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MONIKA BUDZYŃSKA and TADEUSZ KUCZUMOW

A structure of common fixed point sets of commuting holomorphic mappings in finite powers of domains

Abstract. In this paper we consider bounded convex domains D in complex reflexive Banach spaces which are locally uniformly linearly convex in the Kobayashi distance k_D. We show that nonempty common fixed point sets of commuting holomorphic mappings in finite powers of these kind of domains are holomorphic retracts.

1. Introduction. In this paper we apply the notion of a uniform linear convexity of the Kobayashi distance to obtain holomorphic retracts in the finite Cartesian product of bounded convex domains.

2. Basic properties of the Kobayashi distance. Throughout this paper all Banach spaces X will be complex and all domains $D \subset X$ will be bounded and convex.

Let D be a bounded convex domain in a reflexive Banach space $(X, \|\cdot\|)$. In D we have the Kobayashi distance (in fact, this is a definition of the

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Lempert function δ [32] – see also [12], [22])

$$\begin{aligned} k_D(x,y) &= \delta_D(x,y) \\ &= \inf \{k_\Delta(0,\gamma) : \text{there exists } f \in H(\Delta, D) \\ &\qquad\qquad \text{such that } f(0) = x \text{ and } f(\gamma) = y\} \end{aligned}$$

[25], [26]. The Kobayashi distance k_D is always locally equivalent to the norm $\|\cdot\|$. If $x, y, w, z \in D$ and $s \in [0,1]$, then

$$k_D(sx + (1-s)y, sw + (1-s)z) \leq \max\{k_D(x,w), k_D(y,z)\}.$$

Hence each open (closed) k_D-ball in the metric space (D, k_D) is convex [31]. Next, there is the following connection between the Kobayashi distance and the weak topology in a reflexive Banach space X: if $\{x_\lambda\}_{\lambda\in I}$ and $\{y_\lambda\}_{\lambda\in I}$ are nets in D which are weakly convergent to x and y respectively, $x, y \in D$, then

$$k_D(x,y) \leq \liminf_\lambda k_D(x_\lambda, y_\lambda),$$

i.e., the Kobayashi distance is lower semicontinuous with respect to the weak topology in X [28] (see also [9], [23]).

Let us observe that, if D_j is a bounded convex domain in a reflexive Banach space $(X_j, \|\cdot\|_j)$ for $j = 1, 2, ..n$, and $X = \prod_{j=1}^n X_j$ is a finite Cartesian product of X_j with the maximum norm, then

$$k_{\prod_{j=1}^n D_j}(x,y) = \max_{1\leq j\leq n} k_{D_j}(x_j, y_j)$$

for all $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \prod_{j=1}^n D_j$ [20].

We will use the standard definition of strict convexity. A point x on the boundary of a bounded convex set $D \subset X$ is called a real extreme point if $\{x + ty \in X : -1 \leq t \leq 1\} \subset \overline{D}$ implies $y = 0$. If each boundary point of a bounded convex domain D is an extreme point, then D is called a strictly convex domain.

If D is strictly convex, then we can say more about linear convexity of balls in (D, k_D).

Theorem 2.1. [7], [34], [35] (see also [33]). *If D is a strictly convex domain in a reflexive Banach space X, then each k_D-ball is also strictly convex in a linear sense.*

Remark 2.1. More information about strict convexity in a linear sense of k_D-balls can be found in [4] and [9].

Recently, the first author introduced the notion of local uniform linear convexity of the domain D with respect to the Kobayashi distance [4] (earlier, a similar notion in the Hilbert ball B_H was considered by the second author [27]) who gave examples of such domains and applications of these domains in the fixed point theory of holomorphic mappings (see [4], [5], [6], [8], [35]).

Definition 2.1. [5]. Let D be a bounded and convex domain in a reflexive Banach space X. The metric space (D, k_D) is said to be a locally uniformly linearly convex space, if there exist $w \in D$ and the function

$$\delta(w, \cdot, \cdot, \cdot, \cdot, \cdot)$$

such that for all $x, y \in D$, $R_1 > 0$, $z \in D$ with $k_D(w, z) \leq R_1$, $0 < R_2 \leq R \leq R_3$, and $0 < \epsilon_1 \leq \epsilon \leq \epsilon_2 < 2$ we have

$$\delta\left(w, R_1, R_2, R_3, \epsilon_1, \epsilon_2\right) > 0$$

and

$$\left.\begin{array}{r} lk_D\left(z, x\right) \leq R \\ k_D\left(z, y\right) \leq R \\ k_D\left(x, y\right) = \epsilon R \end{array}\right\} \Rightarrow k_D\left(z, \frac{1}{2}x + \frac{1}{2}y\right) \leq \left(1 - \delta\left(w, R_1, R_2, R_3, \epsilon_1, \epsilon_2\right)\right) R.$$

The function $\delta(w, \cdot, \cdot, \cdot, \cdot, \cdot)$ is called a modulus of linear convexity for the Kobayashi distance k_D.

Now we recall the notion of an asymptotic center [14]. Let D be a bounded convex domain in a reflexive Banach space X, $\{x_\lambda\}_{\lambda \in \Lambda}$ a k_D-bounded net in D and C a nonempty, k_D-closed and convex subset of D. Consider the functional

$$r(\cdot, \{x_\lambda\}_{\lambda \in \Lambda}) : D \to [0, \infty)$$

defined by

$$r(x, \{x_\lambda\}_{\lambda \in \Lambda}) = \limsup_{\lambda \in \Lambda} k_D(x, x_\lambda).$$

A point z in C is said to be an asymptotic center of the net $\{x_\lambda\}_{\lambda \in \Lambda}$ with respect to C if

$$r(z, \{x_\lambda\}_{\lambda \in \Lambda}) = \inf\{r(x, \{x_\lambda\}_{\lambda \in \Lambda}) : x \in C\}.$$

The infimum of $r(\cdot, \{x_\lambda\}_{\lambda \in \Lambda})$ over C is called an asymptotic radius of $\{x_\lambda\}_{\lambda \in \Lambda}$ with respect to C and denoted by $r(C, \{x_\lambda\}_{\lambda \in \Lambda})$. Let us observe that the function $r(\cdot, \{x_\lambda\}_{\lambda \in \Lambda})$ is quasi-convex, i.e.,

$$r((1-t)x + ty, \{x_\lambda\}_{\lambda \in \Lambda}) \leq \max(r(x, \{x_\lambda\}_{\lambda \in \Lambda}), r(y, \{x_\lambda\}_{\lambda \in \Lambda}))$$

for all x and y in D and $0 \leq t \leq 1$.

It is easy to prove the following proposition.

Proposition 2.2. [4], [8]. *Let D be a bounded convex domain in a reflexive Banach space X such that the metric space (D, k_D) is locally linearly uniformly convex. Then each k_D-bounded net $\{x_\lambda\}_{\lambda\in\Lambda}$ in D has a unique asymptotic center with respect to any nonempty, k_D-closed and convex subset C of D.*

3. Holomorphic mappings and k_D-nonexpansive mappings. In this section we recall basic properties of holomorphic mappings and k_D-nonexpansive mappings.

Let D be a bounded convex domain in a reflexive Banach space X and C a nonempty and k_D-closed subset of D. We say that a mapping $f : C \to C$ is k_D-nonexpansive if

$$k_D\left(f(x), f(y)\right) \leq k_D\left(x, y\right)$$

for all $x, y \in C$ [17]. Each holomorphic self-mapping $f : D \to D$ is k_D-nonexpansive ([11], [16]). We also have the following useful property of such mappings.

Proposition 3.1.. [10], [21], [28], [30]. *Let D be a bounded convex domain in a reflexive Banach space X. If $\{f_\lambda\}_{\lambda\in\Lambda}$ is a net of k_D-nonexpansive (holomorphic) self-mappings of D which is weakly pointwise convergent to a mapping $f : D \to D$, then f is also k_D-nonexpansive (holomorphic).*

Now, we recall two important facts about k_D-nonexpansive self-mappings of bounded convex domains D in reflexive Banach spaces.

Let C be a nonempty convex and k_D-closed subset of D. If $f : C \to C$ is k_D-nonexpansive, then for each $0 < t < 1$ and $a \in C$ the mapping

$$f_{t,a} = (1-t)a + tf$$

is a contraction. Therefore, for each $x \in C$, the sequence $\{f^n_{t,a}(x)\}$ tends to a unique fixed point $y_{t,a}$ in C. Additionally, we have

$$\lim_{t\to 1^-} \|y_{t,a} - f_{t,a}(y_{t,a})\| = 0$$

[13].

For a k_D-nonexpansive $f : C \to C$, we call a sequence $\{x_n\}$ in C an approximating sequence if

$$\lim_n k_D(x_n, f(x_n)) = 0.$$

So, we are ready to state the following theorem.

Theorem 3.2. [4]. *Let D be a bounded convex domain in a reflexive Banach space X such that the metric space (D, k_D) is locally uniformly linearly convex, C be a nonempty convex and k_D-closed subset of D and let $f : C \to C$ be a k_D-nonexpansive mapping. Then the following statements are equivalent:*

- (i) *f has a fixed point;*
- (ii) *There exists a point x in C such that the sequence of iterates $\{f^n(x)\}$ is k_D-bounded;*
- (iii) *The sequence of iterates $\{f^n(x)\}$ is k_D-bounded for all x in C;*
- (iv) *There exists a k_D-bounded approximating sequence $\{x_n\}$ for f.*

Next we have

Lemma 3.3. [7]. *Let X be a reflexive Banach space and D a bounded convex domain in X such that the metric space (D, k_D) is strictly convex in a linear sense. If $f : D \to D$ is k_D-nonexpansive and has a fixed point, then f has a fixed point in each nonempty, f-invariant, k_D-closed and convex subset C of D.*

Finally, we consider holomorphic (k_D-nonexpansive) retracts. By using the Bruck method ([1], [2]) we can obtain the following theorem about holomorphic (k_D-nonexpansive) retracts.

Theorem 3.4. [7], [9]. *Let D be a bounded strictly convex domain in a reflexive Banach space X. If $f : D \to D$ is k_D-nonexpansive (holomorphic), then the set $\mathrm{Fix}\,(f)$ of fixed points of f is either empty or a k_D-nonexpansive (holomorphic) retract of D.*

For a family of commuting holomorphic (k_D-nonexpansive) mappings in a locally uniformly linearly convex metric space (D, k_D) we have a similar result.

Theorem 3.5. [6]. *Let D be a bounded convex domain in a reflexive Banach space X. Suppose that the metric space (D, k_D) is locally uniformly linearly convex. Then, for every family $\mathcal{F}$ of holomorphic (k_D-nonexpansive) self-mappings of D with a nonempty common fixed point set $\mathrm{Fix}\,(\mathcal{F})$, this set $\mathrm{Fix}\,(\mathcal{F})$ is a holomorphic (k_D-nonexpansive) retract of D.*

One of the main tools in the proof of the above theorem is the following lemma.

Lemma 3.6. [6]. *Let D be a bounded convex domain in a reflexive Banach space X. Suppose that the metric space (D, k_D) is locally linearly uniformly convex. Let $\mathcal{F}$ be a family of holomorphic (k_D-nonexpansive) self-mappings of D with a nonempty common fixed point set $\mathrm{Fix}\,(\mathcal{F})$. If a nonempty k_D-closed convex set $C \subset D$ is $\mathcal{F}$-invariant, then $C \cap \mathrm{Fix}\,(\mathcal{F})$ is nonempty.*

4. A common fixed point set of a family of holomorphic mappings in the finite Cartesian product of domains. We begin this section with the following generalization of Theorem 3.2

Theorem 4.1. *Let D_j be a bounded convex domain in a reflexive Banach space X_j, $j = 1, ..., n$. Suppose that each metric space $\left(D_j, k_{D_j}\right)$ is locally uniformly linearly convex, C is a nonempty, convex and k_D-closed subset of $D = \prod_{j=1}^{n} D_j$ and $f : C \to C$ is a k_D-nonexpansive mapping. Then the following statements are equivalent:*

- (i) *f has a fixed point;*
- (ii) *There exists a point x in C such that the sequence of iterates $\{f^m(x)\}$ is k_D-bounded;*
- (iii) *The sequence of iterates $\{f^m(x)\}$ is k_D-bounded for all x in C;*
- (iv) *There exists a k_D-bounded approximating sequence $\{x_m\}$ for f.*

Proof. To prove this theorem it is sufficient to apply the asymptotic center method and the following facts:

1. Each nonempty, closed, convex, k_D-bounded and f-invariant subset C_0 of C contains a k_D-bounded approximating sequence for f;
2. If $\{x_n\}$ is a k_D-bounded approximating sequence for f, then
$$r(f(y), \{x_n\}) \leq r(y, \{x_n\})$$
for each $y \in C$;
3. If $x \in C$ has the k_D-bounded sequence of iterates $\{f^n(x)\}$, then
$$r(f(y), \{f^n(x)\}) \leq r(y, \{f^n(x)\})$$
for each $y \in C$;
4. By Proposition 2.2 every k_D-bounded sequence $\{x_n\}$ in D has an asymptotic center with respect to any nonempty, k_D-closed and convex subset C of D and this asymptotic center is equal to $\prod_{j=1}^{n} A_j \cap C$, where each A_j is nonempty, closed and convex, and at least one of A_j is a singleton. Hence we can apply Theorem 3.2. and the mathematical induction with respect to n. □

Remark 4.1. Note that in the case of the open unit ball B_H of a Hilbert space H an analogous theorem is known [18], [19], [27], [30].

For our next considerations we need the following generalization of Lemma 3.6.

Lemma 4.2. *Let D_j be a bounded convex domain in a reflexive Banach space X_j, $j = 1, ..., n$. Suppose that each metric space $\left(D_j, k_{D_j}\right)$ is locally uniformly linearly convex, C is a nonempty convex and k_D-closed subset of $D = \prod_{j=1}^{n} D_j$, and $\mathcal{F}$ a family of holomorphic (k_D-nonexpansive) self-mappings of D with a nonempty common fixed point set* $\mathrm{Fix}(\mathcal{F})$. *If a nonempty k_D-closed convex set $C \subset D$ is $\mathcal{F}$-invariant, then $C \cap \mathrm{Fix}(\mathcal{F})$ is nonempty.*

Proof. For $n = 1$ see Lemma 3.6. Assume $n \geq 2$. Let x_0 be a common fixed point of $\mathcal{F}$ and C a nonempty, $\mathcal{F}$-invariant, k_D-closed and convex subset of D. If

$$d = \mathrm{dist}_{k_D}(x_0, C) = \inf_{x \in C} k_D(x_0, x),$$

then the set

$$C_0 = C \cap B_{k_D}(x_0, d+1) = C \cap \{x \in D : k_D(x_0, x) \leq d+1\}$$

is nonempty, convex and weakly compact. Therefore, by the weak lower semicontinuity of the Kobayashi distance k_D and by the weak compactness of k_D-balls, there exists a point $x_1 \in C$ such that

$$k_D(x_0, x_1) = d = \mathrm{dist}_{k_D}(x_0, C).$$

Hence the set

$$\tilde{C} = \{x \in C : k_D(x_0, x) = d = \mathrm{dist}_{k_D}(x_0, C)\}$$

is nonempty and is equal to $\prod_{j=1}^{n} A_j \cap \tilde{C}$, where each A_j is nonempty, closed and convex, and at least one of A_j is a singleton. Choose $f \in \mathcal{F}$. To get $f(\tilde{C}) \subset \tilde{C}$ it is sufficient to observe that

$$k_D(x_0, f(x)) = k_D(f(x_0), f(x)) \leq k_D(x_0, x).$$

for every $x \in \tilde{C}$. Hence we can apply Lemma 3.6 and the mathematical induction with respect to n. This completes the proof. $\square$

5. A structure of a common fixed point set of a family of commuting holomorphic mappings. In this section we state and prove the main theorem of our paper.

Theorem 5.1. *Let D_j be a bounded convex domain in a reflexive Banach space X_j, $j = 1, ..., n$ and $D = \prod_{j=1}^{n} D_j$. Suppose that each metric space $\left(D_j, k_{D_j}\right)$ is locally uniformly linearly convex. Then for every family $\mathcal{F}$ of commuting holomorphic (k_D-nonexpansive) self-mappings of D with a* *nonempty common fixed point set* $\mathrm{Fix}\,(\mathcal{F})$*, this set* $\mathrm{Fix}\,(\mathcal{F})$ *is a holomorphic (*k_D*-nonexpansive) retract of* D*.*

Proof. We will use the Bruck method [2] (see also [1]). We prove this result only in the holomorphic case. Our metric approach works equally well in the k_D-nonexpansive case.

Set

$$\mathcal{N} = \{h : h \text{ is a holomorphic self-mapping of } D,\ \mathrm{Fix}\,(\mathcal{F}) \subset \mathrm{Fix}\,(h)\}$$

and choose $x_0 \in \mathrm{Fix}\,(\mathcal{F})$. Note that

$$\mathcal{N} \subset \prod_{x \in D} \{y \in D : k_D\,(y, x_0) \leq k_D\,(x, x_0)\} = \prod_{x \in D} C_x.$$

If each C_x is equipped with the weak topology, then each C_x is weakly compact and by Tychonoff's Theorem ([15], [24]) the set $\prod_{x \in D} C_x$ is compact in the product topology. The set $\mathcal{N}$ is closed in this topology, i.e., in the topology of coordinate pointwise weak convergence (see Proposition 3.1). Preorder $\mathcal{N}$ by setting $g \leq h$ if and only if

$$k_D\,(g\,(x)\,, w) \leq k_D\,(h\,(x)\,, w)$$

for all $w \in \mathrm{Fix}\,(\mathcal{F})$ and $x \in D$. Let us choose a descending chain $\{g_\lambda\}_{\lambda \in \Lambda}$ in $(\mathcal{N}, \leq)$ and let Λ' be an ultranet in Λ. By the compactness of $\prod_{x \in D} C_x$, a subnet $\{g_{\lambda'}\}_{\lambda' \in \Lambda'}$ is an ultranet which is pointwise weakly convergent and

$$w - \lim_{\lambda'} g_{\lambda'}\,(x) = g\,(x)\,, \quad x \in D.$$

The mapping g is holomorphic (see Proposition 3.1). Since the Kobayashi distance k_D is weakly lower semicontinuous, the following inequalities are valid for each $w \in \mathrm{Fix}\,(\mathcal{F})$ and $x \in D$:

$$k_D\,(g\,(x)\,, w) \leq \lim_{\lambda'} k_D\,(g_{\lambda'}\,(x)\,, w) \leq k_D\,(g_\lambda\,(x)\,, w)\,, \quad \lambda \in \Lambda,$$

and this means that g is a lower bound of the chain. So Zorn's Lemma implies that $\mathcal{N}$ contains a minimal element r. Now, we need to show that r maps D onto $\mathrm{Fix}\,(\mathcal{F})$. Suppose there exists $y \in D$ such that $r\,(y) \notin \mathrm{Fix}\,(\mathcal{F})$. Since $r \circ r \leq r$ and r is minimal,

$$k_D\,(r\,(y_0)\,, w) = k_D\,(y_0, w) > 0$$

for $y_0 = r\,(y)$ and all $w \in \mathrm{Fix}\,(\mathcal{F})$. Let

$$C = \{(g \circ r)(y_0) : g \in \mathcal{N}\}\,.$$

We see that C is k_D-bounded, convex and weakly compact. The definition of $\mathcal{N}$ implies that C is f-invariant for each $f \in \mathcal{F}$ and therefore, by Lemma 4.2, we have

$$C \cap \operatorname{Fix}(\mathcal{F}) \neq \emptyset.$$

Let us choose an arbitrary point $(g \circ r)(y_0) \in C \cap \operatorname{Fix}(\mathcal{F})$. Then we get the following contradiction:

$$\begin{aligned} 0 &= k_D((g \circ r)(y_0), (g \circ r)(y_0)) = k_D((g \circ r)(y_0), (g \circ g \circ r)(y_0)) \\ &= k_D(r(y_0), (g \circ r)(y_0)) > 0. \end{aligned}$$

This completes the proof. □

Finally, we note that the assumption in the above theorem that the common fixed point set $\operatorname{Fix}(\mathcal{F})$ is nonempty, is essential as the example given in [29] shows.

6. The case of a finite commuting family of holomorphic mappings.

We begin with the following simple lemma.

Lemma 6.1. *Let D_j be a bounded convex domain in a reflexive Banach space X_j, $j = 1, ..., n$ and $D = \prod_{j=1}^{n} D_j$. Suppose that each metric space (D_j, k_{D_j}) is locally uniformly linearly convex. If $f : D \to D$ is a holomorphic (k_D-nonexpansive) mapping with $\operatorname{Fix}(f) \neq \emptyset$ and a nonempty set A is invariant under f and is a holomorphic (k_D-nonexpansive) retract of D, then $\operatorname{Fix}(f) \cap A$ is a nonempty holomorphic (k_D-nonexpansive) retract of D.*

Proof. Let r be a holomorphic (k_D-nonexpansive) retraction of D onto A. Let us observe that $f \circ r : D \longrightarrow A$, $(f \circ r)_{|A} = f_{|A}$, $\operatorname{Fix}(f) \neq \emptyset$, and $f : D \to D$ is a holomorphic (k_D-nonexpansive) mapping. Choose $x_0 \in A$. Then by Theorem 4.1 the sequence $\{f \circ r^m(x_0)\} = \{f^m(x_0)\}$ is k_D-bounded and again by Theorem 4.1 this implies that the set $\operatorname{Fix}(f \circ r)$ is nonempty. It is easy to see that

$$\operatorname{Fix}(f) \cap A = \operatorname{Fix}(f \circ r)$$

and therefore by Theorem 5.1, $\operatorname{Fix}(f) \cap A$ is a holomorphic (k_D-nonexpansive) retract. □

The above lemma implies the following theorem.

Theorem 6.2. *Let D_j be a bounded convex domain in a reflexive Banach space X_j, $j = 1, ..., n$ and $D = \prod_{j=1}^{n} D_j$. Suppose that each metric space (D_j, k_{D_j}) is locally uniformly linearly convex. Then, for every finite family $\{f_1, ..., f_l\}$ of commuting holomorphic (k_D-nonexpansive) self-mappings of*

D a common fixed point set $\mathrm{Fix}\,(f_1) \cap \cdots \cap \mathrm{Fix}\,(f_m)$ *is nonempty and a holomorphic (k_D-nonexpansive) retract of D.*

Proof. It is sufficient to apply Theorem 5.1, Lemma 6.1 and the mathematical induction with respect to l. □

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Instytut Matematyki UMCS
pl. M. Curie-Skłodowskiej 1
20-031 Lublin, Poland
e-mail: monikab@golem.umcs.lublin.pl
e-mail: tadek@golem.umcs.lublin.pl

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