizon may be written in the form

$$\begin{aligned} ds^2 \;&=\; -\left(\frac{2M}{T} - 1\right)^{-1} [1 + \varepsilon\eta(T)]\, dT^2 \\ &+\; \left(\frac{2M}{T} - 1\right) [1 + \varepsilon\sigma(T)]\, dx^2 + T^2 d^2 d\Omega^2. \end{aligned} \qquad (44)$$

Now, the semiclassical field equations (18) may be easily solved perturbatively to first order in ε. Indeed, substituting (44) into the semiclassical

Einstein field equations and retaining the $\mathcal{O}(\varepsilon)$ terms one obtains two independent equations governing $\eta(T)$ and $\sigma(T)$:

$$\frac{d}{dT}\left[(2M-T)\,\eta\right] = \frac{8\pi T^2\langle T_x^x\rangle}{\varepsilon}, \tag{45}$$

and

$$\frac{d\sigma}{dT} = -\frac{8\pi T^2\langle T_T^T\rangle}{\varepsilon\,(2M-T)} - \frac{\eta}{2M-T}. \tag{46}$$

Now, we shall assume that the renormalized stress-energy tensor of the quantized massless scalar field inside the black hole event horizon is given by (8-10) with (43). With this tensor the equations (45) and (46) may be easily solved. Elementary quadratures give

$$\eta = \eta_0 + \frac{C_0}{2M-T} \tag{47}$$

and

$$\sigma = \sigma_0 - \frac{C_0}{2M-T} + k_{0,} \tag{48}$$

where

$$\begin{aligned} K\eta_0 \quad = \quad & \frac{13}{3} + \frac{1}{3\,s^2} + \frac{4}{3\,s} + \left(\frac{53}{306}\Theta - \frac{259}{51} + \frac{44}{153}a_4 + \frac{13}{204}a_5\right)s \\ & + \left(\frac{35}{306} - \frac{478}{51}\Theta + \frac{5}{135}a_4 + \frac{5}{63}a_5\right)s^2 \\ & + \left(\frac{7}{153}\Theta - \frac{589}{51} - \frac{49}{153}a_4 - \frac{7}{51}a_5\right)s^3 - \frac{4\,s}{1-s}\ln s, \end{aligned} \tag{49}$$

and

$$\begin{aligned} K\sigma_0 \quad = \quad & \frac{491}{51} + \frac{26}{153}\Theta + \frac{56}{153}a_4 + \frac{31}{306}a_5 + \frac{1}{3\,s^2} + \frac{8}{3\,s} + \\ & - \left(\frac{469}{51} + \frac{29}{306}\Theta + \frac{20}{51}a_4 + \frac{47}{612}a_5\right)s \\ & - \left(\frac{4}{3} + \frac{1}{18}\Theta + \frac{1}{9}a_4 + \frac{1}{12}a_5\right)s^2 + \left(\frac{97}{51} - \frac{1}{51}\Theta + \frac{74}{51}a_4 + \frac{1}{17}a_5\right)s^3 \\ & - 8\,\ln s + \frac{4\,s}{1-s}\ln s. \end{aligned} \tag{50}$$

Here $s = 2M/T$, C_0 and k_0 are the integration constants. Inspection of the Eqs. shows that in $\mathcal{O}(\varepsilon)$ calculations, the constant C_0 may be absorbed into the mass term

$$\tilde{M} = M - \frac{\varepsilon C_0}{2}, \tag{51}$$

thus renormalizing the black hole (bare) mass. As in the exterior case, the black hole's bare mass has no independent meaning in $\mathcal{O}(\varepsilon)$ calculations. In the latter we shall omit the tilde and assume that the black hole mass has been renormalized.

Now, the line element (44) bears the form

$$ds^2 = -\left(\frac{2M}{T}-1\right)^{-1}[1+\varepsilon\eta_0(T)]\,dT^2 + \left(\frac{2M}{T}-1\right)[1+\varepsilon\sigma_0(T)+\varepsilon k_0]\,dx^2 + T^2 d^2 d\Omega^2. \quad (52)$$

Although we are left with one undetermined constant most of the quantities which are of interest to us in this paper are independent of k_0. With $a_4 = a_5 = 0$, and $\Theta = 12$ (49) and (50) reduce to the results obtained in Ref. [29]. However, there is a disagreement with the first term in the right hand side of Eq.(49): we have obtained 13/3 whereas the value cited in the Ref. [29] is 49/3.

Inside the black hole event horizon the natural condition for applicability of the back reaction calculations is, of course, given by (26). Since h_T^T does not depend explicitly on the unknown integration constant one can easily determine the "minimal" value of the time coordinate for which this very condition holds. Taking for example $\varepsilon_0 = 1/10$ and $\varepsilon = 1/10$ ($M = \sqrt{10}M_{PL}$) one has $0.151M$ whereas for $\varepsilon = 1/100$ ($M = 10M_{PL}$) one obtains $0.068M$. In the case of the very massive black holes, the minimal radius of applicability of the linearized back reaction calculations is indeed very small. Results of our numerical calculations are exhibited in Figure 5. The Page stress-energy tensor produces more prominent quantum corrections to the black hole interior, and consequently the minimal radius of applicability of $\mathcal{O}(\varepsilon)$ calculations is everywhere greater as compared to the results obtained within the framework of the improved approximation.

Note that assuming $\varepsilon\,\varepsilon_0 << \varepsilon$ in $\mathcal{O}(\varepsilon)$ calculations, the spatial coordinate x may be rescaled as

$$x \to x' = (1+\varepsilon k_0)^{\frac{1}{2}}x, \quad (53)$$

and hence the spatial fractional correction to $g_{x'x'}$ is given by

$$h_{x'}^{x'} = \sigma_0(T). \quad (54)$$

Repeating calculations with ε and ε_0 considered previously gives respectively $0.091M$ and $0.043M$.

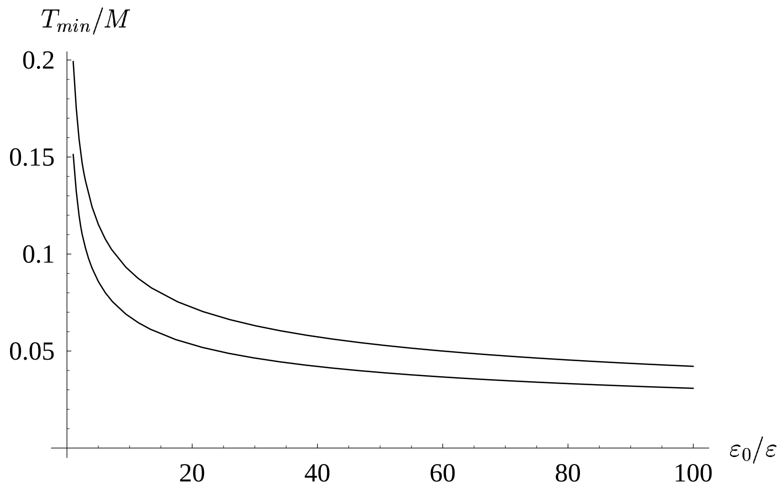

Fig. 5. The minimal "radius" of applicability of the back reaction calculations as a function of $\varepsilon_0/\varepsilon$

As has been shown by Novikov the interior of the Schwarzschild black hole is a highly anisotropic spacetime and near the singularity it takes the form of the Kasner solution. In the exterior region the frame of reference [34] connected with the Schwarzschild coordinates does not deform. This also holds in the case of the quantum corrected spherically symmetric and static exterior of the black hole. On the other hand, however, the interior of the Schwarzschild black hole is not static, and consequently the frame deforms with time [35]. The rate of the deformations of the frame of reference is described by the (spatial) symmetric tensor D_{ij}, which, for the metric (52), has the following nonvanishing components

$$\begin{aligned} D_{xx} &= -M\left(\frac{2M-T}{T^5}\right)^{1/2}\{1+\varepsilon\left[\sigma_0(T)+k_0\right. \\ &\left.\left. -\left(T-\frac{T^2}{M}\right)\frac{d\sigma_0(T)}{dT}-\eta_0(T)\right]\right\} \end{aligned} \tag{55}$$

$$D_{\theta\theta} = \left[T\left(2M-T\right)\right]^{1/2}\left[1-\frac{1}{2}\varepsilon\eta_0(T)\right] \tag{56}$$

and

$$D_{\phi\phi} = \sin^2\theta\, D_{\theta\theta}. \tag{57}$$

Putting $\varepsilon = 0$ in D_{ij} one reproduces the results originally obtained by Novikov [35] for the Schwarzschild interior.

Now, following Ref. [29] we shall investigate the influence of the quantized scalar field on the measure of the anisotropy, which in the case at hand is defined as the ratio of the Hubble expansion rates

$$\alpha = \frac{H_x}{H_\theta} = \frac{1}{g_{xx}}\frac{dg_{xx}}{dT}\left(\frac{1}{g_{\theta\theta}}\frac{dg_{\theta\theta}}{dT}\right)^{-1} \tag{58}$$

Inserting the metric (52) into (58) and retaining $\mathcal{O}(\varepsilon)$ terms one obtains

$$\alpha = \alpha_s + \varepsilon\delta\alpha, \tag{59}$$

where a_s is the anisotropy of the classical Schwarzschild interior and $\delta\alpha$ is its first order quantum perturbation. Since α_s in the interior region of the Schwarzschild solution is everywhere negative, the quantum corrections, $\delta\alpha$, will dampen the anisotropy if its perturbation is positive. From the Figure 6 one can draw a general conclusion that the semiclassical effects caused by semianalytical stress-energy tensor of the massless scalar and conformally coupled field always *isotropises* the black hole interior. This behaviour is in the sharp contrast with the analogous result obtained within the framework of the Page approximation, where $\delta\alpha$ is negative for $T < 0.927M$ and positive near the event horizon. Similarly, the Frolov-Zelnikov approximation gives $\delta\alpha$ which is positive for $T > 0.903M$. One may therefore ask a natural question: which model better describes physical reality inside Schwarzschild black hole event horizon. Although we are unable to give a definite answer, we briefly critically review the main features of the considered models. First, it should be noted that accepting simple form of the stress-energy tensor in the optical spacetime conformally related to the Schwarzschild geometry as given by (1), one ignores curvature effects or assumes that they enter as *(curvature)*^{2}. It may be a safe procedure in the exterior region far from the event horizon, but it is obviously not in the black hole interior. Moreover, one expects that the curvature effects becomes more important as T approaches initial singularity. The “improved” stress-energy tensor reproduces the exact tensor with a great accuracy outside the event horizon, and since it has in the optical metric a general structure

$$\sim 1 + \sum_{i=3}^{8} c_i x^i, \tag{60}$$

presumably it incorporates to certain extend curvature effects in an effective way. However, the main problem now is not the leading terms as $r \to \infty$, but rather the behaviour of the stress-tensor as $T \to 0$. Estimates of Candelas and Jensen are of little help here because they calculated the renormalized field fluctuation not the stress-energy tensor. Moreover, they showed that the Page approximation goes progressively worse as one approaches the singularity. Accepting, on the other hand, the strong thermal hypothesis, i. e. assuming that the curvature effects enter as (*curvature)*^{2}, the simplest approximation gives $\delta\alpha$ positive for $T > 1.005M$. To this end, we remark here that the process of dissipation of the anisotropy by quantum effects is well known and well established in the Bianchi I type spacetimes.

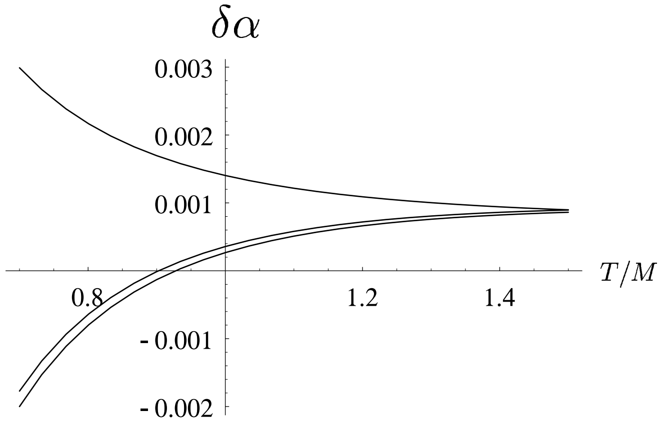

Fig. 6. This graph shows $\delta\alpha$ as a function of T/M. Top to bottom the functions are for the improved and Page approximation respectively. Dots represent numerical values

Finally, we shall analyse the quantum corrections to the trace anomaly and, as a by-product, the behavior of the Kretschmann scalar. Corrections to the trace anomaly caused by the back reaction are given by a purely geometric terms constructed from the $\mathcal{O}(\varepsilon)$ quantum-corrected metric. It could be easily shown that for the metric (52) both $R^{\mu\nu}R_{\mu\nu}$ and R^2 are $\mathcal{O}(\varepsilon^2)$, and in the first order calculations $H = G$, and therefore

$$\langle T^{\mu}_{\mu}\rangle = \frac{2}{3}a\left(Riem^2 + \Box R\right). \tag{61}$$

Remaining $\mathcal{O}(\varepsilon)-$ terms in the trace anomaly are given by

$$\begin{aligned} R^{\mu\nu\sigma\tau}R_{\mu\nu\sigma\tau} &= C^{\mu\nu\sigma\tau}C_{\mu\nu\sigma\tau} = \frac{48M^2}{T^6} \\ &+\varepsilon\left[\frac{\eta}{T^5}\left(16M - \frac{96M^2}{T^2}\right) + \frac{1}{T^4}\frac{d\sigma}{dT}\left(\frac{24M^2}{T} - 8M\right)\right. \end{aligned}$$

$$+ \frac{d\sigma}{dT}\frac{M}{T^4}\left(8 + \frac{40M}{T}\right) - \frac{d^2\sigma}{dT^2}\frac{M}{T^3}\left(8 + 16\frac{M}{T}\right)\Bigg] \tag{62}$$

and

$$\begin{aligned}\Box R &= \varepsilon\left[\frac{\eta}{T^4}\left(4 - \frac{16M}{T}\right) - \frac{d\eta}{dT}\frac{1}{T^3}\left(4 - \frac{2M}{T} - \frac{24M^2}{T^2}\right)\right.\\ &+ \frac{d\sigma}{dT}\frac{1}{T^4}\left(6M - \frac{8M^2}{T}\right) + \frac{d^2\eta}{dT^2}\frac{1}{T^2}\left(2 - \frac{7M}{T} - \frac{18M^2}{T^2}\right)\\ &- \frac{d^2\sigma}{dT^2}\frac{1}{T^3}\left(6M + \frac{2M}{T}\right) + \frac{d^3\eta}{dT^3}\frac{1}{T}\left(2 - \frac{7M}{T} + \frac{6M^2}{T^2}\right)\\ &- \left.\frac{d^3\sigma}{dT^3}\frac{1}{T}\left(4 - \frac{7M}{T} - \frac{2M^2}{T^2}\right) + \frac{d^4\sigma}{dT^4}\left(\frac{4M}{T} - \frac{4M^2}{T^2} - 1\right)\right]\end{aligned} \tag{63}$$

Making use of (49) and (50) one obtains

$$\begin{aligned}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48\,M^2}{T^6} + \frac{\varepsilon}{\pi}\left[\frac{1346\,M^5}{255\,T^9} - \frac{47\,M^4}{170\,T^8} - \frac{64\,M^3}{255\,T^7} - \frac{131\,M^2}{510\,T^6}\right.\\ + \frac{M}{40\,T^5} + \frac{1}{480\,T^4} - \frac{22\,M^5\,\Theta}{765\,T^9} - \frac{19\,M^4\,\Theta}{1020\,T^8} - \frac{7\,M^3\,\Theta}{1020\,T^7} + \frac{41\,M^2\,\Theta}{12240\,T^6}\\ + \frac{154\,M^5\,a_4}{765\,T^9} - \frac{22\,M^4\,a_4}{255\,T^8} - \frac{37\,M^3\,a_4}{1530\,T^7} + \frac{13\,M^2\,a_4}{1530\,T^6} + \frac{22\,M^5\,a_5}{255\,T^9}\\ - \left.\frac{107\,M^4\,a_5}{2040\,T^8} + \frac{7\,M^3\,a_5}{6120\,T^7} + \frac{43\,M^2\,a_5}{24480\,T^6} - \frac{M^2}{10\,T^6}\ln\left(\frac{2M}{T}\right)\right]\end{aligned} \tag{64}$$

and

$$\Box R = \frac{\varepsilon}{\pi}\left(\frac{48M^5}{5T^9} - \frac{4M^4}{T^8}\right). \tag{65}$$

Note that $\Box R$ is equal to its equivalent evaluated within the framework of the Page approximation.

The anomalous trace is rather complicated function of T, and therefore instead of considering $\langle T^{\mu}_{\mu}\rangle$ itself, we shall analyse the fractional trace anomaly, Δ, defined as

$$\Delta = \frac{\langle T^{\mu}_{\mu}\rangle - \langle \tilde{T}^{\mu}_{\mu}\rangle}{\langle \tilde{T}^{\mu}_{\mu}\rangle}, \tag{66}$$

where in $\langle \tilde{T}^{\mu}_{\mu}\rangle = M^2/(60\pi^2 T^6)$ one recognizes the pure conformal anomaly of the scalar field in the Schwarzschild spacetime. Evaluated at the event horizon, the fractional correction is independent of a_4 and a_5, and is given by a simple expression

$$\Delta = \frac{\varepsilon}{\pi}\frac{56 - \Theta}{5760}. \tag{67}$$

With $\Theta = 12$, it reduces to Δ obtained earlier by York [21]. From (67) one sees that the corrections to the Page approximation tend to increase Δ.

5. CONCLUDING REMARKS

In this paper we have constructed the renormalized stress-energy tensor of the massless, conformally invariant scalar field in the Schwarzschild spacetime that reproduces the exact numerical results with a great accuracy. We employed it in the various back reaction calculations, that helped us to understand nature of the quantum corrected geometry in the outer and inner region of the black hole. As the stress-energy tensor in the Hartle-Hawking state approaches for large r that of gas of massless particles at fixed temperature it is necessary to include the system consisting of the black hole and quantized radiation in a cavity. The radius of the cavity has been determined the perturbation of the metric. Inside the event horizon the quantum corrections caused by the improved stress-energy tensor tend to dissipate the anisotropy of the spacetime. However, near the singularity the linearized semiclassical Einstein field equations are meaningless, and, unfortunately, we are unable to address the important question of influence of the quantized field upon the singularity.

Apart from obvious generalizations of our results to other phenomena in the effective spacetime, it is of great interest to extend the analyses to massless quantized scalar field with the general curvature coupling and to massless fields of higher helicities. Finally, let us mention an interesting and important direction for future work. It is the problem of the construction of the stress-energy tensor of the quantized *massive* field in the curved spacetime beyond the first-order renormalized effective action. Such calculations would naturally extend the results of Refs. [38] [39]. The calculations of this type are expected to be extremely complex as they require knowledge of the functional derivative of the effective action constructed form the fifth Hadamard-DeWitt coefficient. We intend to return to this group of problems elsewhere.

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