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On a functional equation

Abstract. The existence of continuous solutions of the functional equation

$$\phi(\phi(x)) = 2\phi(x) - x + p$$

is studied.

The object of the paper is to investigate the functional equation of the form

$$\phi(\phi(x)) = 2\phi(x) - x + p, \tag{1}$$

where ϕ is the unknown function and p a real constant. The equation (1) is a particular case of the equation $\phi(\phi(x)) = g(x, \phi(x))$, which has been studied in [K1, p. 282], [K2], [K3] and [F]. Here we look for continuous solutions of (1) defined on the whole real line $\mathbb{R}$. We show that this equation has a solution only when $p = 0$. In this case the only solutions are $\phi(x) = x + \alpha$, $\alpha \in \mathbb{R}$.

To prove our main result we will need the following lemmas.

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Lemma 1. *If $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous solution of (1), then ϕ is strictly increasing and $\phi(\mathbb{R}) = \mathbb{R}$.*

Proof. Observe first that any solution of (1) (even not continuous) must be a one-to-one map. Indeed, if ϕ satisfies (1) and $\phi(x) = \phi(y)$, then

$$2\phi(x) - x + p = \phi(\phi(x)) = \phi(\phi(y)) = 2\phi(y) - y + p = 2\phi(x) - y + p.$$

Thus $x = y$. Consequently, each continuous solution ϕ of (1) is either strictly decreasing or strictly increasing. In the former case, one of the following conditions holds.

1) $\lim\limits_{x\to-\infty} \phi(x) = +\infty$ and $\lim\limits_{x\to\infty} \phi(x) = -\infty$,
2) $\lim\limits_{x\to-\infty} \phi(x) = +\infty$ and $\lim\limits_{x\to\infty} \phi(x) = b > -\infty$,
3) $\lim\limits_{x\to-\infty} \phi(x) = a > -\infty$ and $\lim\limits_{x\to\infty} \phi(x) = -\infty$,
4) $\lim\limits_{x\to-\infty} \phi(x) = a > -\infty$ and $\lim\limits_{x\to\infty} \phi(x) = b > -\infty, \quad a > b.$

We claim that any of 1) - 4) is not possible. Indeed, if 1) holds, then we would get

$$\lim_{x\to-\infty} \phi(\phi(x)) = \lim_{x\to-\infty} (2\phi(x) - x + p) = +\infty.$$

On the other hand, by the continuity of ϕ,

$$\lim_{x\to-\infty} \phi(\phi(x)) = \phi(\lim_{x\to-\infty} \phi(x)) = -\infty,$$

a contradiction. In case 2) we have

$$-\infty \neq \phi(b) = \phi(\lim_{x\to+\infty} \phi(x)) = \lim_{x\to\infty} \phi(\phi(x)) = \lim_{x\to\infty} (2\phi(x) - x + p) = -\infty,$$

a contradiction. Similar analysis can be applied to show that neither 3) nor 4) is possible. This proves that ϕ cannot be strictly decreasing. So it is strictly increasing. Our next claim is that $\phi(\mathbb{R}) = \mathbb{R}$. The following four cases are possible:

5) $\lim\limits_{x\to-\infty} \phi(x) = -\infty$ and $\lim\limits_{x\to\infty} \phi(x) = \infty$,
6) $\lim\limits_{x\to-\infty} \phi(x) = -\infty$ and $\lim\limits_{x\to\infty} \phi(x) = b < \infty$,
7) $\lim\limits_{x\to-\infty} \phi(x) = a > -\infty$ and $\lim\limits_{x\to\infty} \phi(x) = +\infty$,
8) $\lim\limits_{x\to-\infty} \phi(x) = a > -\infty$ and $\lim\limits_{x\to\infty} \phi(x) = b < \infty, \quad a < b.$

As above one can show that the last three cases give a contradiction. To see that 5) is possible, we rewrite (1) in the form

$$\phi(\phi(x)) + x = 2\phi(x) + p. \quad \square$$

Lemma 2. *Equation (1) has a solution if and only if* $p = 0$.

Proof. If $p = 0$, then the identity function is a solution of (1). Suppose now that (1) has a solution ϕ and let $x_0 \in \mathbb{R}$ be arbitrarily chosen. If $\phi(x_0) = x_0$, then

$$x_0 = \phi(x_0) = \phi(\phi(x_0)) = 2\phi(x_0) - x_0 + p = x_0 + p.$$

Thus $p = 0$. If $\phi(x_0) \neq x_0$, then we construct a sequence $\{x_n\}_{-\infty}^{\infty}$ by setting $x_n = \phi^n(x_0)$, where ϕ^n denotes the nth iterate of ϕ, that is,

$$\phi^0(x) = x, \quad \phi^{n+1}(x) = \phi(\phi^n(x)), \quad \phi^{n-1}(x) = \phi^{-1}(\phi^n(x)),$$

where ϕ^{-1} is the inverse of ϕ. Since $x_1 \neq x_0$, we have $x_1 = x_0 + r$ with some nonzero r. Then the sequence $\{x_n\}_{-\infty}^{\infty}$ is, by Lemma 1, strictly increasing if $r > 0$, and strictly decreasing if $r < 0$. Moreover, for each integer n,

$$x_{n+2} = 2x_{n+1} - x_n + p,$$

or, equivalently,

$$x_{n+2} - x_{n+1} = x_{n+1} - x_n + p.$$

By induction, we obtain

$$x_n = x_0 + n\left(r + \frac{(n-1)}{2}p\right), \quad n = 0, \pm 1, \ldots. \tag{2}$$

So, if $p \neq 0$, then for sufficiently large $|n|$ the terms x_n are either greater than x_0 or less then x_0, which contradicts strict monotonicity of $\{x_n\}_{-\infty}^{\infty}$. Thus $p = 0$. $\square$

Now we are ready to prove our main result.

Theorem. *The only functions continuous on* $\mathbb{R}$ *and satisfying the equation*

$$\phi(\phi(x)) = 2\phi(x) - x \tag{3}$$

are

$$\phi(x) = x + \alpha, \quad \alpha \in \mathbb{R}.$$

Proof. It is clear that the identity function $\phi(x) = x$ is a solution of (3). Now, suppose that ϕ, different from identity, is a solution of (3). Then there exists an x_0 such that $\phi(x_0) \neq x_0$. As in Lemma 2 we define the sequence $\{x_n\}_{-\infty}^{\infty}$ with $x_n = \phi^n(x_0)$. By (2),

$$x_{n+1} = x_n + r \quad \text{and} \quad x_n = x_0 + nr,$$

where $r = x_1 - x_0 = \phi(x_0) - x_0$. Choose y_0 from the open interval with end points x_0 and x_1 and consider the sequence $\{y_n\}_{-\infty}^{\infty}$ with $y_n = \phi^n(y_0)$. If we set $\rho = y_1 - y_0$, then by (2),

$$y_{n+1} = y_n + \rho \quad \text{and} \quad y_n = y_0 + n\rho.$$

Since ϕ, as a solution of (3), is strictly increasing, we see that each y_n is between x_n and x_{n+1}. Note that the sequences $\{x_n\}_{-\infty}^{\infty}$ and $\{y_n\}_{-\infty}^{\infty}$ are both either strictly decreasing or strictly increasing. Suppose, for example, that the latter case holds. If $\rho \neq r$, then for some n, x_n would not be in the interval (x_n, x_{n+1}), a contradiction. An analogous reasoning can be applied in the other case. Thus we see that $\rho = r$. Consequently, if ϕ is a continuous solution of (3), then $\phi(x) - x = \alpha$ with some real constant α. □

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