ANNALES
UNIVERSITATIS MARIAE CURIE – SKŁODOWSKA
LUBLIN – POLONIA

VOL. LVII, 10 SECTIO A 2003

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On upper semicontinuity of geometric difference of multifunctions

ABSTRACT. The short proof of upper semicontinuity of geometric difference of multifunctions is given.

Let X and Y be two topological spaces. *A multifunction* (or a set-valued map) $F : X \to Y$ is a mapping from X to the nonempty subsets of Y; thus, for each $x \in X, F(x)$ is a nonempty set in Y.

We say that F is *upper semicontinuous (usc) at* $x \in X$ if for any open set V containing $F(x)$ there exists a neighborhood U of x such that $F(y) \subset V$ for any $y \in U$. F is usc on X if it is usc at each $x \in X$.

We say that F is *lower semicontinuous (lsc) at* $x \in X$ if for any open set V which meets $F(x)$ there exists a neighborhood U of x such that $F(y) \cap V \neq \emptyset$ for every $y \in U$. F is lsc on X if it is lsc at any $x \in X$.

If a multifunction $F : X \to Y$ is *compact-valued*, i.e. if for every $x \in X$, the set $F(x)$ is a compact set in Y, and if X and Y satisfy the "first axiom of countability", then we have the following useful conditions, which are equivalent to usc and lsc, respectively.

2000 *Mathematics Subject Classification.* 47H04, 49J53.

Key words and phrases. Lower and upper semicontinuous multifunction, geometric difference of multifunctions.

Proposition 1. ([4, Proposition 4.1, p. 48]). *A multifunction $F : X \to Y$ is usc at $x \in X$ if and only if for any sequence $\{x_n\}$ in X converging to x and for any sequence $\{y_n\}$ of elements of $F(x_n)$ there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to $y \in F(x)$.*

Proposition 2. ([3, Proposition II-2-1, p. 15]). *A multifunction $F : X \to Y$ is lsc at $x \in X$ if and only if for any $y \in F(x)$ and for any sequence $\{x_n\}$ in X converging to x there exists a sequence $\{y_n\}$ of elements of $F(x_n)$ converging to y.*

Now let Y be a linear topological space. For $A \subset Y, B \subset Y$ and $\lambda \in \mathbb{R}$ we put

$$A + B = \{a + b : a \in A, b \in B\},$$

$$\lambda A = \{\lambda a : a \in A\},$$

$$A - B = A + (-1)B.$$

The geometric difference (or Minkowski subtraction [1], [2], [5]) of the set A and B is denoted by $A \stackrel{*}{-} B$ and defined by setting

$$A \stackrel{*}{-} B = \{y \in Y : y + B \subset A\}.$$

Remark. It is worth noting here that the set $A \stackrel{*}{-} B$ is different from $A - B$.

In [1] the following theorem is proved

Theorem 1. ([1, Theorem 2.1, p. 165]). *Let X be a complete metric space, Y a separable Banach space and let $F, G : X \to Y$ be weakly compact-valued multifunction. If $F : X \to Y$ is weakly usc, G weakly lsc and a multifunction $H : X \to Y$ is defined by $H(x) = F(x) \stackrel{*}{-} G(x) \neq \emptyset$ for any $x \in X$, then the multifunction H is weakly usc, provided $H(X)$ is contained in some weakly compact set in Y.*

We will give a certain generalisation of this result. Moreover, our proof seems to be shorter and simpler.

Theorem 2. *Let X be a topological space with "the first axiom of countability", Y a metrisable linear topological space and let $F, G : X \to Y$ be compact-valued multifunctions. If F is usc, G is lsc then the multifunction $H : X \to Y$ defined by $H(z) = F(x) \stackrel{*}{-} G(x) \neq \emptyset$ for any $x \in X$ is usc.*

Proof. Obviously the multifunction $H = F \stackrel{*}{-} G$ is compact-valued. Therefore, by Proposition 1, it suffices to show that for every $x \in X$ and for any sequence $\{x_n\} \subset X$ converging to x and for any sequence $\{y_n\} \subset Y$ such that $y_n \in H(x_n)$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which converges to $y \in H(x)$.

So, let $x \in X$ and suppose that $\{x_n\} \subset X$ converges to x. Let $\{y_n\} \subset Y$ be such that $y_n \in H(x_n)$. We have $y_n + G(x_n) \subset F(x_n)$. From lower semicontinuity of G at x it follows (by Proposition 2) that for each $z \in G(x)$ there exists a sequence $\{z_n\} \subset Y$ with $z_n \in G(x_n)$ which converges to z. Thus we have $u_n = y_n + z_n \in F(x_n)$. Since F is usc at x, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging to $u \in F(x)$.

Hence the subsequence $\{y_{n_k}\}$ of $\{y_n\}$, where $y_{n_k} = u_{n_k} - z_{n_k}$, converges to $y = u - z$ and $y + z = u \in F(x)$.

Since $z \in G(x)$ was chosen arbitrarily, $y + G(x) \subset F(x)$, which gives $y \in F(x) \overset{*}{-} G(x) = H(x)$.

By Proposition 2, the multifunction $H = F \overset{*}{-} G : X \to Y$ is usc at x and the proof of Theorem 2 is complete. □

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Received November 18, 2003