ANNALES
UNIVERSITATIS MARIAE CURIE – SKŁODOWSKA
LUBLIN – POLONIA

VOL. LVII, 6 SECTIO A 2003

JAN KUREK and WŁODZIMIERZ M. MIKULSKI

Liftings of horizontal 1-forms to some vector bundle functors on fibered fibered manifolds

Abstract. Let $F : \mathcal{F}^2\mathcal{M} \to \mathcal{VB}$ be a vector bundle functor on fibered fibered manifolds. We classify all natural operators

$$T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$$

transforming $\mathcal{F}^2\mathcal{M}$-projectable vector fields on Y to functions on the dual bundle $(FY)^*$ for any (m_1, m_2, n_1, n_2)-dimensional fibered fibered manifold Y. Next, under some assumption on F we study natural operators

$$T^*_{\mathcal{F}^2\mathcal{M}-hor|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^*(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$$

lifting $\mathcal{F}^2\mathcal{M}$-horizontal 1-forms on Y to 1-forms on $(FY)^*$ for any Y as above. As an application we classify natural operators

$$T^*_{\mathcal{F}^2\mathcal{M}-hor|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^*(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$$

for a particular vector bundle functor F on fibered fibered manifolds.

0. Introduction. The concept of fibered fibered manifolds was introduced in [16]. Fibered fibered manifolds are fibered surjective submersions between

2000 *Mathematics Subject Classification.* 58A20.
Key words and phrases. Fibered fibered manifold, bundle functor, natural operator.

fibered manifolds. They appear naturally in differential geometry if we consider transverse natural bundles in the sense of R. Wolak [18]. Product preserving bundle functors on fibered fibered manifolds are studied in [17].

In this paper we consider the following categories over manifolds: the category $\mathcal{M}f$ of manifolds and maps, the category $\mathcal{M}f_m$ of m-dimensional manifolds and embeddings, the category $\mathcal{FM}$ of fibered manifolds and fibered maps, the category $\mathcal{FM}_{m,n}$ of fibered manifolds of dimension (m, n) (i.e. with m-dimensional bases and n-dimensional fibers) and fibered embeddings, the category $\mathcal{F}^2\mathcal{M}$ of fibered fibered manifolds and their fibered fibered maps, the category $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ of fibered fibered manifolds of dimension (m_1, m_2, n_1, n_2) and fibered fibered embeddings, the category $\mathcal{VB}$ of vector bundles and vector bundle maps.

The notions of bundle functors and natural operators can be found in the fundamental monograph [4].

In [7], given a vector bundle functor $F : \mathcal{M}f \to \mathcal{VB}$ we classified all natural operators $A : T_{|\mathcal{M}f_m} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{M}f_m})^*$ transforming vector fields Z on m-dimensional manifolds M into functions $A(Z) : (FM)^* \to \mathbf{R}$ on the dual vector bundle $(FM)^*$ and proved that every natural operator $B : T^*_{|\mathcal{M}f_m} \rightsquigarrow T^*(F_{|\mathcal{M}f_m})^*$ transforming 1-forms ω from m-manifolds M into 1-forms $B(\omega)$ on $(FM)^*$ is of the form $B(\omega) = a\omega^V + \lambda$ for some uniquely determined canonical map $a : (FM)^* \to \mathbf{R}$ and some canonical 1-form λ on $(FM)^*$. These results were generalizations of [1],[6].

In [8], we studied similar problems for a vector bundle functor $F : \mathcal{FM} \to \mathcal{VB}$ on fibered manifolds instead of on manifolds. For natural numbers m and n we classified all natural operators $A : T_{proj|\mathcal{FM}_{m,n}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{FM}_{m,n}})^*$ transforming projectable vector fields Z on (m, n)-dimensional fibered manifolds Y into functions $A(Z) : (FY)^* \to \mathbf{R}$ on the dual vector bundle $(FY)^*$ and proved (under some assumption on F) that every natural operator $B : T^*_{hor|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(F_{|\ \mathcal{FM}_{m,n}})^*$ transforming horizontal 1-forms ω from (m, n)-dimensional fibered manifolds Y into 1-forms $B(\omega)$ on $(FY)^*$ is of the form $B(\omega) = a\omega^V + \lambda$ for some uniquely determined canonical map $a : (FY)^* \to \mathbf{R}$ and some canonical 1-form λ on $(FY)^*$.

In the present paper we study similar problems for a vector bundle functor $F : \mathcal{F}^2\mathcal{M} \to \mathcal{VB}$ on fibered fibered manifolds instead of on manifolds or on fibered manifolds. For natural numbers m_1, m_2, n_1 and n_2 we classify all natural operators $A: T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$ transforming $\mathcal{F}^2\mathcal{M}$-projectable vector fields Z on (m_1, m_2, n_1, n_2)-dimensional fibered fibered manifolds Y into functions $A(Z) : (FY)^* \to \mathbf{R}$ on the dual vector bundle $(FY)^*$ and prove (under an assumption on F) that every natural operator

$$B : T^*_{\mathcal{F}^2\mathcal{M}-hor|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^*(F_{|\ \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$$

transforming $\mathcal{F}^2\mathcal{M}$-horizontal 1-forms ω from $(m_1, m_2,\ n_1, n_2)$-dimensional fibered fibered manifolds Y into a 1-form $B(\omega)$ on $(FY)^*$ is of the form $B(\omega) = a\omega^V + \lambda$ for some uniquely determined canonical map $a : (FY)^* \to \mathbf{R}$ and some canonical 1-form λ on $(FY)^*$. As an application we classify all natural operators $T^*_{\mathcal{F}^2\mathcal{M}-hor|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^*(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$ for a particular vector bundle functor F on fibered fibered manifolds.

Natural operators lifting functions, vector fields and 1-form to some bundle functors were used practically in all papers in which problem of prolongations of geometric structures was studied, e.g. [19]. That is why such natural operators have been classified, see [1], [3]—[14], etc.

From now on the usual coordinates on $\mathbf{R}^{m_1,m_2,n_1,n_2} = \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ will be denoted by $x^1, ..., x^{m_1}, y^1, ..., y^{m_2}, w^1, ..., w^{n_1}, v^1, ..., v^{n_2}$.

All manifolds are assumed to be finite dimensional and smooth, i.e. of class $\mathcal{C}^\infty$. Maps between manifolds are assumed to be smooth.

1. Fibered fibered manifolds. The concept of fibered fibered manifolds was introduced in [16]. A fibered fibered manifold is a fibered surjective submersion $\pi : Y \to X$ between fibered manifolds, i.e. a surjective submersion which sends fibers into fibers such that the restricted and corestricted maps are submersions. (We will write Y instead of π if π is clear.) If $\overline{\pi} : \overline{Y} \to \overline{X}$ is another fibered fibered manifold, a morphism $\pi \to \overline{\pi}$ is a fibered map $f : Y \to \overline{Y}$ such that there is a fibered map $f_o : X \to \overline{X}$ with $\overline{\pi} \circ f = f_o \circ \pi$. Thus all fibered fibered manifolds form a category which will be denoted by $\mathcal{F}^2\mathcal{M}$. This category is over manifolds, local and admissible in the sense of [4].

Fibered fibered manifolds appear naturally in differential geometry. To see this, we consider a fibered manifold $p : X \to M$. Then X has the foliated structure $\mathcal{F}$ by fibres. Its normal bundle $Y = \mathcal{N}(X, \mathcal{F}) = TX/T\mathcal{F}$ has the induced foliation, [18]. This foliation is by the fibered manifold $[Tp] : Y \to TM$, the quotient map of the differential $Tp : TX \to TM$. Clearly, the projection $\pi : Y \to X$ of the normal bundle is a fibered fibered manifold. Considering other transverse natural bundles in the sense of [18] instead of $\mathcal{N}(X, \mathcal{F})$, we can produce many fibered fibered manifolds.

A fibered fibered manifold $\pi : Y \to X$ has dimension (m_1, m_2, n_1, n_2) if fibered manifold Y has dimension $(m_1 + n_1, m_2 + n_2)$ and fibered manifold X has dimension (m_1, m_2). All fibered fibered manifolds of dimension (m_1, m_2, n_1, n_2) and their local isomorphisms form a subcategory $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2} \subset \mathcal{F}^2\mathcal{M}$. Every $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object is locally isomorphic to $\mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \to \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}$, the projection, where $\mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ (or $\mathbf{R}^{m_1} \times \mathbf{R}^{m_2}$) is over $\mathbf{R}^{m_1} \times \mathbf{R}^{n_1}$ (or $\mathbf{R}^{m_1}$).

2. A classification of natural operators $T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{F}\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$. Let $F : \mathcal{F}^2\mathcal{M} \to \mathcal{VB}$ be a vector bundle functor. Let $m_1, m_2, n_1, n_2 \in \mathbf{N}$. In this section we classify natural operators $A : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$ transforming $\mathcal{F}^2\mathcal{M}$-projectable vector fields Z on (m_1, m_2, n_1, n_2)-dimensional fibered fibered manifolds Y into functions $A(Z) : (FY)^* \to \mathbf{R}$ on the dual vector bundle $(FY)^*$.

We recall (see [16]) that a $\mathcal{F}^2\mathcal{M}$-projectable vector field on a fibered fibered manifold $\pi : Y \to X$ is a projectable vector field Z on fibered manifold Y such that there exists a π-related (with Z) projectable vector field Z_o on fibered manifold X. If Z is $\mathcal{F}^2\mathcal{M}$-projectable then its flow is formed by local $\mathcal{F}^2\mathcal{M}$-isomorphisms.

Example 1. Let $v \in F_0(\mathbf{R}^{1,0,0,0})$. Consider a $\mathcal{F}^2\mathcal{M}$-projectable vector field Z on an (m_1, m_2, n_1, n_2)-dimensional fibered fibered manifold $\pi : Y \to X$. We define $A^v(Z) : (FY)^* \to \mathbf{R}$, $A^v(Z)_\eta = \langle \eta, F(\Phi^Z_y)(v)\rangle$, $\eta \in (F_yY)^*$, $y \in Y_x$, $x \in X$. Here $\Phi^Z_y : (\epsilon, \epsilon) \to Y$, $\Phi^Z_y(t) = \mathrm{Exp}(tZ)_y$, $t \in (-\epsilon, \epsilon)$, $\epsilon > 0$. We consider Φ^X_y as fibered fibered map $\mathbf{R}^{1,0,0,0} \to Y$. The correspondence $A^v : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$ is a natural operator.

Proposition 1. *Let $v_1, ..., v_L \in F_0\mathbf{R}^{1,0,0,0}$ be a basis. Every natural operator $A : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$ is of the form*

$$A = H(A^{v_1}, ..., A^{v_L})$$

for some uniquely determined smooth map $H \in \mathcal{C}^\infty(\mathbf{R}^L)$.

Proof. Let $v_1^*, ..., v_L^* \in (F_0\mathbf{R}^{1,0,0,0})^*$ be the dual basis. Let $q = x^1 : \mathbf{R}^{m_1,m_2,n_1,n_2} \to \mathbf{R}$ be the projection onto the first factor. It is a fibered fibered map $\mathbf{R}^{m_1,m_2,n_1,n_2} \to \mathbf{R}^{1,0,0,0}$. For A as above we define $H : \mathbf{R}^L \to \mathbf{R}$,

$$H(t_1, ..., t_L) = A\left(\frac{\partial}{\partial x^1}\right)_{(F_0q)^*(\sum_{s=1}^L t_s v_s^*)} .$$

We prove that $A = H(A^{v_1}, ..., A^{v_L})$. Since any $\mathcal{F}^2\mathcal{M}$-projectable vector field Z on an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object Y such that its underlying projectable vector field has non-vanishing underlying vector field is locally $\frac{\partial}{\partial x^1}$ in some local fibered fibered coordinates on Y, it is sufficient to show that $A(\frac{\partial}{\partial x^1})_\eta = H(A^{v_1}(\frac{\partial}{\partial x^1})_\eta, ..., A^{v_L}(\frac{\partial}{\partial x^1})_\eta)$ for any $\eta \in (F_0\mathbf{R}^{m_1,m_2,n_1,n_2})^*$. By the invariance of A and A^{v_s} with respect to $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-morphisms $(x^1, \frac{1}{t}x^2, ..., \frac{1}{t}x^m, \frac{1}{t}y^1, ..., \frac{1}{t}y^{m_2}, \frac{1}{t}w^1, ..., \frac{1}{t}w^{n_1}, \frac{1}{t}v^1, ..., \frac{1}{t}v^{n_2}) : \mathbf{R}^{m_1,m_2,n_1,n_2} \to \mathbf{R}^{m_1,m_2,n_1,n_2}$ for $t \neq 0$ and next putting $t \to 0$, we can assume that

$\eta = (F_0 q)^*(\sum_{s=1}^{L} t_s v_s^*)$. Now, it remains to observe that $A^{v_s}(\frac{\partial}{\partial x^1})_\eta = t_s$ for $s = 1, ..., L$.

The uniqueness of H is clear as $(A^{v_s}(\frac{\partial}{\partial x^1}))_{s=1}^{L}$ is a surjection onto $\mathbf{R}^L$. □

We have functors $i_\alpha : \mathcal{M}f \to \mathcal{FM}$, $i_1(M) = (\mathrm{id}_M : M \to M)$, $i_2(M) = (M \to pt)$, $i_\alpha(f) = f : i_\alpha(M) \to i_\alpha(N)$, $\alpha = 1, 2$, $M \in obj(\mathcal{M}f)$, $f : M \to N$ is a map, pt is one point manifold. We have also a functor $j : \mathcal{M}f \to \mathcal{F}^2\mathcal{M}$, $j(M) = (\mathrm{id}_M : i_1(M) \to i_2(M))$, $j(f) = f : j(M) \to j(N)$, $M \in obj(\mathcal{M}f)$, $f : M \to N$ a map.

Thus we have a vector bundle functor $F \circ j : \mathcal{M}f \to \mathcal{VB}$. So, by [2], we can choose a basis $v_1, ..., v_L \in F_0\mathbf{R}^{1,0,0,0} = (F \circ j)_0\mathbf{R}$ such that v_s is homogeneous of weight $n_s \in \mathbf{N} \cup \{0\}$, i.e. $F(\tau \mathrm{id})(v_s) = \tau^{n_s} v_s$ for any $\tau \in \mathbf{R}$.

(*) By a permutation we assume that $v_1, ..., v_{k_1}$ are of weight $0, v_{k_1+1}, ...,$ v_{k_2} are of weight 1, etc.

Then $A^{v_1}(Z), ..., A^{v_{k_1}}(Z)$ do not depend on Z, i.e. $A^{v_1}, ..., A^{v_{k_1}}$ are natural functions on $(FY)^*$. Moreover $A^{v_{k_1+1}}(Z), ..., A^{v_{k_2}}(Z)$ depend linearly on Z, i.e. $A^{v_{k_1+1}}, ..., A^{v_{k_2}}$ are linear operators.

Corollary 1. *Every natural (canonical) function G on $(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$ is of the form*

$$G = K(A^{v_1}, ..., A^{v_{k_1}})$$

for some uniquely determined $K \in \mathcal{C}^\infty(\mathbf{R}^{k_1})$. If $F \circ j$ has the point property, i.e. $F \circ j(pt) = pt$, then $G = const$.

Corollary 2. *Let $A\colon T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{FM}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$ be a natural linear operator. Then*

$$A = \sum_{s=k_1+1}^{k_2} K_s(A^{v_1}, ..., A^{v_{k_1}}) A^{v_s}$$

for some uniquely determined $K_s \in \mathcal{C}^\infty(\mathbf{R}^{k_1})$.

Proof. The corollaries are consequences of Proposition 1 and the homogeneous function theorem, [4]. □

3. A decomposition proposition. Let F and $v_1, ..., v_L$ be as in Section 1 with the assumption (*). Let $j : \mathcal{M}f \to \mathcal{F}^2\mathcal{M}$ be the functor as in Section 2.

Let $\pi : Y \to X$ be a fibered fibered manifold. A 1-form $\omega : TY \to \mathbf{R}$ on Y is called $\mathcal{F}^2\mathcal{M}$-horizontal if $\omega|VY = 0$ and $\omega|\tilde{V}Y = 0$, where VY is the vertical bundle of the fibered manifold Y and $\tilde{V}Y$ is the vertical bundle of fibered manifold $\pi : Y \to X$.

In this section we study natural operators $B : T^*_{\mathcal{F}^2\mathcal{M}-hor|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^*(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$ transforming $\mathcal{F}^2\mathcal{M}$-horizontal 1-forms ω on fibered fibered manifolds Y of dimension (m_1, m_2, n_1, n_2) into 1-forms $B(\omega)$ on the dual vector bundle $(FY)^*$.

Example 2. If $\omega : TY \to \mathbf{R}$ is a $\mathcal{F}^2\mathcal{M}$-horizontal 1-form on a fibered fibered manifold $\pi : Y \to X$, we have its vertical lifting $B^V(\omega) = \omega \circ T\pi^F : T(FY)^* \to \mathbf{R}$ to $(FY)^*$, where $\pi^F : (FY)^* \to Y$ is the bundle projection. The correspondence $B^V : T^*_{\mathcal{F}^2\mathcal{M}-hor|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^*(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$ is a natural operator.

Assumption 1. From now on we assume that there exists a basis $w_1, ..., w_K \in F_0\mathbf{R}^{m_1,m_2,n_1,n_2}$ such that w_s is homogeneous of weight $n_s \in \mathbf{N} \cup \{0\}$. It means that $F(\tau\mathrm{id})(w_s) = \tau^{n_s} w_s$ for any $\tau \in \mathbf{R}$.

Remark 1. It seems that every vector bundle functor $F : \mathcal{F}^2\mathcal{M} \to \mathcal{VB}$ satisfies Assumption 1.

Proposition 2 (Decomposition Proposition). *Consider a natural operator $B : T^*_{\mathcal{F}^2\mathcal{M}-hor|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^*(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$. Under Assumption 1 there exists the uniquely determined natural function a on $(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$ such that*

$$B = aB^V + \lambda$$

for some canonical 1-form λ on $(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$.

Lemma 1.

(a) *We have $(B(\omega)-B(0))|(V(F\mathbf{R}^{m_1,m_2,n_1,n_2})^*)_0 = 0$ for any $\mathcal{F}^2\mathcal{M}$-horizontal 1-form ω on $\mathbf{R}^{m_1,m_2,n_1,n_2}$, where $(V(F\mathbf{R}^{m_1,m_2,n_1,n_2})^*)_0$ is the fiber over $0 \in \mathbf{R}^{m_1,m_2,n_1,n_2}$ of the π^F-vertical subbundle in $T(F\mathbf{R}^{m_1,m_2,n_1,n_2})^*$.*

(b) *If $F \circ j$ has the point property then $B(\omega)|(V(F\mathbf{R}^{m_1,m_2,n_1,n_2})^*)_0 = 0$ for any $\mathcal{F}^2\mathcal{M}$-horizontal 1-form ω on $\mathbf{R}^{m_1,m_2,n_1,n_2}$.*

Proof.

ad (a) We use the invariance of $(B(\omega)-B(0))|(V(F\mathbf{R}^{m_1,m_2,n_1,n_2})^*)_0$ with respect to the homotheties $\frac{1}{t}\mathrm{id}_{\mathbf{R}^{m_1,m_2,n_1,n_2}}$ for $t \neq 0$ and apply the homogeneous function theorem. We obtain that $(B(\omega)-B(0))|(V(F\mathbf{R}^{m_1,m_2,n_1,n_2})^*)_0$ is independent of ω. This ends the proof of the part (a).

ad (b) We observe that if $F \circ j$ has the point property then $(F_0\mathbf{R}^{m_1,m_2,n_1,n_2})^*$ has no non-zero homogeneous elements of weight 0. Next, we use the invariance of $B(\omega)|(V(F\mathbf{R}^{m_1,m_2,n_1,n_2})^*)_0$ with respect to the homotheties $\frac{1}{t}\mathrm{id}_{\mathbf{R}^{m_1,m_2,n_1,n_2}}$ for $t \neq 0$ and put $t \to 0$. □

Proof of Proposition 2. Clearly, $B(0)$ is a canonical 1-form. Then replacing B by $B - B(0)$ we have $B(0) = 0$ and $B(\omega)|(V(F\mathbf{R}^{m_1,m_2,n_1,n_2})^*)_0 = 0$. Then B is determined by the values $\langle B(\omega)_\eta, F^*(\frac{\partial}{\partial x^1})_\eta\rangle$ for all $\mathcal{F}^2\mathcal{M}$-horizontal 1-forms $\omega = \sum_{i=1}^{m_1}\omega_i dx^i$ on $\mathbf{R}^{m_1,m_2,n_1,n_2}$ and $\eta \in (F_0\mathbf{R}^{m_1,m_2,n_1,n_2})^*$, where $F^*(\frac{\partial}{\partial x^1})$ is the complete lifting (flow prolongation) of $\frac{\partial}{\partial x^1}$ to $(F\mathbf{R}^{m_1,m_2,n_1,n_2})^*$.

Using the invariance of B with respect to the homotheties $\frac{1}{t}\mathrm{id}_{\mathbf{R}^{m_1,m_2,n_1,n_2}}$ for $t \neq 0$ we get the homogeneity condition

$$t\langle B(\omega)_\eta, F^*\left(\frac{\partial}{\partial x^1}\right)_\eta\rangle = \langle B((t\,\mathrm{id}_{\mathbf{R}^{m_1,m_2,n_1,n_2}})^*\omega)_{F(\frac{1}{t}\mathrm{id}_{\mathbf{R}^{m_1,m_2,n_1,n_2}})^*(\eta)},$$
$$F^*\left(\frac{\partial}{\partial x^1}\right)_{F(\frac{1}{t}\mathrm{id}_{\mathbf{R}^{m_1,m_2,n_1,n_2}})^*(\eta)}\rangle$$

Then by the non-linear Petree theorem [4], the homogeneous function theorem and $B(0) = 0$ we deduce that $\langle B(\omega)_\eta, F^*(\frac{\partial}{\partial x^1})_\eta\rangle$ is a linear combination of $\omega_1(0), ..., \omega_{m_1}(0)$ with coefficients being smooth maps in homogeneous coordinates of η of weight 0.

Then using the invariance of B with respect to $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-morphisms $(x^1, \frac{1}{t}x^2, ..., \frac{1}{t}x^m, \frac{1}{t}y^1, ..., \frac{1}{t}y^{m_2}, \frac{1}{t}w^1, ..., \frac{1}{t}w^{n_1}, \frac{1}{t}v^1, ..., \frac{1}{t}v^{n_2}) : \mathbf{R}^{m_1,m_2,n_1,n_2} \to \mathbf{R}^{m_1,m_2,n_1,n_2}$ for $t \neq 0$ and put $t \to 0$ we end the proof. □

4. On canonical 1-forms on $(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$.

Proposition 3. *Every canonical 1-form λ on $(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$ induces a linear natural operator*

$$A^{(\lambda)} : T_{\mathcal{F}^2\mathcal{M}-proj|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$$

such that $A^{(\lambda)}(Z)_\eta = \langle\lambda_\eta, F^*(Z)_\eta\rangle$, $\eta \in (FY)^*$, Z is a $\mathcal{F}^2\mathcal{M}$-projectable vector field on Y, where $F^*(Z)$ is the complete lifting (flow operator) of Z to $(FY)^*$. If $F \circ j$ has the point property, then (under Assumption 1) the correspondence "$\lambda \to A^{(\lambda)}$" is a linear injection.

Proof. The injectivity is a consequence of Lemma 1 (b). □

5. A corollary.

Let $j : \mathcal{M}f \to \mathcal{F}^2\mathcal{M}$ be the functor as in Section 2.

Corollary 3. *Assume that $F\circ j$ has the point property and there are no non-zero elements from $F_0\mathbf{R}^{1,0,0,0}$ of weight 1. (For example, let $F = F_1 \otimes F_2 : \mathcal{F}^2\mathcal{M} \to \mathcal{VB}$ be the tensor product of two vector bundle functors F_1, F_2 :*

$\mathcal{F}^2\mathcal{M} \to \mathcal{VB}$ such that $F_1 \circ j, F_2 \circ j$ have the point property.) Then (under Assumption 1) every natural operator $B : T^*_{\mathcal{F}^2\mathcal{M}-hor|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^*(F_{|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}})^*$ is a constant multiple of the vertical lifting.

Proof. Since there are no non-zero elements from $F_0\mathbf{R}^{1,0,0,0}$ of weight 1, we see that every canonical 1-form on $(F|\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})^*$ is zero because of Corollary 2 and Proposition 3. Then Proposition 2 together with Corollary 1 ends the proof. □

6. An application. Let $r_1, r_2, ..., r_8 \in \mathbf{N}$ be such that $r_8 \geq r_4 \leq r_5 \geq r_3$ and $r_8 \geq r_6 \leq r_7 \geq r_2$ and $r_1 \leq r_i$ for $i = 2, 3, ..., 8$.

The concept of r-jets and (r, s, q)-jets can be generalized as follows. Let $\pi : Y \to X$ be a fibered fibered manifold being surjecive fibered submersion between fibered manifolds $p^Y : Y \to \underline{Y}$ and $p^X : X \to \underline{X}$. Let $\pi' : Y' \to X'$ be another fibered fibered manifold being surjective fibered submersion between $p^{Y'} : Y' \to \underline{Y}'$ and $p^{X'} : X' \to \underline{X}'$. Let $y \in Y$ be a point and $\underline{y} = p^Y(y) \in \underline{Y}$, $x = \pi(y) \in X$ and $\underline{x} = p^X(x) \in \underline{X}$ be its underlying points. Let $f, g : Y \to Y'$ be two fibered fibered maps and $\underline{f}, \underline{g} : \underline{Y} \to \underline{Y}'$, $f_o, g_o : X \to X'$ and $\underline{f}_o, \underline{g}_o : \underline{X} \to \underline{X}'$ be their underlying maps. We say that f, g determine the same $(r_1, ..., r_8)$-jet $j^{(r_1,...,r_8)}_y f = j^{(r_1,...,r_8)}_y g$ at $y \in Y$ if $j^{r_1}_y f = j^{r_1}_y g$, $j^{r_2}_y(f|Y_x) = j^{r_2}_y(g|Y_x)$, $j^{r_3}_y(f|Y_{\underline{y}}) = j^{r_3}_y(g|Y_{\underline{y}})$, $j^{r_4}_x(f_o) = j^{r_4}_x(g_o)$, $j^{r_5}_x(f_o|X_{\underline{x}}) = j^{r_5}_x(g_o|X_{\underline{x}})$, $j^{r_6}_{\underline{y}}(\underline{f}) = j^{r_6}_{\underline{y}}(\underline{g})$, $j^{r_7}_{\underline{y}}(\underline{f}|\underline{Y}_{\underline{x}}) = j^{r_7}_{\underline{y}}(\underline{g}|\underline{Y}_{\underline{x}})$ and $j^{r_8}_{\underline{x}}(\underline{f}_o) = j^{r_8}_{\underline{x}}(\underline{g}_o)$. The space of all $(r_1, r_2, ..., r_8)$-jets of Y into Y' is denoted by $J^{(r_1,...,r_8)}(Y, Y')$. The composition of fibered fibered maps induces the composition of $(r_1, ..., r_8)$-jets.

The (described in [4] and [5],[15]) vector bundle functors $T^{(r)} = (J^r(., \mathbf{R})_0)^* : \mathcal{M}f \to \mathcal{VB}$ and $T^{(r,s,q)} = (J^{(r,s,q)}(., \mathbf{R}^{1,1})_0)^* : \mathcal{FM} \to \mathcal{VB}$ can be generalized as follows. The space $J^{(r_1,...,r_8)}(Y, \mathbf{R}^{1,1,1,1})_0$, $0 \in \mathbf{R}^4$, has an induced structure of a vector bundle over Y. Every fibered fibered map $f : Y \to Y'$, $f(y) = y'$, induces a linear map $\lambda(j^{(r_1,...,r_8)}_y f) : J^{(r_1,...,r_8)}_{y'}(Y', \mathbf{R}^{1,1,1,1})_0 \to J^{(r_1,...,r_8)}_y(Y, \mathbf{R}^{1,1,1,1})_0$ by means of the jet composition. If we denote by $T^{(r_1,...,r_8)}Y$ the dual vector bundle of $J^{(r_1,...,r_8)}(Y, \mathbf{R}^{1,1,1,1})_0$ and define $T^{(r_1,...,r_8)}f : T^{(r_1,...,r_8)}Y \to T^{(r_1,...,r_8)}Y'$ by using the dual maps to $\lambda(j^{(r_1,...,r_8)}_y f)$, we obtain a vector bundle functor $T^{(r_1,...,r_8)} : \mathcal{F}^2\mathcal{M} \to \mathcal{VB}$.

Example 3. We have 1-forms $\lambda^{(r_1,...,r_8)}_\alpha : TJ^{(r_1,...,r_8)}(Y, \mathbf{R}^{1,1,1,1})_0 \to \mathbf{R}$ on $J^{(r_1,...,r_8)}(Y, \mathbf{R}^{1,1,1,1})_0$, $\alpha = 1, 2, 3, 4$, $\lambda^{(r_1,...,r_8)}_\alpha(v) = d\gamma_\alpha(T\tilde{\pi}(v))$, $v \in T_w J^{(r_1,...,r_8)}(Y, \mathbf{R}^{1,1,1,1})_0$, $w = j^{(r_1,...,r_8)}_y(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, $y \in Y$, $\tilde{\pi} : J^{(r_1,...,r_8)}(Y, \mathbf{R}^{1,1,1,1})_0 \to Y$ is the bundle projection.

Corollary 4. *Every natural operator*

$$B : T^*_{\mathcal{F}^2\mathcal{M}-hor|\mathcal{FM}_{m_1,m_2,n_1,n_2}} \rightsquigarrow T^*(J^{(r_1,\dots,r_8)}(.,\mathbf{R}^{1,1,1,1})_0)$$

is a linear combination of the vertical lifting B^V *and the canonical* 1*-forms* $\lambda_\alpha^{(r_1,\dots,r_8)}$ *for* $\alpha = 1,2,3,4$ *with real coefficients.*

Proof. The vector bundle functor $T^{(r_1,\dots,r_8)}$ satisfies Assumption 1. Moreover, $T^{(r_1,\dots,r_8)} \circ j$ has the point property and the subspace of elements from $T_0^{(r_1,\dots,r_8}\mathbf{R}^{1,0,0,0}$ of weight 1 is 4-dimensional. Then by Proposition 3 together with Corollaries 1 and 2, the space of canonical 1-forms on $J^{(r_1,\dots,r_8)}(.,\mathbf{R}^{1,1,1,1})_0$ is at most 4-dimensional. Now, Proposition 2 ends the proof. □

References

[1] Doupovec, M., J. Kurek, *Liftings of tensor fields to the cotangent bundle*, Differential Geometry and Applications, Proceedings of the International Conference Brno (1995), 141–150.

[2] Epstein, D.B.A., *Natural Vector Bundles*, Category Theory, Homology Theory and Their Applications III, Lecture Notes in Mathematics, vol. 99, Springer-Verlag, Berlin–New York, 1969.

[3] Gancarzewicz, J., *Liftings of functions and vector fields to natural bundles*, Dissert. Math. CCXII, Warszawa, 1983.

[4] Kolář, I., P.W. Michor and J. Slovák, *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993.

[5] Kolář, I., W.M. Mikulski, *Contact elements on fibered manifolds*, Czechoslovak Math. J. (to appear).

[6] Kurek, J., W.M. Mikulski, *The natural operators lifting 1-forms to the r-jet prolongation of the cotangent bundle*, Proc. Conf. on Diff. Geom. and Appl. Opava 2001, Math. Publ. Silesian Univ. Opava (2001), 215–227.

[7] Kurek, J., W.M. Mikulski, *The natural operators lifting 1-forms to some vector bundle functors*, Colloq. Math. **93(2)** (2002), 259–265.

[8] Kurek, J., W.M. Mikulski, *The natural operators lifting horizontal 1-forms to some vector bundle functors on fibered manifolds*, Colloq. Math. (to appear).

[9] Mikulski, W.M., *Natural transformations transforming functions and vector fields to functions on some natural bundles*, Math. Bohemica **117** (1992), 217–223.

[10] Mikulski, W.M., *The natural operators lifting 1-forms on manifolds to the bundles of A-velocities*, Monatsh. Math. **119** (1995), 63–77.

[11] Mikulski, W.M., *The natural operators lifting 1-forms to r-jet prolongation of the tangent bundle*, Geometriae Dedicata **68** (1997), 1–20.

[12] Mikulski, W.M., *Liftings of 1-forms to the linear r-tangent bundle*, Arch. Math. Brno **31(2)** (1995), 97–111.

[13] Mikulski, W.M., *Liftings of 1-forms to the bundle of affinors*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **55** (2001), 109–113.

[14] Mikulski, W.M., *Liftings of* 1*-form to* $(J^rT^*)^*$, Colloq. Math. **91(1)** (2002), 69–77.
[15] Mikulski, W.M., *Natural affinors on* $(J^{r,s,q}(.,\mathbf{R}^{1,1})_0)^*$, Comment. Math. Univ. Carolinae **42(4)** (2001), 655–663.
[16] Mikulski, W.M., *The jet prolongations of fibered fibered manifolds and the flow operator*, Publ. Math. Debrecen **59(3-4)** (2001), 441–458.
[17] Mikulski, W.M., J. Tomaš, *Product preserving bundle functors on fibered fibered manifolds*, Colloq. Math. (to appear).
[18] Wolak, R., *On transverse structures of foliations*, Suppl. Rend. Circ. Mat. Palermo **9** (1985), 227–243.
[19] Yano, K., S. Ishihara, *Tangent and Cotangent Bundles*, Marcel Dekker, Inc., New York, 1973.

Institute of Mathematics
Maria Curie-Skłodowska University
pl. M. Curie-Skłodowskiej 1
20-031 Lublin, Poland
e-mail: kurek@golem.umcs.lublin.pl

Institute of Mathematics
Jagiellonian University
ul. Reymonta 4
Kraków, Poland
e-mail: mikulski@im.uj.edu.pl

Received May 14, 2003