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# Plane convex sets via distributions

Abstract. We will establish the correspondence between convex compact subsets of $\mathbb{R}^2$ and $2\pi$–periodic distributions in $\mathbb{R}$. We also give the necessary and sufficient condition for the positively homogeneous extension $\widetilde{u} : \mathbb{R}^n \to \mathbb{R}$ of $u : S^{n-1} \to \mathbb{R}$ to be a convex function.

**1. Introduction.** We say that a $2\pi$–periodic function $p : \mathbb{R} \to \mathbb{R}$ is a support function if there exists a convex compact set $C \subset \mathbb{R}^2$ such that

$$p(t) = \max_{x \in C} \langle x, e(t) \rangle, \; t \in \mathbb{R},$$

where $e(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$ and $\langle x, y \rangle$ stands for the scalar product of vectors $x, y \in \mathbb{R}^2$.

We refer to Rademacher's test for convexity (see [7], and [1, p. 28]) as a necessary and sufficient condition for $p$ to be a support function. There are also other tests, one of them was proposed by Gelfond ([5, p. 132]), and another one by Firey ([3, p. 239, Lemma]).

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Gelfond's test. *A $2\pi$–periodic function $p:\mathbb{R}\to\mathbb{R}$ is a support function iff*

$$\det\begin{bmatrix}\cos t_1 & \sin t_1 & p(t_1)\\ \cos t_2 & \sin t_2 & p(t_2)\\ \cos t_3 & \sin t_3 & p(t_3)\end{bmatrix}\geq 0$$

*for all $0\leq t_1\leq t_2\leq t_3\leq 2\pi$, such that $t_2-t_1\leq\pi$ and $t_3-t_2\leq\pi$.*

Let

$$S^{n-1}=\{x\in\mathbb{R}^n:\|x\|=1\}.$$

We say that $p:S^{n-1}\to\mathbb{R}$ is a support function if there exists a convex compact set $C\subset\mathbb{R}^n$ such that

$$p(u)=\max_{x\in C}\langle x,u\rangle,\ u\in S^{n-1}.$$

Firey's test. *Let $\{a_1,a_2,\dots,a_n\}$ be a fixed orthonormal basis in $\mathbb{R}^n$ . A function $p:S^{n-1}\to\mathbb{R}$ is a support function iff*

$$\det\begin{bmatrix}\langle u_1,a_1\rangle & \dots & \langle u_1,a_n\rangle & p(u_1)\\ \dots & \dots & \dots & \dots\\ \langle u_n,a_1\rangle & \dots & \langle u_n,a_n\rangle & p(u_n)\\ \langle u_{n+1},a_1\rangle & \dots & \langle u_{n+1},a_n\rangle & p(u_{n+1})\end{bmatrix}$$

$$\times\det\begin{bmatrix}\langle u_1,a_1\rangle & \dots & \langle u_1,a_n\rangle\\ \dots & \dots & \dots\\ \langle u_n,a_1\rangle & \dots & \langle u_n,a_n\rangle\end{bmatrix}\leq 0$$

*for all $u_1,\dots,u_{n+1}\in S^{n-1}$, such that $u_{n+1}=\sum_{i=1}^{n}t_iu_i$, $t_i\geq 0$, $i=1,2,\dots,n$.*

In this paper we propose another test for convexity involving distributional derivatives of the function $p$.

**2. Main result.** In this section we will present the main result of the paper.

The symbol $D'(\mathbb{R})$ will stand for the space of all distributions in $\mathbb{R}$ and $\mathcal{L}^1$ will denote the Lebesgue measure in $\mathbb{R}$. Distribution theory will be the main tool used in the sequel.

**Theorem 1.** *Let $C\subset\mathbb{R}^2$ be a nonempty convex compact subset of $\mathbb{R}^2$. Define $p_C:\mathbb{R}\to\mathbb{R}$,*

$$p_C(t)=\max_{x\in C}\langle x,e(t)\rangle,$$

*where $e(t)=(\cos t,\sin t)$, $t\in\mathbb{R}$. Under these assumptions, the distribution $p_C+p_C''$ is a $2\pi$–periodic non-negative Radon measure in $\mathbb{R}$.*

**Theorem 2.** *Given a $2\pi$–periodic non-negative Radon measure $\varrho$ in $\mathbb{R}$, satisfying the condition*

$$\int_0^{2\pi} e(t)\,\varrho(dt) = 0.$$

*Let $p \in D'(\mathbb{R})$ be a distributional solution of the differential equation*

$$p + p'' = \varrho. \tag{1}$$

*Under these assumptions*
*(a) $p$ is a $2\pi$–periodic Lipschitz function,*
*(b) for each $t \in \mathbb{R}$,*

$$p(t) = \max_{x \in C_p} \langle x, e(t) \rangle$$

*where $C_p$ is the closure of the convex hull of all points of the form*

$$p(t)\,e(t) + p'(t)\,e'(t),$$

*(c) if $q \in D'(\mathbb{R})$ is another solution of (1) then*

$$C_q = C_p + w$$

*for some $w \in \mathbb{R}^2$.*

Theorems 1 and 2 establish a "local" version of the Rademacher–Gelfond's test for convexity. Proofs of Theorem 1 and Theorem 2 will be presented in sections 3 and 4.

## 3. From set to measure.

**A.** Let $C \subset \mathbb{R}^2$ be a nonempty convex compact set. Define

$$u(y) = \max_{x \in C} \langle x, y \rangle, \; y \in \mathbb{R}^2.$$

Clearly

$$p_C(t) = u(e(t)), \; t \in \mathbb{R}.$$

Since $u$ is Lipschitz and positively homogeneous, there exists a set $E \subset \mathbb{R}$ such that $\mathcal{L}^1(\mathbb{R} \setminus E) = 0$ and for each $t \in E$, $u$ has a usual derivative $u'$ at $e(t)$ and $e(t)$ is a Lebesgue point of $u'$. Indeed, if $u'(e(t))$ does not exist then $u'(\lambda e(t))$ does not exist for all $\lambda > 0$. Therefore, if the measurable set $\{t \in \mathbb{R} : u'(e(t)) \text{ does not exist}\}$ has a positive measure, then the set $\left\{x \in \mathbb{R}^2 : u'(x) \text{ does not exist}\right\}$ has a positive measure which contradicts Rademacher's theorem.

Moreover,

$$\langle u'(e(t)), e(t)\rangle = u(e(t)),\ t \in E. \tag{2}$$

**B.** Let us fix $\psi \in C_0^\infty(\mathbb{R}^2)$ with the following properties

$$\psi \geq 0,$$
$$\operatorname{supp} \psi \subset B[0,1] = \left\{x \in \mathbb{R}^2 : \|x\| \leq 1\right\},$$
$$\int_{\mathbb{R}^2} \psi(x)\, dx = 1.$$

Next, for each $\varepsilon > 0$, $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$, define

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon^2}\psi\left(\frac{x}{\varepsilon}\right)$$
$$u_\varepsilon(x) = \int_{\mathbb{R}^2} u(x-y)\,\psi_\varepsilon(y)\, dy$$
$$p_\varepsilon(t) = u_\varepsilon(e(t)).$$

Obviously, $u_\varepsilon$ is convex, both $u_\varepsilon$ and $p_\varepsilon$ are $C^\infty$ functions and $p_\varepsilon \to p_C$ uniformly in $\mathbb{R}$. Since $e'' = -e$ we have

$$p_\varepsilon''(t) = \langle u_\varepsilon''(e(t))\, e'(t), e'(t)\rangle - \langle u_\varepsilon'(e(t)), e(t)\rangle,\ t \in \mathbb{R}.$$

Consequently, for each $\varphi \in C_0^\infty(\mathbb{R})$

$$\begin{aligned}\langle p_\varepsilon + p_\varepsilon'', \varphi\rangle_{L^2} &= \int_{\mathbb{R}} (p_\varepsilon(t) + p_\varepsilon''(t))\,\varphi(t)\, dt \\ &= \int_{\mathbb{R}} \langle u_\varepsilon''(e(t))\, e'(t), e'(t)\rangle\, \varphi(t)\, dt \\ &\qquad + \int_{\mathbb{R}} (p_\varepsilon(t) - \langle u_\varepsilon'(e(t)), e(t)\rangle)\,\varphi(t)\, dt.\end{aligned}$$

By (2), see e.g. [2, Theorem 1 (iv), (v), p. 123],

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} (p_\varepsilon(t) - \langle u_\varepsilon'(e(t)), e(t)\rangle)\,\varphi(t)\, dt = 0.$$

Thus, when $\varphi \geq 0$,

$$\langle p_C + p_C'', \varphi\rangle_{L^2} = \lim_{\varepsilon \downarrow 0} \langle p_\varepsilon + p_\varepsilon'', \varphi\rangle_{L^2} \geq 0. \tag{3}$$

**C.** Clearly, $p_C + p''_C$ is $2\pi$–periodic. It follows from (3), see e.g. [6, Theorems 2.1.7, 2.1.8, 2.1.9], that $p_C + p''_C$ is a non-negative Radon measure in $\mathbb{R}$.

## 4. From measure to set.

**D.** Every solution to (1) has the form (see e.g. [4, p. 28])

$$p(t) = a\cos t + b\sin t + S(t),$$

where $a, b \in \mathbb{R}$ and

$$S(t) = \int_0^t \sin(t-s)\,\varrho(ds),\ t \in \mathbb{R}.$$

It is easy to verify that

$$\langle S', \varphi\rangle_{L^2} = \langle C, \varphi\rangle_{L^2},\ \varphi \in C_0^\infty(\mathbb{R}),$$

where

$$C(t) = \int_0^t \cos(t-s)\,\varrho(ds),\ t \in \mathbb{R}.$$

Therefore, see [2, Theorem 5, p. 131], $S$ is Lipschitz.

**E.** Let $p$ be a solution to (1). Denote by $E$ the set of all $t \in \mathbb{R}$ for which the usual derivative $p'$ exists. Let

$$z(t) \overset{\text{def}}{=} p(t)\,e(t) + p'(t)\,e'(t),\ t \in E,$$
$$Z \overset{\text{def}}{=} \{z(t) : t \in E\}.$$

We claim that

$$p(\tau) = \sup_{t\in E} \langle z(t), e(\tau)\rangle,\ \tau \in E.$$

Indeed, for $t \in E$, we have

$$\langle z(t), e(\tau)\rangle = \langle p(t)\,e(t) + p'(t)\,e'(t), e(\tau)\rangle$$

and

$$\lim_{t\to\tau} \langle z(t), e(\tau)\rangle = p(\tau).$$

On the other hand, in the sense of distribution theory,

$$\begin{aligned}\frac{d}{dt}\langle z(t), e(\tau)\rangle &= \langle p'e + pe' + p''e + p'e'', e(\tau)\rangle \\ &= (p + p'')\,\langle e'(t), e(\tau)\rangle = \varrho\sin(\tau - t).\end{aligned}$$

It follows from [6, Theorem 4.1.6], that $\langle z(t), e(\tau)\rangle$ is nondecreasing in $(\tau-\pi,\tau)$ and nonincreasing in $(\tau,\tau+\pi)$. Consequently, since $p$ is $2\pi$–periodic, we have

$$p(t)=\sup_{t\in E}\langle z(t), e(\tau)\rangle,\ \tau\in E,$$

as claimed.

**F.** Let $C_p$ be the closure of the convex hull of $Z$. Obviously,

$$p(\tau)=\max_{x\in C_p}\langle x, e(\tau)\rangle,\ \tau\in E.$$

Since $p$ and $e$ are continuous and $E$ is dense in $\mathbb{R}$, we have,

$$p(t)=\max_{x\in C_p}\langle x, e(t)\rangle,\ t\in \mathbb{R}.$$

**5. Convex extension.** In this section a simple application of Theorem 1 and Theorem 2 will be given. We will prove the necessary and sufficient condition for the positively homogeneous extension $\widetilde{u}:\mathbb{R}^n\to\mathbb{R}$ of $u: S^{n-1}\to\mathbb{R}$ to be a convex function.

Let $S^{n-1}=\{x\in\mathbb{R}^n:\|x\|=1\}$ and let $u:S^{n-1}\to\mathbb{R}$ be a function. For each $a,b\in S^{n-1}$ satisfying $\langle a,b\rangle=0$, define $e_{a,b}:\mathbb{R}\to S^{n-1}$, $u_{a,b}:\mathbb{R}\to\mathbb{R}$ and $\widetilde{u}:\mathbb{R}^n\to\mathbb{R}$

$$\begin{aligned} e_{a,b}(t)&=a\cos t+b\sin t,\\ u_{a,b}(t)&=u\left(e_{a,b}(t)\right),\\ \widetilde{u}(x)&=\begin{cases}\|x\|\cdot u\left(\frac{x}{\|x\|}\right), & x\neq 0\\ 0, & \text{x=0.}\end{cases}\end{aligned}$$

Recall that $u:\mathbb{R}^n\to\mathbb{R}$ is positively homogeneous if

$$u(\alpha x)=\alpha\cdot u(x)$$

for all $x\in\mathbb{R}^n$ and $\alpha>0$.

**Theorem 3.** *If $u:\mathbb{R}^n\to\mathbb{R}$ is convex and positively homogeneous then $u_{a,b}+u''_{a,b}$ is a $2\pi$–periodic, non-negative Radon measure on $\mathbb{R}$ for all $a,b\in S^{n-1}$, where $\langle a,b\rangle=0$.*

**Proof.** Fix $a,b\in S^{n-1}$, $\langle a,b\rangle=0$. Let $v:\mathbb{R}^2\to\mathbb{R}$,

$$v(x_1,x_2)=u(x_1a+x_2b)$$

be a restriction of $u$ to $\operatorname{lin}\{a, b\}$. Obviously, $v$ is convex. The set

$$C = \left\{x \in \mathbb{R}^2 : \forall_{y \in \mathbb{R}^2} \langle x, y\rangle \leq v(y)\right\}$$

is a convex compact subset of $\mathbb{R}^2$ and

$$v(y) = \max_{x \in C} \langle x, y\rangle$$

for all $y \in \mathbb{R}^2$, see e.g. [8, Corollary 13.2.1]. Consider

$$u_{a,b}(t) = u(e_{a,b}(t)) = v(e(t))$$

and apply Theorem 1 to show that $u_{a,b} + u''_{a,b}$ is a $2\pi$–periodic, non-negative Radon measure on $\mathbb{R}$. □

**Theorem 4.** *If $u : S^{n-1} \to \mathbb{R}$ is continuous and $u_{a,b} + u''_{a,b}$ is a $2\pi$ – periodic, non-negative Radon measure on $\mathbb{R}$, satisfying*

$$\int_0^{2\pi} e_{a,b}(t) \left(u_{a,b} + u''_{a,b}\right)(dt) = 0$$

*for all $a, b \in S^{n-1}$, where $\langle a, b\rangle = 0$, then $\widetilde{u} : \mathbb{R}^n \to \mathbb{R}$ is convex.*

**Proof.** Let $z, y \in \mathbb{R}^n$ be fixed. There exist $a, b \in S^{n-1}$, $\langle a, b\rangle = 0$, such that $z, y \in \operatorname{lin}\{a, b\}$. Applying Theorem 2 to the function $u_{a,b}$ , we have

$$\begin{aligned}\widetilde{u}(z+y) &= \|z+y\| \max_{x \in C} \left\langle x, \frac{z+y}{\|z+y\|}\right\rangle \\ &\leq \|z\| \max_{x \in C} \left\langle x, \frac{z}{\|z\|}\right\rangle + \|y\| \max_{x \in C} \left\langle x, \frac{y}{\|y\|}\right\rangle \\ &= \widetilde{u}(z) + \widetilde{u}(y).\end{aligned}$$

for some convex compact set $C \subset \operatorname{lin}\{a, b\}$. Therefore $\widetilde{u}$ is convex. □

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