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Note on random partitions of the segment

Abstract. Let (X_n) be a sequence of independent random variables uniformly distributed on the interval $[0,1]$. R_n stands for the diameter of the partition of $[0,1]$ by the random points $X_1, X_2, \ldots, X_{n-1}$. It was shown by R. Jajte that the sequence $(nR_n/\log n)$ converges to 1 in probability. We prove the convergence in p-th mean, $p > 0$, of the sequence $(nR_n/\log n)$ to 1. We are also interested in the rate of convergence in probability of this sequence. Almost sure convergence of $(nR_n/\log n)$ to 1 is also obtained.

1. Introduction. Let (X_n) be a sequence of independent random variables uniformly distributed on the interval $[0,1]$ and let R_n stand for the diameter of the partition of $[0,1]$ by the random points $X_1, X_2, \ldots, X_{n-1}$. The distribution of R_n is presented in [3]. It is easily seen that $\lim_{n\to\infty} R_n = 0$ with probability 1, but it gives no information about the asymptotic behaviour of the sequence (nR_n). It is shown in [5] by the Laplace transform technique that the sequence $(nR_n/\log n)$ converges in probability to 1.

We prove that the sequence $(nR_n/\log n)$ converges to 1 in mean of order p, $p > 0$. Hence we estimate the rate of convergence in probability of this sequence. Moreover, we show that the sequence $(nR_n/\log n)$ converges almost surely to 1.

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2. Preliminaries. We start with some moment properties of the diameter R_n. It is known (cf. [5]) that the r-th moment of R_n is equal to

$$ER_n^r = \frac{r!}{n^{(r)}}\,\gamma_n^r, \tag{1}$$

where

$$\gamma_n^r = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k^{-r}$$

and $x^{(r)}$ (the notation from [7]) denotes the rising factorial, i.e.

$$x^{(r)} = x(x+1)\dots(x+r-1).$$

In [5] it was also shown that the quantity γ_n^r can be written as

$$\begin{aligned} \gamma_n^1 &= \sum_{i=1}^{n} \frac{1}{i}, \\ \gamma_n^r &= \sum_{1 \le i_1 \le \ldots \le i_r \le n} \frac{1}{i_1 \cdot \ldots \cdot i_r}, \quad r = 2, 3, \ldots . \end{aligned} \tag{2}$$

The numbers γ_n^r in (2) are inconvenient for evaluations, so we represent them in a different form.

Define $a_r \equiv a_r(\alpha_1, \dots, \alpha_n)$, $r = 1, 2, \dots, n$, the *elementary symmetric function* of weight r, and $h_r \equiv h_r(\alpha_1, \dots, \alpha_n)$, $r = 1, 2, \dots$, the so-called *homogeneous product sum symmetric function* of weight r (cf. [6], pp. 47, 93) by the equations

$$\begin{aligned} &1/(1-\alpha_1 x)(1-\alpha_2 x)(1-\alpha_3 x)\dots(1-\alpha_n x) \\ &\qquad = 1/(1 - a_1 x + a_2 x^2 + \ldots + (-1)^n a_n x^n) \\ &\qquad = 1 + h_1 x + h_2 x^2 + \ldots + h_r x^r + \ldots . \end{aligned}$$

For instance

$$\begin{aligned} a_1(\alpha_1, \dots, \alpha_n) &= \alpha_1 + \alpha_2 + \ldots + \alpha_n \\ a_2(\alpha_1, \dots, \alpha_n) &= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots + \alpha_{n-1}\alpha_n \end{aligned}$$

and

$$\begin{aligned} h_1(\alpha_1, \dots, \alpha_n) &= \alpha_1 + \alpha_2 + \ldots + \alpha_n \\ h_2(\alpha_1, \dots, \alpha_n) &= \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2 + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots + \alpha_{n-1}\alpha_n). \end{aligned}$$

The generating function of the sequence $\{\gamma_n^r,\, r \geq 0\}$ in (2) has the form

$$G_n(z) = \sum_{r=0}^{\infty} \gamma_n^r z^r = \frac{1}{(1-z)\left(1-\frac{z}{2}\right)\ldots\left(1-\frac{z}{n}\right)} \qquad \text{(cf. [4]).}$$

Thus, we see that

$$\gamma_n^r = h_r\left(1, \frac{1}{2}, \ldots, \frac{1}{n}\right). \tag{3}$$

It is known that the *homogeneous product sum symmetric function* $h_r(\alpha_1, \ldots, \alpha_n)$ satisfies

$$r!h_r(\alpha_1, \ldots, \alpha_n) = C_r(s_1, \ldots, s_r), \quad \text{(cf. [6], p. 119),} \tag{4}$$

where s_i denotes the so-called *power sum symmetric function* given by

$$s_i = \sum_{j=1}^{n} \alpha_j^i, \tag{5}$$

and C_r is the so-called *cycle indicator* of the symmetric group defined by

$$C_r(s_1, \ldots, s_r) = \sum_{a_1+2a_2+\ldots+ra_r=r} (r; a_1, \ldots, a_r)^* s_1^{a_1} \ldots s_r^{a_r}, \tag{6}$$

(cf. [6], p. 68), with the notation from [1]

$$(r; a_1, \ldots, a_r)^* = \frac{r!}{1^{a_1} a_1! 2^{a_2} a_2! \ldots r^{a_r} a_r!}\ . \tag{7}$$

The sum in (6) is over all non-negative integer values of a_i, $1 \leq i \leq r$, such that $a_1 + 2a_2 + \ldots + ra_r = r$, or equivalently, over all partitions of n. For instance

$$\begin{aligned} C_1(s_1) &= s_1 \\ C_2(s_1, s_2) &= s_1^2 + s_2 \\ C_3(s_1, s_2, s_3) &= s_1^3 + 3s_1s_2 + 2s_3 \\ C_4(s_1, s_2, s_3, s_4) &= s_1^4 + 6s_1^2s_2 + 3s_2^2 + 8s_1s_3 + 6s_4 \end{aligned}$$

(cf. [6], the table on p. 69).

Letting in (5)

$$\alpha_j = \frac{1}{j}, \quad 1 \leq j \leq n,$$

we write s_r, $r \geq 1$, as the harmonic number of order r

$$H_n^{(r)} = \sum_{i=1}^{n} \frac{1}{i^r}, \quad r \geq 1, \quad \text{(cf. [4])}. \tag{8}$$

We are interested in positive integer values of r in (8). If $r = 1$ then

$$\log n < H_n^{(1)} \leq \log n + 1, \quad n \geq 1, \tag{9}$$

and for $r \geq 2$ we use the notation of the Riemann's ζ-function

$$\zeta(r) = H_\infty^{(r)} = \sum_{i=1}^{\infty} \frac{1}{i^r}.$$

Combining (3), (4), (5) and (6) we deduce that the quantity γ_n^r can be written as

$$\gamma_n^r = \frac{1}{r!} \sum_{a_1+2a_2+\ldots+ra_r=r} (r; a_1, \ldots, a_r)^* \left(H_n^{(1)}\right)^{a_1} \ldots \left(H_n^{(r)}\right)^{a_r}. \tag{10}$$

The following recurrence relation for γ_n^r permits us to derive the recurrence formula for the moments of R_n.

Lemma 1. *The numbers $\{\gamma_n^r,\, r \geq 0\}$ satisfy the recurrence equation*

$$\gamma_n^{r+1} = \frac{1}{r+1} \sum_{j=0}^{r} H_n^{(j+1)} \gamma_n^{r-j}, \quad r = 0, 1, 2 \ldots \tag{11}$$

and $\gamma_n^0 = 1$.

Proof. Knowing that the generating function of the sequence $\{\gamma_n^r,\, r \geq 0\}$ is

$$G_n(z) = \sum_{r=0}^{\infty} \gamma_n^r z^r = \frac{1}{(1-z)\left(1-\frac{z}{2}\right)\ldots\left(1-\frac{z}{n}\right)},$$

we have

$$G_n'(z) = \sum_{r=0}^{\infty} (r+1)\gamma_n^{r+1} z^r.$$

On the other hand,

$$\begin{aligned} \frac{G_n'(z)}{G_n(z)} = \frac{d}{dz} \log G_n(z) &= \sum_{i=1}^{n} \frac{1}{i} \frac{1}{1-\frac{z}{i}} \\ &= \sum_{i=1}^{n} \frac{1}{i} \sum_{j=0}^{\infty} \left(\frac{z}{i}\right)^j \\ &= \sum_{j=0}^{\infty} H_n^{(j+1)} z^j. \end{aligned}$$

Therefore

$$G_n'(z) = G_n(z)\sum_{j=0}^{\infty} H_n^{(j+1)} z^j \ ,$$

or

$$\sum_{r=0}^{\infty}(r+1)\gamma_n^{r+1} z^r = \sum_{r=0}^{\infty}\sum_{j=0}^{r} H_n^{(j+1)}\gamma_n^{r-j} z^r.$$

From this equality we conclude that (11) holds. □

Now putting (10) into (1) we get

$$(12)\quad ER_n^r = \frac{1}{n^{(r)}} \sum_{a_1+2a_2+\ldots+ra_r=r} (r; a_1, \ldots, a_r)^* \left(H_n^{(1)}\right)^{a_1} \ldots \left(H_n^{(r)}\right)^{a_r}.$$

The recurrence relation for ER_n^r is given by

Proposition 1. *The moments ER_n^r satisfy the following recurrence relation*

$$(13)\quad ER_n^{r+1} = \sum_{j=0}^{r} \frac{r!}{(r-j)!}\frac{1}{(n+r-j)^{(j+1)}} H_n^{(j+1)} ER_n^{r-j}, \quad r = 1, 2, \ldots,$$

and

$$ER_n = \frac{1}{n} H_n^{(1)}.$$

Proof. From (1) and (11) we have

$$\begin{aligned} ER_n^{r+1} &= \frac{(r+1)!\gamma_n^{r+1}}{n^{(r+1)}} \\ &= \frac{r!}{n^{(r+1)}} \sum_{j=0}^{r} H_n^{(j+1)}\gamma_n^{r-j} \\ &= \sum_{j=0}^{r} \frac{r!}{(r-j)!}\frac{1}{(n+r-j)^{(j+1)}} H_n^{(j+1)} ER_n^{r-j} \end{aligned}$$

which gives (13). □

3. L_p–convergence. We see that by (1)

$$E\left(\frac{nR_n}{\log n}\right) = \frac{H_n^{(1)}}{\log n}.$$

Taking the limit as $n \to \infty$ and using (9) we get

$$\lim_{n\to\infty} E\left(\frac{nR_n}{\log n}\right) = \lim_{n\to\infty}\frac{H_n^{(1)}}{\log n} = 1.$$

Now, taking into account that $E\left(\frac{nR_n}{\log n}\right) \to 1$ as $n \to \infty$, it is sufficient to estimate $E\left(\frac{nR_n}{\log n} - \frac{nER_n}{\log n}\right)^{2k}$.

Proposition 2. *For $k \in \mathbb{N}$ and sufficiently large n*

$$E(R_n - ER_n)^{2k} \le \frac{C(k)}{n^{2k}}, \tag{14}$$

where

$$C(k) = \sum_{p=0}^{2k} \sum_{2a_2+\ldots+pa_p=p} \frac{(2k)!}{2^{a_2}a_2!\ldots p^{a_p}a_p!} \zeta^{a_2}(2)\ldots\zeta^{a_p}(p). \tag{15}$$

Proof. By the binomial formula

$$E\left(R_n - ER_n\right)^{2k} = \sum_{r=0}^{2k} \binom{2k}{r} (-1)^{2k-r} ER_n^r (ER_n)^{2k-r}.$$

Hence by (12)

$$E\left(R_n - ER_n\right)^{2k} = \sum_{r=0}^{2k} \binom{2k}{r} (-1)^{2k-r} \frac{1}{n^{(r)}}$$
$$\times \sum_{a_1+2a_2+\ldots+ra_r=r} (r; a_1, \ldots, a_r)^* \left(H_n^{(1)}\right)^{a_1} \ldots \left(H_n^{(r)}\right)^{a_r} \frac{1}{n^{2k-r}} \left(H_n^{(1)}\right)^{2k-r}.$$

Now, taking the sum with respect to a_1 we get

$$E\left(R_n - ER_n\right)^{2k} = \frac{1}{n^{2k}} \sum_{r=0}^{2k} \binom{2k}{r} (-1)^{2k-r} \frac{n^r}{n^{(r)}} \left(H_n^{(1)}\right)^{2k-r}$$
$$\times \sum_{p=0}^{r} \sum_{p+2a_2+\ldots+ra_r=r} (r; p, \ldots, a_r)^* \left(H_n^{(1)}\right)^{p} \ldots \left(H_n^{(r)}\right)^{a_r}.$$

Using the identity

$$\sum_{r=0}^{2k} \sum_{p=0}^{r} a(r,p) = \sum_{p=0}^{2k} \sum_{r=p}^{2k} a(r, r-p)$$

we obtain

$$E(R_n - ER_n)^{2k} = \frac{1}{n^{2k}} \sum_{p=0}^{2k} \sum_{r=p}^{2k} \binom{2k}{r} (-1)^{2k-r} \frac{n^r}{n^{(r)}} \left(H_n^{(1)}\right)^{2k-r}$$
$$\times \sum_{r-p+2a_2+\ldots+ra_r=r} (r; r-p, \ldots, a_r)^* \left(H_n^{(1)}\right)^{r-p} \ldots \left(H_n^{(r)}\right)^{a_r}$$
$$= \frac{1}{n^{2k}} \sum_{p=0}^{2k} \sum_{r=p}^{2k} \binom{2k}{r} (-1)^{2k-r} \frac{n^r}{n^{(r)}} \left(H_n^{(1)}\right)^{2k-r}$$
$$\times \sum_{2a_2+\ldots+ra_r=p} (r; r-p, \ldots, a_r)^* \left(H_n^{(1)}\right)^{r-p} \ldots \left(H_n^{(r)}\right)^{a_r}.$$

The sum

$$\sum_{2a_2+\ldots+ra_r=p} (r; r-p, \ldots, a_r)^* \left(H_n^{(1)}\right)^{r-p} \ldots \left(H_n^{(r)}\right)^{a_r}$$

can be written as

$$\sum_{2a_2+\ldots+pa_p=p} (r; r-p, \ldots, a_p)^* \left(H_n^{(1)}\right)^{r-p} \ldots \left(H_n^{(p)}\right)^{a_p}$$

as $a_{p+1} = \ldots = a_r = 0$ and by (7)

$$\begin{aligned}(r; r-p, \ldots, a_r)^* &= (r; r-p, \ldots, a_p, \underbrace{0, \ldots, 0}_{r-p})^* \\ &= \frac{r!}{(r-p)! 2^{a_2} a_2! \ldots p^{a_p} a_p!} \\ &= (r; r-p, \ldots, a_p)^*.\end{aligned}$$

Therefore

$$\begin{aligned}E(R_n - ER_n)^{2k} &= \frac{1}{n^{2k}} \sum_{p=0}^{2k} \sum_{r=p}^{2k} \binom{2k}{r} (-1)^{2k-r} \frac{n^r}{n^{(r)}} \left(H_n^{(1)}\right)^{2k-p} \\ &\times \sum_{2a_2+\ldots+pa_p=p} (r; r-p, \ldots, a_p)^* \left(H_n^{(2)}\right)^{a_2} \ldots \left(H_n^{(p)}\right)^{a_p} \\ &= \frac{1}{n^{2k}} \sum_{p=0}^{2k-1} \left(H_n^{(1)}\right)^{2k-p} \sum_{2a_2+\ldots+pa_p=p} \frac{1}{2^{a_2} a_2! \ldots p^{a_p} a_p!} \left(H_n^{(2)}\right)^{a_2} \ldots \left(H_n^{(p)}\right)^{a_p} \\ &\times \sum_{r=p}^{2k} \binom{2k}{r} (-1)^{2k-r} \frac{r!}{(r-p)!} \frac{n^r}{n^{(r)}} \\ &+ \frac{1}{n^{(2k)}} \sum_{2a_2+\ldots+2ka_{2k}=2k} \frac{(2k)!}{2^{a_2} a_2! \ldots (2k)^{a_{2k}} a_{2k}!} \left(H_n^{(2)}\right)^{a_2} \ldots \left(H_n^{(2k)}\right)^{a_{2k}} \\ &:= A(n) + B(n),\end{aligned}$$

say, where

$$\begin{aligned}A(n) = \frac{1}{n^{2k}} \sum_{p=0}^{2k-1} \left(H_n^{(1)}\right)^{2k-p} &\times \sum_{2a_2+\ldots+pa_p=p} \frac{1}{2^{a_2} a_2! \ldots p^{a_p} a_p!} \left(H_n^{(2)}\right)^{a_2} \ldots \left(H_n^{(p)}\right)^{a_p} \\ &\times \sum_{r=p}^{2k} \binom{2k}{r} (-1)^{2k-r} \frac{r!}{(r-p)!} \frac{n^r}{n^{(r)}}\end{aligned}$$

and

$$B(n) = \frac{1}{n^{(2k)}} \sum_{2a_2+\ldots+2ka_{2k}=2k} \frac{(2k)!}{2^{a_2} a_2! \ldots (2k)^{a_{2k}} a_{2k}!} \left(H_n^{(2)}\right)^{a_2} \ldots \left(H_n^{(2k)}\right)^{a_{2k}}.$$

Taking into account that

$$\sum_{r=p}^{2k} \binom{2k}{r} (-1)^{2k-r} \frac{r!}{(r-p)!} \frac{n^r}{n^{(r)}}$$
$$= \frac{(2k)!}{(2k-p)!} \frac{n^p}{n^{(2k)}} \sum_{r=0}^{2k-p} \binom{2k-p}{r} (-1)^{2k-p-r} n^r (n+r+p)^{(2k-r-p)},$$

we see that

$$A(n) = \frac{1}{n^{2k}} \sum_{p=0}^{2k-1} \frac{n^p}{n^{(2k)}} \left(H_n^{(1)}\right)^{2k-p} \frac{a(n)}{(2k-p)!}$$
$$\times \sum_{2a_2+\ldots+pa_p=p} \frac{(2k)!}{2^{a_2} a_2! \ldots p^{a_p} a_p!} \left(H_n^{(2)}\right)^{a_2} \ldots \left(H_n^{(p)}\right)^{a_p},$$

where

$$a(n) := \sum_{r=0}^{2k-p} \binom{2k-p}{r} (-1)^{2k-p-r} n^r (n+r+p)^{(2k-r-p)}.$$

But the order of the quantity $a(n)$ is less than or equal to n^{2k-p-1} since the coefficient of n^{2k-p} in $a(n)$ is equal to

$$\sum_{r=0}^{2k-p} \binom{2k-p}{r} (-1)^{2k-p-r} = 0.$$

Thus $|a(n)| \leq c(p) n^{2k-p-1}$, where $c(p)$ is a positive constant independent of n. Hence

$$n^{2k}|A(n)| \leq \sum_{p=0}^{2k-1} \frac{1}{n} \left(H_n^{(1)}\right)^{2k-p} \frac{c(p)}{(2k-p)!}$$
$$\times \sum_{2a_2+\ldots+pa_p=p} \frac{(2k)!}{2^{a_2} a_2! \ldots p^{a_p} a_p!} \zeta(2)^{a_2} \ldots \zeta(p)^{a_p}$$
$$\leq \sum_{p=0}^{2k-1} \frac{(\log n + 1)^{2k-p}}{n} \frac{c(p)}{(2k-p)!}$$
$$\times \sum_{2a_2+\ldots+pa_p=p} \frac{(2k)!}{2^{a_2} a_2! \ldots p^{a_p} a_p!} \zeta(2)^{a_2} \ldots \zeta(p)^{a_p},$$

as $H_n^{(1)}$ satisfies (9).

Then we get

$$\lim_{n\to\infty} n^{2k} A(n) = 0,$$

so for n sufficiently large

$$n^{2k}|A(n)| \le \sum_{p=0}^{2k-1} \sum_{2a_2+\ldots+pa_p=p} \frac{(2k)!}{2^{a_2}a_2!\ldots p^{a_p}a_p!}\,\zeta(2)^{a_2}\ldots\zeta(p)^{a_p}. \tag{16}$$

Moreover, we conclude that

$$\begin{aligned}&\lim_{n\to\infty} n^{2k} B(n)\\ &\quad= \sum_{2a_2+\ldots+2ka_{2k}=2k} \frac{(2k)!}{2^{a_2}a_2!\ldots(2k)^{a_{2k}}a_{2k}!}\,\zeta^{a_2}(2)\ldots\zeta^{a_{2k}}(2k).\end{aligned} \tag{17}$$

Therefore by (16) and (17) we obtain (14). □

Remark 1. The properties of the moments of R_n allow us to give estimates in the cases $k=1$ and $k=2$ valid for all $n\in\mathbb{N}$. Namely, we have

$$\sigma^2 R_n \le \frac{\pi^2}{6n^2} \tag{18}$$

and

$$E(R_n - ER_n)^4 \le \frac{3}{n^4}\left(\frac{16}{e^2}+\frac{\pi^4}{20}\right), \tag{19}$$

respectively, $e = 2,71\ldots$.

Proof. For the variance of R_n we have

$$\sigma^2 R_n = ER_n^2 - (ER_n)^2.$$

Using the recurrence relation for ER_n^r and (12) we get

$$\sigma^2 R_n = \frac{1}{n(n+1)}H_n^{(2)} - \frac{1}{n^2(n+1)}\left(H_n^{(1)}\right)^2 \le \frac{\zeta(2)}{n^2} = \frac{\pi^2}{6n^2}.$$

To prove the second inequality we also use the recurrence relation for ER_n^r and formula (12). By the binomial formula it follows

$$E(R_n - ER_n)^4 = \sum_{r=0}^{4}\binom{4}{r}(-1)^{4-r}ER_n^r(ER_n)^{4-r}.$$

Using the formula for the r-th moment of $R_n$we get

$$E(R_n - ER_n)^4 \le \frac{3n}{n^4(n+1)(n+2)(n+3)}\left(H_n^{(1)}\right)^4 + \frac{3}{n(n+1)(n+2)(n+3)}\left(H_n^{(2)}\right)^2 + \frac{6}{n(n+1)(n+2)(n+3)}H_n^{(4)}.$$

Hence by (9)

$$n^4E(R_n - ER_n)^4 \le 3\left(\frac{(\log n+1)^4}{n^2} + \zeta^2(2) + 2\zeta(4)\right).$$

The function $f(x) = \frac{(\log x+1)^4}{x^2}$, $x > 1$, attains the maximum value $\frac{16}{e^2}$ for $x = e$. Moreover, note that $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$, which immediately yields the desired result. □

The following theorem is an easy consequence of Proposition 2.

Theorem 1. *For* $p > 0$

$$\frac{nR_n}{\log n} \xrightarrow{L_p} 1, \quad n \to \infty.$$

By Markov's inequality and Proposition 2 we get the rate of convergence in probability of the sequence $(nR_n/\log n)$ to 1 stated in [5].

Theorem 2. *Let* $k \in \mathbb{N}$. *Then for any given* $\varepsilon > 0$

$$P\left[\left|\frac{nR_n}{\log n} - 1\right| \ge \varepsilon\right] \le \frac{C(k)}{\varepsilon^{2k}\log^{2k} n}, \tag{20}$$

for sufficiently large n, *where* $C(k)$ *is given by* (15).

Proof. From Markov's inequality it follows that

$$P\left[\left|\frac{nR_n}{\log n} - \frac{nER_n}{\log n}\right| \ge \varepsilon\right] \le \frac{n^{2k}}{\varepsilon^{2k}\log^{2k} n}\, E\left(R_n - ER_n\right)^{2k}. \tag{21}$$

Hence by (21) and (14) we immediately get (20). □

Remark 2. Using Remark 1 we have

$$P\left[\left|\frac{nR_n}{\log n} - \frac{nER_n}{\log n}\right| \ge \varepsilon\right] \le \frac{\pi^2}{6\varepsilon^2\log^2 n}$$

and

$$P\left[\left|\frac{nR_n}{\log n}-\frac{nER_n}{\log n}\right|\geq\varepsilon\right]\leq\frac{3}{\varepsilon^4\log^4 n}\left(\frac{\pi^4}{20}+\frac{16}{e^2}\right).$$

Remark 3. For any given $\varepsilon>0$

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left[\left|\frac{nR_n}{\log n}-\frac{nER_n}{\log n}\right|\geq\varepsilon\right]\leq C\sum_{n=1}^{\infty}\frac{1}{n\log^{2k}n}<\infty,$$

where C is a positive constant not depending on n.

4. Almost sure convergence. Following an idea of Etemadi (cf. [2]) we prove that the sequence $(nR_n/\log n)$ converges to unity almost surely.

Theorem 3.

$$\frac{nR_n}{\log n}\xrightarrow{a.s.}1,\quad n\to\infty. \tag{22}$$

Proof. Let $\varepsilon>0$, $\alpha>1$ and $m_n=\lceil\alpha^n\rceil$ for $n\geq 1$, where

$\lceil x\rceil$ = the smallest integer greater than or equal to x
(the notation from [4]),

i.e. $\lceil x\rceil$ denotes the ceiling function of x. In what follows, C denotes a finite positive constant that can vary from step to step.

Then using Theorem 2, for all $k\in\mathbb{N}$

$$\begin{aligned}\sum_{n=1}^{\infty}P\left[\left|\frac{m_nR_{m_n}}{\log m_n}-\frac{m_nER_{m_n}}{\log m_n}\right|\geq\varepsilon\right]&\leq C\sum_{n=1}^{\infty}\frac{1}{\log^{2k}m_n}\\&\leq C\sum_{n=1}^{\infty}\frac{1}{n^{2k}}<\infty.\end{aligned}$$

The Borel-Cantelli lemma implies

$$\frac{m_nR_{m_n}}{\log m_n}\xrightarrow{a.s.}1,\quad n\to\infty.$$

Let $p(n)$ be such that $m_{p(n)}\leq n<m_{p(n)+1}$, for $n\geq 1$. Since R_n as a function n is non-increasing, we have

$$\begin{aligned}\liminf_{n\to\infty}\frac{nR_n}{\log n}&\geq\liminf_{n\to\infty}\frac{m_{p(n)+1}R_{m_{p(n)+1}}}{\log m_{p(n)+1}}\frac{m_{p(n)}}{m_{p(n)+1}}\\&\geq\frac{1}{\alpha}\lim_{n\to\infty}\frac{m_{p(n)+1}R_{m_{p(n)+1}}}{\log m_{p(n)+1}}=\frac{1}{\alpha}.\end{aligned}$$

Similarly, we can get an analogous relation for the upper limit, namely

$$\begin{aligned}\limsup_{n\to\infty}\frac{nR_n}{\log n} &\le \limsup_{n\to\infty}\frac{m_{p(n)}R_{m_{p(n)}}}{\log m_{p(n)}}\frac{m_{p(n)+1}}{m_{p(n)}}\\ &\le \alpha\lim_{n\to\infty}\frac{m_{p(n)}R_{m_{p(n)}}}{\log m_{p(n)}}=\alpha.\end{aligned}$$

Since $\alpha > 1$ was arbitrary, letting $\alpha \to 1$ we obtain (22). □

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