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Universal linearly invariant families and Bloch functions in the unit ball

Abstract. In this note we consider universal linearly invariant families of mappings defined in the unit ball. We give a connection of such families with Bloch functions, as well as with Bloch mappings.

1. Preliminaries. Connections between linearly invariant families of functions on the unit disk ([P]) and Bloch functions were studied in several papers (see for example [CCP], [GS1]). In the case of the unit polydisk similar results were obtained in [GS2], [GS3]. In this paper we connect the universal linearly invariant families of locally biholomorphic mappings in the unit ball of $\mathbb{C}^n$ ([Pf2]) with Bloch functions ([H1], [H2], [T1], [T2]) or Bloch mappings ([L]).

Let $\mathbb{C}^n$ denote n-dimensional complex space of all ordered n-tuples $z = (z_1, z_2, \dots, z_n)$ of complex numbers with the inner product $\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$. The unit ball $\mathbf{B}^n$ of $\mathbb{C}^n$ is then the set of all $z \in \mathbb{C}^n$ with $\|z\| = (\langle z, z\rangle)^{\frac{1}{2}} < 1$. For a vector-valued, holomorphic mapping $f(z) = (f^1(z), \dots, f^n(z))$ let $f^j_k(z) = \frac{\partial f^j(z)}{\partial z_k}$ and $f^j_{ik}(z) = \frac{\partial^2 f^j(z)}{\partial z_i \partial z_k}$. Then the derivative $\mathrm{D}\, f(z)$ of f at z is represented by a matrix $(f^j_k(z))$ and let the

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second derivative operator be given by the following formula $\mathrm{D}^2 f(z)(w,\cdot) = (\sum_{k=1}^n f_{ik}^j(z)w_k)$ and the identity matrix by $\mathbf{I}$. The (complex) Jacobian of f at z can be defined by $J_f(z) = \det \mathrm{D} f(z)$. Let

$$\mathcal{LS}_n = \{f : f \text{ is holomorphic in } \mathbf{B}^n, \\ J_f(z) \neq 0 \text{ for } z \in \mathbf{B}^n, f(\mathbb{O}) = \mathbb{O}, \mathrm{D} f(\mathbb{O}) = \mathbf{I}\}$$

be the family of normalized, locally biholomorphic mappings of $\mathbf{B}^n$. The operator on $\mathcal{LS}_n$ that defines the linear invariance is the Koebe transform

$$\Lambda_\phi(f)(z) = (\mathrm{D}\,\phi(\mathbb{O}))^{-1}((\mathrm{D}\,f)(\phi(\mathbb{O})))^{-1}\{f(\phi(z)) - f(\phi(\mathbb{O}))\},$$

where ϕ belongs to the set $\mathcal{A}$ of biholomorphic authomorphisms of $\mathbf{B}^n$ and $f \in \mathcal{LS}_n$. Up to multiplication by an unitary matrix, the biholomorphic automorphisms of $\mathbf{B}^n$ are

$$\phi(z) = \phi_a(z) = \frac{a - P_a z - sQ_a z}{1 - \langle z, a\rangle}, \quad a \in \mathbf{B}^n,$$

where $P_{\mathbb{O}} = \mathbb{O}$ and $P_a z = \frac{\langle z,a\rangle}{\langle a,a\rangle} a$ for $a \neq \mathbb{O}$, $Q_a = \mathbf{I} - P_a$ and $s = (1 - \|a\|^2)^{1/2}$. For details see [R]. The following definitions are known ([Pf2],[BFG]).

Definition 1.1. A family $\mathcal{F}$ is called linearly invariant if
(i) $\mathcal{F} \subset \mathcal{LS}_n$,
(ii) $\Lambda_\phi(f) \in \mathcal{F}$ for all $f \in \mathcal{F}$ and $\phi \in \mathcal{A}$.

Let the trace of a matrix will be denoted by tr. The number

$$\begin{aligned} \operatorname{ord} \mathcal{F} &= \sup_{g\in\mathcal{F}} \sup_{\|w\|=1} \left| \operatorname{tr}\left\{ \frac{1}{2} \mathrm{D}^2 g(\mathbb{O})(w,\cdot) \right\} \right| \\ &= \sup_{g\in\mathcal{F}} \sup_{\|w\|=1} \left| \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n g_{jk}^j(\mathbb{O}) w_k \right| \end{aligned} \tag{1.1}$$

is called ([Pf2]) the order of a linearly invariant family $\mathcal{F}$. Let us introduce the notion of the order of a function.

Definition 1.2. For $f \in \mathcal{LS}_n$ the number

$$\operatorname{ord} f = \sup_{\phi\in\mathcal{A}} \sup_{\|w\|=1} \frac{1}{2} |\operatorname{tr}\{\mathrm{D}^2 g(\mathbb{O})(w,\cdot)\}|,$$

where $g(z) = \Lambda_\phi(f)(z)$, is called the order of f.

Definition 1.3. The family

$$\mathcal{U}_\alpha = \cup\{f \in \mathcal{LS}_n : \operatorname{ord} f \le \alpha\}$$

is called the universal linearly invariant family.

In the paper we will use the following results. If $\mathcal{F} \subset \mathcal{LS}_n$ is a linearly invariant family of order α and $f \in \mathcal{F}$ then

$$(1.2)\qquad \frac{(1-\|z\|)^{\alpha-\frac{n+1}{2}}}{(1+\|z\|)^{\alpha+\frac{n+1}{2}}} \le |J_f(z)| \le \frac{(1+\|z\|)^{\alpha-\frac{n+1}{2}}}{(1-\|z\|)^{\alpha+\frac{n+1}{2}}}, \quad z \in \mathbf{B}^n, \quad ([Pf2])$$

$$(1.3)\qquad |\log((1-\|z\|^2)^{\frac{n+1}{2}}|J_f(z)|)| \le \alpha \log\frac{1+\|z\|}{1-\|z\|}, \quad z \in \mathbf{B}^n, \quad ([Pf2])$$

$$(1.4)\qquad \begin{aligned}&\frac{d}{d\rho}\log(J_f(\rho w))\\ &= \operatorname{tr}\{(\mathrm{D}\, f(\rho w))^{-1}\, \mathrm{D}^2 f(\rho w)(w,\cdot)\},\ \rho \in [0,1),\ w \in \overline{\mathbf{B}^n}. \quad ([Pf1])\end{aligned}$$

The above inequalities are rendered by the mappings

$$K_\alpha(z) = (k_\alpha(z_1), z_2\sqrt{k'_\alpha(z_1)}, \dots, z_n\sqrt{k'_\alpha(z_1)}), \quad ([Pf2],[LS2])$$

where

$$k_\alpha(z_1) = \frac{n+1}{4\alpha}\left[\left(\frac{1+z_1}{1-z_1}\right)^{\frac{2\alpha}{n+1}} - 1\right].$$

In [GLS] it was proved the following theorem.

Theorem A. *The family $\mathcal{U}_\alpha$ coincides with the set of all functions satisfying the conditions of Definition 1.1 and the right hand side inequality in (1.2).*

2. Bloch functions. R. Timoney studied ([T1], [T2]) Bloch functions in several complex variables and he gave several equivalent definitions (see also [H1], [H2]). In this paper we will use the following one.

Definition 2.1. A holomorphic function $h : \mathbf{B}^n \to \mathbb{C}$ is called a Bloch function if its norm

$$\|h\|_{\mathcal{B}} = |h(\mathbb{O})| + \sup_{\phi \in \mathcal{A}} \|\nabla(h \circ \phi)(\mathbb{O})\|$$

is finite.

Now let

$$Q_h(z) = \sup_{\mathbb{C}^n \ni x \neq \mathbb{O}} \frac{|\langle \nabla h(z), \bar{x} \rangle|}{H_z(x,x)^{1/2}},$$

where $H_z(u,v) = \frac{n+1}{2}[(1-\|z\|^2)\langle u,v\rangle + \langle u,z\rangle\langle z,v\rangle]/(1-\|z\|^2)^2$, $u,v \in \mathbb{C}^n, z \in \mathbf{B}^n$, is the Bergman metric. Then from Lemma 1 of [H1] it follows that $Q_{h\circ\phi}(z) = Q_h(\phi(z))$ for every automorphism $\phi \in \mathcal{A}$. Therefore $\sup_{a\in\mathbf{B}^n} Q_h(a) = \frac{2}{n+1} \sup_{\phi\in\mathcal{A}, \|x\|=1} |\langle \nabla(h\circ\phi)(\mathbb{O}), x\rangle| = \frac{2}{n+1} \sup_{\phi\in\mathcal{A}} \|\nabla(h\circ\phi)(\mathbb{O})\|$. Thus Definition 2.1 is equivalent to the following definition of Bloch functions given in [H2]: $\sup_{a\in\mathbf{B}^n} Q_h(a) < \infty$. Timoney in [T1] proved that quantities $\sup_{a\in\mathbf{B}^n} Q_h(a)$ and $\sup_{\|w\|\le 1} [(1-\|w\|^2)\langle \nabla h(w), \bar{w}\rangle|]$ are equivalent. In this way the norms $\|h\|_{\mathcal{B}}$ and

$$\|h\|_X = |h(0)| + \sup_{w\in\mathbf{B}^n} (1-\|w\|^2)|\langle \nabla h(w), \bar{w}\rangle| \tag{2.1}$$

are equivalent. The family of all Bloch functions will be denoted by $\mathcal{B} = \mathcal{B}(\mathbf{B}^n)$. In the next theorem we give a new condition which is equivalent to the definition of a Bloch function.

Theorem 2.1. *A holomorphic function $h : \mathbf{B}^n \to \mathbb{C}$ belongs to $\mathcal{B}$ if and only if there exists a mapping $f \in \bigcup_{\alpha<\infty} \mathcal{U}_\alpha$ such that*

$$h(z) - h(\mathbb{O}) = \log(J_f(z)), \quad z \in \mathbf{B}^n.$$

Moreover, if $h(z) - h(\mathbb{O}) = \log(J_f(z)) \in \mathcal{B}$ and $\operatorname{ord} f = \alpha$, *then*

$$2\left(\alpha - \frac{n+1}{2}\right) \le \|h - h(\mathbb{O})\|_X \le 2\left(\alpha + \frac{n+1}{2}\right)$$

and

$$2\left(\alpha - \frac{n+1}{2}\right) \le \|h - h(\mathbb{O})\|_{\mathcal{B}} \le 2\left(\alpha + \frac{n+1}{2}\right).$$

The inequalities are sharp.

Proof. For $\rho \in [0,1)$, $w \in \partial\mathbf{B}^n$ define $h(\rho w) = \log(J_f(\rho w))$, where $\operatorname{ord} f = \alpha$. Observe that we have

$$\frac{\mathrm{d}}{\mathrm{d}\rho} h(\rho w) = \langle (\nabla h)(\rho w), \bar{w}\rangle = \frac{\mathrm{d}}{\mathrm{d}\rho} \log(J_f(\rho w))$$

and

$$\langle(\nabla h)(\rho w), \overline{\rho w}\rangle = \rho \frac{\mathrm{d}}{\mathrm{d}\rho} \log(J_f(\rho w)). \tag{2.2}$$

Pfaltzgraff showed ([Pf2]) that for $g(z) = \Lambda_\phi(f)(z)$, $\phi = \phi_a$ and $a = \rho w$
(2.3)

$$\rho \frac{\mathrm{d}}{\mathrm{d}\rho} \log(J_f(\rho w)) = (n+1)\frac{\|\rho w\|^2}{1-\|\rho w\|^2} + \operatorname{tr}\left\{\mathrm{D}^2 g(\mathbb{O})\left(\frac{-\rho w}{1-\|\rho w\|^2}, \cdot\right)\right\}$$

$$= (n+1)\frac{\rho^2}{1-\rho^2} + \operatorname{tr}\left\{\mathrm{D}^2 g(\mathbb{O})\left(\frac{-\rho w}{1-\rho^2}, \cdot\right)\right\}.$$

Therefore by (2.2) we get

$$|\langle(\nabla h)(\rho w), \overline{\rho w}\rangle| \le (n+1)\frac{\rho^2}{1-\rho^2} + \left|\operatorname{tr}\left\{\mathrm{D}^2 g(\mathbb{O})\left(\frac{-\rho w}{1-\rho^2}, \cdot\right)\right\}\right|$$

and thus

$$\begin{aligned}(1-\rho^2)|\langle(\nabla h)(\rho w), \overline{\rho w}\rangle| &\le (n+1)\rho^2 + \rho|\operatorname{tr}\{\mathrm{D}^2 g(\mathbb{O})(w,\cdot)\}| \\ &\le (n+1)\rho^2 + 2\rho\alpha \le ((n+1)+2\alpha)\rho.\end{aligned}$$

By (2.1) the function h belongs to the Bloch class $\mathcal{B}$ and $\|h - h(\mathbb{O})\|_X \le 2(\alpha + \frac{n+1}{2})$.

Conversely, let $h \in \mathcal{B}$ and let $f \in \mathcal{LS}_n$, such that $\log J_f(z) = h(z) - h(\mathbb{O})$. In $\mathcal{LS}_n$ there is such mapping, for example

$$f(z) = (z_1, \ldots, z_{n-1}, \int_0^{z_n} \exp[h(z_1, \ldots, z_{n-1}, s) - h(\mathbb{O})]ds).$$

Let $z = w\rho$, where $\rho \in [0,1), \|w\| = 1$. Let $\phi \in \mathcal{A}$ be fixed. Then let $g(z) = \Lambda_\phi(f)(z)$. Now combining (2.2) and (2.3) we get

$$(1-\rho^2)\langle(\nabla h)(\rho w), \rho\bar{w}\rangle = (n+1)\rho^2 - \rho \operatorname{tr}\{\mathrm{D}^2 g(\mathbb{O})(w,\cdot)\}.$$

Thus by (2.1) we obtain

$$\begin{aligned}\frac{1}{2}\rho|\operatorname{tr}\{\mathrm{D}^2 g(\mathbb{O})(w,\cdot)\}| &\le \frac{n+1}{2}\rho^2 + \frac{1}{2}(1-\rho^2)|\langle(\nabla h)(\rho w), \rho\bar{w}\rangle| \\ &\le \frac{n+1}{2} + \frac{1}{2}\|h - h(\mathbb{O})\|_X.\end{aligned}$$

For $\rho \to 1$ we get

$$\frac{1}{2}|\operatorname{tr}\{\mathrm{D}^2 g(\mathbb{O})(w,\cdot)\}| \leq \frac{n+1}{2} + \frac{1}{2}\|h - h(\mathbb{O})\|_X.$$

Therefore f belongs to a class $\mathcal{U}_\alpha$. Moreover

$$\alpha = \operatorname{ord} f = \frac{1}{2}\sup_{\|w\|=1}|\operatorname{tr}\{\mathrm{D}^2 g(\mathbb{O})(w,\cdot)\}| \leq \frac{n+1}{2} + \frac{1}{2}\|h - h(\mathbb{O})\|_X.$$

Thus $2\alpha - (n+1) \leq \|h - h(\mathbb{O})\|_X$. In the above inequality the equality is attained for $f_0(z) = z$, $h(z) = \log J_f(z) \equiv 0$; $(\operatorname{ord} f_0 = \frac{n+1}{2})$. In the inequality $\|h - h(\mathbb{O})\|_X \leq 2\alpha + n + 1$ the equality is attained for $h = h_\alpha = \log J_{K_\alpha}$, where $K_\alpha(z)$ was defined before; $(\operatorname{ord} K_\alpha = \alpha$, [Pf2]). Since

$$J_{K_\alpha}(z) = (k'_\alpha(z_1))^{(n+1)/2} = \frac{(1+z_1)^{\alpha-(n+1)/2}}{(1-z_1)^{\alpha+(n+1)/2}},$$

we have $\nabla h_\alpha(z) = (\frac{2\alpha+(n+1)z_1}{1-z_1^2}, 0, \dots, 0)$ and

$$\|h_\alpha - h_\alpha(\mathbb{O})\|_X = \sup_{|z_1|<1}\left[(1-|z_1|^2)|z_1|\left|\frac{2\alpha+(n+1)z_1}{1-z_1^2}\right|\right] = 2\alpha + n + 1.$$

Now we will prove suitable inequalities for $\|\cdot\|_{\mathcal{B}}$. Let $\operatorname{ord} f = \alpha$, $g = \Lambda_\phi(f)$, $\phi \in \mathcal{A}$ and $h = \log J_f$. Then $J_g(z) = CJ_f(\phi(z))J_\phi(z)$, where C is a constant. Therefore

$$\nabla(\log J_g)(\mathbb{O}) = \nabla(h \circ \phi)(\mathbb{O}) + \frac{(\nabla J_\phi)(\mathbb{O})}{J_\phi(\mathbb{O})}.$$

For a holomorphic function $q(z)$ in $\mathbf{B}^n$ we have $\frac{\partial \operatorname{Re} q}{\partial z_k} = \frac{\partial(q(z)+\overline{q(z)})}{2\partial z_k} = \frac{1}{2}\frac{\partial q}{\partial z_k}$. Thus $\nabla \operatorname{Re} q = \frac{1}{2}\nabla q$. Moreover $|J_\phi(z)| = (\frac{1-\|a\|^2}{|1-\langle z,a\rangle|^2})^{(n+1)/2}$, for $a \in \mathbf{B}^n$ (see [R]), and then

$$(\nabla \log J_\phi)(\mathbb{O}) = 2(\nabla \log|J_\phi|)(\mathbb{O}) = (n+1)\bar{a},$$

where a is an arbitrary element in $\mathbf{B}^n$ for arbitrary $\phi \in \mathcal{A}$.

It is known (see for example [S]) that for a matrix $(f_{k,j}(z))_{k,j=1}^n$, where $f_{k,j}(z)$ are analytic functions in a domain,

$$\frac{\mathrm{d}}{\mathrm{dz}}\det(f_{k,j})_{k,j=1}^n = \sum_{k=1}^n \det\begin{pmatrix} f_{11}(z) & \dots & f_{1n}(z) \\ \vdots & \vdots & \vdots \\ f'_{k1}(z) & \dots & f'_{kn}(z) \\ \vdots & \vdots & \vdots \\ f_{n1}(z) & \dots & f_{nn}(z) \end{pmatrix}.$$

From the normalization of $g(z) = (g^1, \dots, g^n)$ it follows that

$$(\nabla J_g)(\mathbb{O}) = \left(\sum_{k=1}^{n} g_{1k}^k(\mathbb{O}), \dots, \sum_{k=1}^{n} g_{nk}^k(\mathbb{O}) \right)$$

and $\langle (\nabla J_g)(\mathbb{O}), \bar{w} \rangle = \operatorname{tr}\{\mathrm{D}^2\, g(0)(w,\cdot)\}$. Therefore

$$\langle \nabla(\log J_g)(\mathbb{O}), \bar{w} \rangle = \operatorname{tr}\{\mathrm{D}^2\, g(0)(w,\cdot)\} = \langle \nabla(h\circ\phi)(\mathbb{O}), \bar{w}\rangle + (n+1)\langle a, \bar{w}\rangle,$$

where a depends on ϕ and

$$\begin{aligned} \sup_{\phi\in\mathcal{A}, \|w\|=1} &|\langle \nabla(h\circ\phi)(\mathbb{O}), \bar{w}\rangle| - (n+1)\cdot \sup_{a\in\mathbf{B}^n, \|w\|=1} |\langle a, \bar{w}\rangle| \\ &\le 2\alpha = \sup_{\phi\in\mathcal{A}, \|w\|=1} |\operatorname{tr}\{\mathrm{D}^2\, g(0)(w,\cdot)\}| \\ &\le \sup_{\phi\in\mathcal{A}} \|\nabla(h\circ\phi)(\mathbb{O})\| + (n+1) \sup_{a\in\mathbf{B}^n} \|a\|, \end{aligned}$$

which is equivalent to the following inequalities

$$2\alpha - n - 1 \le \|h - h(\mathbb{O})\|_{\mathcal{B}} \le 2\alpha + n + 1.$$

For $h \equiv 0$ we have the equality in the left inequality. Similarly as before for $h = h_\alpha$ we have the equality in the right inequality. It is sufficient to prove that $\sup_{a\in\mathbf{B}^n} \|\nabla(h_\alpha \circ \phi_a)(\mathbb{O})\| = 2\alpha + n + 1$. Indeed

$$h_\alpha \circ \phi_a = \left(\alpha - \frac{n+1}{2}\right) \log(1+\phi_a^1) - \left(\alpha + \frac{n+1}{2}\right) \log(1-\phi_a^1),$$

$$\nabla(h_\alpha \circ \phi_a)(\mathbb{O}) = \frac{2\alpha + a_1(n+1)}{1-a_1^2} \nabla\phi_a^1(\mathbb{O}), \quad a = (a_1, \dots, a_n).$$

Since (see [R])

$$\phi_a^1(z) = \frac{a_1 - a_1 \frac{\langle z,a\rangle}{\|a\|^2} - s(z_1 - a_1 \frac{\langle z,a\rangle}{\|a\|^2})}{1 - \langle z, a\rangle}, \quad s = \sqrt{1 - \|a\|^2},$$

we get

$$\nabla\phi_a^1(\mathbb{O}) = \left(\dots, a_1 \bar{a}_k \frac{s}{s+1} - s\delta_k^1, \dots\right), \quad 1 \le k \le n,$$

where δ_k^i denotes the Kronecker delta. Therefore

$$\|\nabla(h_\alpha \circ \phi_a)(\mathbb{O})\| = \frac{|2\alpha + a_1(n+1)|}{|1-a_1^2|}\|\nabla\phi_a^1(\mathbb{O})\| = \frac{|2\alpha + a_1(n+1)|}{|1-a_1^2|}\sqrt{\frac{1-\|a\|^2}{1-|a_1|^2}}$$

and

$$\|h_\alpha\|_{\mathcal{B}} \geq \sup_{a\in\mathbf{B}^n}\left[\frac{|2\alpha + a_1(n+1)|}{|1-a_1^2|}\sqrt{(1-\|a\|^2)(1-|a_1|^2)}\right] = 2\alpha + n + 1.$$

This proves the exactness of the inequality $\|h - h(\mathbb{O})\|_{\mathcal{B}} \leq 2\alpha + n + 1$. □

It was proved in [LS1] that for every f from $\mathcal{U}_\alpha$ and every $v \in \mathbb{C}^n$, $\|v\| = 1$, the quantities

$$|J_f(rv)|\frac{(1-r)^{\alpha+(n+1)/2}}{(1+r)^{\alpha-(n+1)/2}} \quad \text{and} \quad \max_{\|v\|=1}|J_f(rv)|\frac{(1-r)^{\alpha+(n+1)/2}}{(1+r)^{\alpha-(n+1)/2}}$$

are decreasing with respect to $r \in [0,1)$ and for $r \to 1^-$ they have limits which belong to the interval $[0,1]$. From the above and Theorem 2.1 the next result follows.

Corollary 2.1. *For every function $h \in \mathcal{B}$ and every $v \in \mathbb{C}^n$, $\|v\| = 1$ the quantities*

$$\mathrm{Re}[h(rv) - h(\mathbb{O})] + \left(\alpha + \frac{n+1}{2}\right)\log(1-r) - \left(\alpha - \frac{n+1}{2}\right)\log(1+r)$$

and

$$\max_{\|v\|=1}\mathrm{Re}[h(rv) - h(\mathbb{O})] + \left(\alpha + \frac{n+1}{2}\right)\log(1-r) - \left(\alpha - \frac{n+1}{2}\right)\log(1+r)$$

are decreasing with respect to $r \in [0,1)$ and for $r \to 1^-$ they have non-positive limits, where $\alpha = \operatorname{ord} f$ for $f \in \cup_{\alpha<\infty}\mathcal{U}_\alpha$ such that $h(z) - h(\mathbb{O}) = \log J_f(z)$.

Since order of $e^{i\lambda}h$ is changing with $\lambda \in \mathbb{R}$ note that it is not possible to replace the real part by the modulus sign in the last corollary.

Theorem 2.2. *A holomorphic function* $h : \mathbf{B}^n \to \mathbb{C}$ *belongs to* $\mathcal{B}$ *if and only if there exists a positive constant* C *such that for all* $z \in \mathbf{B}^n$

$$\begin{aligned} &\sup_{\phi\in\mathcal{A}} \left| \mathrm{Re}[h(\phi(z)) - h(\phi(\mathbb{O}))] + \log\left|\frac{J_\phi(z)}{J_\phi(\mathbb{O})}\right| + \log(1-\|z\|^2)^{\frac{n+1}{2}} \right| \\ &\le C \log \frac{1+\|z\|}{1-\|z\|}, \end{aligned} \tag{2.4}$$

where the best value (the smallest) of C *is equal to* $\operatorname{ord} f$*, for a mapping* f *from* $\mathcal{LS}_n$ *such that* $\log J_f(z) = h(z) - h(\mathbb{O})$*.*

Proof. Let $h \in \mathcal{B}$. We can assume that $h(\mathbb{O}) = 0$. Then by Theorem 2.1 there exists a mapping $f \in \cup_{\alpha<\infty}\mathcal{U}_\alpha$ such that $h(z) = \log(J_f(z))$. For $g(z) = \Lambda_\phi(f)(z)$ we get

$$\mathrm{D}\, g(z) = (\mathrm{D}\,\phi(\mathbb{O}))^{-1}((\mathrm{D}\, f)(\phi(\mathbb{O})))^{-1}(\mathrm{D}\, f)(\phi(z))\,\mathrm{D}\,\phi(z).$$

Moreover, it is clear that

$$\log|J_g(z)| = \mathrm{Re}[h(\phi(z)) - h(\phi(\mathbb{O}))] - \log|J_\phi(\mathbb{O})| + \log|J_\phi(z)|.$$

By (1.3) we have

$$|\log((1-\|z\|^2)^{\frac{n+1}{2}}|J_g(z)|)| \le \alpha \log\frac{1+\|z\|}{1-\|z\|},$$

where $\alpha = \operatorname{ord} f$. The equality is attained for $g = K_\alpha$ and $z = (z_1, 0, \dots, 0) \in \mathbf{B}^n$. Thus we get (2.4). The equality is attained for $g = K_\alpha$, $z = (z_1, 0, \dots, 0) \in \mathbf{B}^n$.

Conversely, suppose that a holomorphic function h satisfies inequality (2.4). Now, let $f(z) = (z_1, \dots, z_{n-1}, \int_0^{z_n} \exp[h(z_1, \dots, z_{n-1}, s) - h(\mathbb{O})]\, ds)$. Note that f belongs to $\mathcal{LS}_n$ and $J_f(z) = \exp[h(z) - h(\mathbb{O})]$. Thus for an automorphism $\phi \in \mathcal{A}$ we get

$$\exp[h(\phi(z)) - h(\phi(\mathbb{O}))] = \frac{J_f[\phi(z)]}{J_f[\phi(\mathbb{O})]}.$$

As in the first part the proof, for $g(z) = \Lambda_\phi(f)(z)$ we have

$$J_g(z) = \frac{J_f[\phi(z)] \cdot J_\phi(z)}{J_\phi(\mathbb{O}) \cdot J_f[\phi(\mathbb{O})]}.$$

Observe that

$$\begin{aligned}\log|J_g(z)| &= \log\left|\frac{J_f[\phi(z)]}{J_f[\phi(\mathbb{O})]}\right| + \log\left|\frac{J_\phi(z)}{J_\phi(\mathbb{O})}\right| \\ &= \operatorname{Re}[h(\phi(z)) - h(\phi(\mathbb{O}))] + \log\left|\frac{J_\phi(z)}{J_\phi(\mathbb{O})}\right|.\end{aligned} \tag{2.5}$$

Thus by (2.4) we obtain

$$\left|\log|J_g(z)| + \frac{n+1}{2}\log(1-\|z\|^2)\right| \le C\log\frac{1+\|z\|}{1-\|z\|}, \quad z \in \mathbf{B}^n.$$

Hence for $z = \rho w$, $\rho \in [0,1)$, $w \in \partial\mathbf{B}^n$,

$$-C\log\frac{1+\rho}{1-\rho} \le \operatorname{Re}\left[\log J_g(\rho w) + \frac{n+1}{2}\log(1-\rho^2)\right] \le C\log\frac{1+\rho}{1-\rho}.$$

For $\rho = 0$ the equality holds in the above inequalities. Therefore, after differentiation with respect to ρ at $\rho = 0$ we get (using (1.4))

$$-2C \le \operatorname{Re}[\operatorname{tr}(\mathrm{D}\, g(\mathbb{O}))^{-1}\,\mathrm{D}^2 g(\mathbb{O})(w,\cdot)] \le 2C.$$

Since $\mathrm{D}\, g(\mathbb{O}) = \mathbf{I}$, we have

$$|\operatorname{Re}[\operatorname{tr}\{\mathrm{D}^2 g(\mathbb{O})(w,\cdot)\}]| \le 2C.$$

For fixed $u \in \mathbb{C}^n$ we have

$$\|u\| \le \sup_{\|w\|=1} \operatorname{Re}\langle w,u\rangle \le \sup_{\|w\|=1} |\langle w,u\rangle| \le \|u\|.$$

Therefore

$$\sup_{\|w\|=1} |\langle w,u\rangle| = \sup_{\|w\|=1} \operatorname{Re}\langle w,u\rangle.$$

Note that $\operatorname{tr}\{\mathrm{D}^2 g(\mathbb{O})(w,\cdot)\} = \langle w,u\rangle$ for some $u \in \mathbb{C}^n$. Then

$$\max_{\|w\|=1} |\operatorname{Re}[\operatorname{tr}\{\mathrm{D}^2 g(\mathbb{O})(w,\cdot)\}]| = \max_{\|w\|=1} |\operatorname{tr}\{\mathrm{D}^2 g(\mathbb{O})(w,\cdot)\}| \le 2C.$$

Thus $f \in \mathcal{U}_C$ and (by Theorem 2.1) $h \in \mathcal{B}$.

Now let us observe that from the proof it follows that $\alpha = \operatorname{ord} f \le C$. Thus from the first part of the proof we get that $C = \operatorname{ord} f = \alpha$ is the best constant in (2.4). □

Remark 2.1. ([GLS]) *From Theorem A and the fact that*

$$J_{\Lambda_\phi(f)}(z) = \frac{J_f(\phi(z))J_\phi(z)}{J_f(\phi(\mathbb{O}))J_\phi(\mathbb{O})},$$

it follows that for $f_1, f_2 \in \mathcal{LS}_n$ *with* $J_{f_1}(z) = J_{f_2}(z)$ *we have* $\operatorname{ord} f_1 = \operatorname{ord} f_2$.

3. Bloch mappings. In this section we will consider Bloch mappings from the unit ball $\mathbf{B}^n$ into $\mathbb{C}^n$ and their connections with linearly invariant families of mappings. Now we give a definition of Bloch mappings (see [L]).

Definition 3.1. A holomorphic mapping $h : \mathbf{B}^n \to \mathbb{C}^n$ is called a Bloch mapping if it has a finite Bloch norm

$$\|h\|_{\mathcal{B}(n)} = \|h(\mathbb{O})\| + \sup_{\phi\in\mathcal{A}} \|\operatorname{D}(h \circ \phi)(\mathbb{O})\|,$$

where $\|\operatorname{D} h(z)\|$ denotes the norm of linear operator $\operatorname{D} h(z)$.

The family of all such mappings will be denoted by $\mathcal{B}(n)$. Let functions f_k belong to $\mathcal{U}_\alpha$, for $k = 1, \dots, n$. Then by (1.2) we have

$$\log|J_{f_k}(z)| \le \left(\alpha - \frac{n+1}{2}\right)\log(1+\|z\|) - \left(\alpha + \frac{n+1}{2}\right)\log(1-\|z\|),$$

$k = 1, \dots, n$. The next theorem gives a relationship between $\mathcal{B}(n)$ and $\mathcal{U}_\alpha$.

Theorem 3.1. *A holomorphic mapping* $h : \mathbf{B}^n \to \mathbb{C}^n$ *belongs to* $\mathcal{B}(n)$ *if and only if there exist mappings* $f_1, \dots, f_n \in \cup_{\alpha<\infty}\mathcal{U}_\alpha$ *such that*

$$h(z) - h(\mathbb{O}) = (\log J_{f_1}(z), \dots, \log J_{f_n}(z)).$$

Moreover, if $\alpha_k = \operatorname{ord} f_k$, $k = 1, \dots, n$ *then*

$$2\sqrt{\sum_{k=1}^{n}\left(\alpha_k - \frac{n+1}{2}\right)^2} \le \|h - h(\mathbb{O})\|_{\mathcal{B}(n)} \le 2\sqrt{\sum_{k=1}^{n}\left(\alpha_k + \frac{n+1}{2}\right)^2};$$

and both inequalities are best possible.

Proof. Let $h = (h^1, \dots, h^n) = (\log J_{f_1}, \dots, \log J_{f_n})$ and let for every $k = 1, \dots, n$ $\operatorname{ord} f_k = \alpha_k < \infty$. Then by Theorem 2.1

$$\|h^k\|_{\mathcal{B}} = |h^k(\mathbb{O})| + \sup_{\phi\in\mathcal{A}} \|\nabla(h^k \circ \phi)(\mathbb{O})\| \le 2\alpha_k + n + 1,$$

for every $k = 1, \dots, n$ and $h^k \in \mathcal{B}$. Because $\mathrm{D}(h \circ \phi)(\mathbb{O}) = (\frac{\partial(h^j \circ \phi)}{\partial z_k}(\mathbb{O}))_{j,k=1}^n$, then for every $\phi \in \mathcal{A}$, we have

$$\begin{aligned}\|\mathrm{D}(h \circ \phi)(\mathbb{O})\| &= \sup_{\|w\|=1} \|\mathrm{D}(h \circ \phi)(\mathbb{O})w\| \\ &= \sup_{\|w\|=1} \|(\langle \nabla(h^1 \circ \phi)(\mathbb{O}), \bar{w}\rangle, \dots, \langle \nabla(h^n \circ \phi)(\mathbb{O}), \bar{w}\rangle\| \\ &\le \sqrt{\sum_{k=1}^n \|\nabla(h^k \circ \phi)(\mathbb{O})\|^2} \le \sqrt{\sum_{k=1}^n (2\alpha_k + n + 1)^2}.\end{aligned}$$

By the above we get that $h \in \mathcal{B}(n)$ and

$$\|h - h(\mathbb{O})\|_{\mathcal{B}(n)} \le \sqrt{\sum_{k=1}^n (2\alpha_k + n + 1)^2}.$$

From the proof of Theorem 1 exactness of the last inequality follows. The equality is attained for the mapping $h = (h_{\alpha_1}, \dots, h_{\alpha_n})$, where h_{α_k} were defined in Theorem 2.1.

Conversely, let $h \in \mathcal{B}(n)$, $h = (h^1, \dots, h^n) = (\log J_{f_1}, \dots, \log J_{f_n})$, where (similarly as in the proof of Theorem 2.1)

$$f_k(z) = \left(z_1, \dots, z_{n-1}, \int_0^{z_n} \exp\left[h^k(z_1, \dots, z_{n-1}, s) - h^k(\mathbb{O})\right] ds\right) \in \mathcal{LS}_n,$$

$k = 1, \dots, n$.

Then by Definition 3.1 there is a constant $C = C(h)$ such that for every automorphism $\phi \in \mathcal{A}$ holds $\|\mathrm{D}(h \circ \phi)(\mathbb{O})\| \le C$, which is equivalent to

$$\sup_{\|w\|=1, \phi \in \mathcal{A}} \|(\langle \nabla(h^1 \circ \phi)(\mathbb{O}), \bar{w}\rangle, \dots, \langle \nabla(h^n \circ \phi)(\mathbb{O}), \bar{w}\rangle\| \le C.$$

Thus for every $k = 1, \dots, n$ $\sup_{\phi \in \mathcal{A}} \|\nabla(h^k \circ \phi)(\mathbb{O})\| \le C$, or equivalently $h^k \in \mathcal{B}$ by Definition 2.1. By Theorem 2.1 $\operatorname{ord} f_k = \alpha_k < \infty$, which means that $f_1, \dots, f_n \in \cup_{\alpha < \infty} \mathcal{U}_\alpha$. Then we obtain

$$2\alpha_k - n - 1 \le \sup_{\phi \in \mathcal{A}} \|\nabla(h^k \circ \phi)(\mathbb{O})\| = \sup_{\phi \in \mathcal{A}, \|w\|=1} |\langle \nabla(h^k \circ \phi)(\mathbb{O}), \bar{w}\rangle|,$$

and therefore

$$\begin{aligned}\|h - h(\mathbb{O})\|_{\mathcal{B}(n)} &= \sup_{\phi \in \mathcal{A}, \|w\|=1} \|\mathrm{D}(h \circ \phi)(\mathbb{O})w\| \\ &= \sup_{\|w\|=1, \phi \in \mathcal{A}} \|(\langle \nabla(h^1 \circ \phi)(\mathbb{O}), \bar{w}\rangle, \dots, \langle \nabla(h^n \circ \phi)(\mathbb{O}), \bar{w}\rangle)\| \\ &\ge \sqrt{\sum_{k=1}^n (2\alpha_k - n - 1)^2}.\end{aligned}$$

The equality holds for $h(z) \equiv \mathbb{O}$. $\square$

Remark 3.1. *A holomorphic mapping $h = (h_1, \dots, h_n)$ belongs to $\mathcal{B}(n)$ if and only if for every $k = 1, \dots, n$ a function h_k belongs to $\mathcal{B}$.*

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