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A note on mappings with nonexpansive square

Dedicated to W.A. Kirk on the occasion of
His Honorary Doctorate of
Maria Curie-Skłodowska University

Abstract. Let C be a bounded, closed and convex subset of a Banach space X. We present here some observations on the existence of fixed points for lipschitzian mappings $T : C \to C$ having nonexpansive square T^2. We list some problems connected with this class of mappings.

1. Preliminaries. The aim of this note is to present some, in our opinion unnoticed, facts from the nonexpansive mappings theory.

Let $(X, \|\cdot\|)$ be a Banach space with the closed unit ball B and the unit sphere S. Let C be a bounded, closed and convex subset of X. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

The fundamental problem of the nonexpansive mapping theory is to identify geometric conditions which imposed on C guarantee that any nonexpansive mapping $T : C \to C$ has a fixed point. In other words, under which

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condition $\operatorname{Fix} T \neq \emptyset$ for all nonexpansive T. If this is the case we say that C has the *fixed point property for nonexpansive mappings* (shortly FPP). Extensive discussion of the present state of this theory can be found in [7]. The classical here is Kirk's result [6], stating that C has FPP if C is weakly compact and has *the normal structure*.

In this note we shall use two more terms connected to FPP.

First: We say that the space X has the *fixed point property for spheres*, (FPPS for short), if any closed and convex subset D of the unit sphere S has FPP. Obviously in such spaces, closed and convex subsets of spheres of any center and radius have FPP.

The two types of spaces having this property are; strictly convex spaces, in which the only convex subsets of S are singletons, and spaces with Kadec-Klee property, in which all the closed and convex subsets of S are compact. There are also other spaces sharing this property. The usefulness of the fixed point property for spheres lies in the following observation.

Suppose $T : C \to C$ is nonexpansive and let x, y be two fixed points of T, $x = Tx$, $y = Ty$. Let $d = \|x - y\|$. Consider the set $D = B\left(x, \frac{d}{2}\right) \cap B\left(y, \frac{d}{2}\right)$. Thus D is contained in a sphere of radius $\frac{d}{2}$. If $z \in D$ then

$$\|Tz - x\| = \|Tz - Tx\| \leq \|z - x\| = \frac{d}{2},$$
$$\|Tz - y\| = \|Tz - Ty\| \leq \|z - y\| = \frac{d}{2},$$

show that D is T-invariant, $T : D \to D$. Then D must contain a fixed point of T say $u = Tu$ with $\|x - u\| = \|y - u\| = \frac{d}{2}$. This means that the fixed point sets of nonexpansive mappings T in spaces having FPPS are *metrically convex* in the sense of Menger [8]. Consequently for any $x, y \in \operatorname{Fix} T$ there exists an arc joining x and y isometric to the interval $[0, d]$ where $d = \|x - y\|$.

In case of X being strictly convex there is only one such arc, the segment $[x, y]$. In this case the fixed point set is convex.

Second: We say that the set C has the *hereditary fixed point property* (HFPP) if any nonexpansive mapping $T : C \to C$ has a fixed point in each of the closed and convex subsets $D \subset C$ which is T-invariant, $T(D) \subset D$.

This notion has been first studied by Bruck [2]. It is known that under assumption that C is weakly compact HFPP implies that for any nonexpansive mapping $T : C \to C$ the fixed point set $\operatorname{Fix} T$ is a nonexpansive retract of C. This means that there exists a nonexpansive mapping $R : C \to \operatorname{Fix} T$ such that $Rx = x$ for all $x \in \operatorname{Fix} T$.

Let us close with recalling that if the space X has FPPS then any convex weakly compact set C having FPP also has HFPP. Also note that there are sets having FPP but lacking HFPP. The unit ball in l^1 serves as the classical example.

For more information concerning FPP and conditions implying it we advise reader to books [4], [7].

2. Strict and uniform convexity coefficients. Let us recall some basic tools used for scaling convexity of unit balls of Banach spaces.

A Banach space $(X, \|\cdot\|)$ is said to be strictly convex if the unit sphere does not contain any segment of positive length. In other words if the following implication holds:

$$\left.\begin{array}{c} \|x\| = 1 \\ \|y\| = 1 \\ x \neq y \end{array}\right\} \Longrightarrow \left\| \frac{x+y}{2} \right\| < 1.$$

More precise information about the convexity are given by the *modulus of convexity.* This is the function $\delta_X : [0,2] \to [0,1]$ defined by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \varepsilon \right\}.$$

The modulus of convexity is nondecreasing and continuous on $[0,2)$. A point of discontinuity can appear at $\varepsilon = 2$. Obviously $\delta_X(0) = 0$, but δ_X can vanish also for some positive ε. The *coefficient of uniform convexity* is defined as

$$\varepsilon_0(X) = \sup \{\varepsilon : \delta(\varepsilon) = 0\}.$$

The coefficient $\varepsilon_0(X)$ measures the maximal length of segments which can be imbedded in the unit ball B and placed arbitrarily close to the unit sphere S.

Strict convexity of the space X is characterized by the condition $\delta_X(2) = 1$. The space is said to be uniformly convex if $\varepsilon_0(X) = 0$. Of course uniformly convex spaces are strictly convex but not vice versa. There are strictly convex spaces with $\varepsilon_0(X) > 0$. It is also known that

$$\lim_{\varepsilon \to 2^-} \delta_X(\varepsilon) = 1 - \frac{1}{2} \varepsilon_0(X).$$

Strict convexity has not been used much in the fixed point theory for nonexpansive mappings. The main fact which is the consequence of strict convexity was mentioned above. This is the convexity of the fixed point set $\operatorname{Fix} T$. We are going to show some other consequences of strict convexity. Let us introduce first the natural *coefficient of strict convexity* $\eta_0(X)$ by

$$\eta_0(X) = \sup \{d : S \text{ contains a segment of length } d\}.$$

Obviously X is strictly convex if and only if $\eta_0(X) = 0$ and $\eta_0(X) \leq \varepsilon_0(X)$. The strict inequality may hold in some cases. In extreme case it may happen that $\varepsilon_0(X) = 2$ and the space X is strictly convex, $\eta_0(X) = 0$, $\delta_X(2) = 1$.

The coefficient $\eta_0(X)$ can be also described in terms of the modulus of convexity. Indeed if $x, y \in B$ with $\|x - y\| = d$ and $\frac{1}{2}(x + y) \in S$, then $x, y \in S$ and for $z = -y$ we have $\|x - z\| = 2$ and $\left\|\frac{x+z}{2}\right\| = \frac{d}{2}$. This with the converse reasoning leads to

$$\delta_X(2) = 1 - \frac{1}{2}\eta_0(X),$$

or

$$\eta_0(X) = 2(1 - \delta_X(2)).$$

A simple observation will be needed in the next section. Suppose $x, y \in X$ with $\|x - y\| = d > 0$. Consider the *equidistant set*

$$E(x, y) = B\left(x, \frac{d}{2}\right) \cap B\left(y, \frac{d}{2}\right) = S\left(x, \frac{d}{2}\right) \cap S\left(y, \frac{d}{2}\right).$$

Obviously $u = \frac{1}{2}(x + y) \in E(x, y)$. For any $z \in E(x, y)$ the point $2u - z = x + y - z$ symmetric to z with respect to u belongs to $E(x, y)$. Since $E(x, y)$ is a convex set contained in the sphere of radius $\frac{d}{2}$, for any $z \in E(x, y)$ we have

$$\left\|z - \frac{x + y}{2}\right\| \le \frac{1}{2}\eta_0(X)\frac{d}{2} = \frac{1}{4}\eta_0(X)d.$$

3. Mappings with the nonexpansive square. The mapping $T : C \to C$ is said to be lipschitzian if there exists $k \ge 0$ such that

$$\|Tx - Ty\| \le k\|x - y\|$$

holds for all $x, y \in C$. If k is fixed we use to say that T is k-lipschitzian.

The smallest k for which the above holds is said to be the Lipschitz constant of T and is denoted by $k(T)$. For any two lipschitzian mappings T_1, T_2 we have $k(T_1 \circ T_2) \le k(T_1)k(T_2)$ and especially $k(T^n) \le k(T)^n$.

The class of lipschitzian mappings is sometimes denoted $\mathcal{L}$ and is divided into subclasses $\mathcal{L}(k)$ of k-lipschitzian mappings. Nonexpansive mappings are 1-lipschitzian, of class $\mathcal{L}(1)$.

If $T : C \to C$ is nonexpansive, so nonexpansive are all iterates T^n, $n = 1, 2, \ldots$. However, converse is not true. There are lipschitzian mappings with $k(T) > 1$ having one of the iterates nonexpansive. Especially it may happen that T^2 is nonexpansive, $k(T^2) \le 1$. For any two nonnegative constants k_1, k_2 denote by $\mathcal{L}(k_1, k_2)$ the class of mappings T for which $k(T) \le k_1$ and $k(T^2) \le k_2$. We are mostly interested with the class of mappings having nonexpansive square. It means mappings of a class $\mathcal{L}(k, 1)$.

Obviously for $k \geq 1$, $\mathcal{L}(1) \subset \mathcal{L}(k,1)$. In sake of completeness let us present a scheme showing that in general $\mathcal{L}(1)$ forms the proper subfamily of the class $\mathcal{L}(k,1)$.

Example 1. There are various lipschitzian functions $\beta : \mathbb{R} \to \mathbb{R}$ satisfying $\beta \circ \beta(t) = t$ for all $t \in \mathbb{R}$. The typical k-lipschitzian one is

$$\beta_k(t) = \begin{cases} -kt & \text{for } t \leq 0, \\ -\frac{1}{k}t & \text{for } t > 0. \end{cases}$$

All the asymmetric intervals $[-a, ka]$, $a > 0$ are β_k-invariant. A similar situation is observed for many domains $D \subset \mathbb{R}^n$. It may happen that there exists a lipschitzian mapping $\alpha : D \to D$, $\alpha(D) = D$ satisfying $\alpha \circ \alpha(x) = x$ for all $x \in D$. Let now $(X, \|\cdot\|)$ be a normed function space consisting of functions $f : D \to \mathbb{R}$. At this point we do not assume anything about the type of the norm. Let $\alpha : D \to D$, $\beta : \mathbb{R} \to \mathbb{R}$ be two lipschitzian functions with the properties described above. Assume now that the space X has the property that for each $f \in X$ also $\beta \circ f \circ \alpha \in X$. For concrete norm $\|\cdot\|$ the convolution operator $T_{\alpha\beta}$ defined by

$$T_{\alpha\beta} = \beta \circ f \circ \alpha$$

is usually lipschitzian. But the Lipschitz constant $k(T_{\alpha\beta})$ depends on the character of the norm and is in a way related to the Lipschitz constants of the defining functions α and β. For example if $X = C[D]$ with the standard uniform norm then $k(T_{\alpha\beta})$ coincides with the Lipschitz constant of β. In general we often have $k(T_{\alpha\beta}) > 1$. But obviously we have

$$T_{\alpha\beta}^2 f = \beta \circ \beta \circ f \circ \alpha \circ \alpha = f$$

for all $f \in X$. In many cases it is not difficult to identify closed convex sets which are $T_{\alpha\beta}$-invariant. Obviously since $T_{\alpha\beta}^2 = I$, $T_{\alpha\beta} \in \mathcal{L}(k(T_{\alpha\beta}), 1)$ on each such invariant set but $T_{\alpha\beta}$ is not necessarily nonexpansive.

Each mapping $T : C \to C$ of class $\mathcal{L}(k,1)$ generates a metric ρ_T on C by

$$\rho_T(x,y) = \|x-y\| + \|Tx - Ty\|.$$

The metric ρ_T is equivalent to the norm metric since we have

$$\|x-y\| \leq \rho_T(x,y) \leq (1+k)\|x-y\|.$$

Any mapping $T \in \mathcal{L}(k,1)$ is nonexpansive with respect to ρ_T and conversely if T is nonexpansive with respect to ρ_T then T^2 is norm nonexpansive. Indeed, it comes from

$$\begin{aligned} \rho_T(Tx,Ty) &= \|Tx-Ty\| + \left\|T^2x - T^2y\right\| \\ &\leq \|Tx-Ty\| + \|x-y\| = \rho_T(x,y). \end{aligned}$$

Our main observation is the following.

Theorem 1. *Suppose the Banach space X has fixed point property for spheres. If C is a closed and convex subset of X having fixed point property for nonexpansive mappings, then C has the fixed point property with respect to all classes $\mathcal{L}(k,1)$ satisfying*

$$k < 2\left(1 - \frac{1}{4}\eta_0(X)\right). \tag{1}$$

Proof. Let $T : C \to C$ be of class $\mathcal{L}(k,1)$. Since C has FPP for nonexpansive mappings, we have $\operatorname{Fix} T^2 \neq \emptyset$. Let $x \in \operatorname{Fix} T^2$, $x = T^2x$. Then also $Tx \in \operatorname{Fix} T^2$. If $x = Tx$ there is nothing to prove. Otherwise, if $x \neq Tx$ put $d = \|x - Tx\| > 0$. In view of T^2 being nonexpansive the set $E(x, Tx)$ is T^2-invariant. Since X has FPPS, there exists a point $y \in E(x, Tx)$ being a fixed point of T^2, $y = T^2y$. Then we have

$$\left\| y - \frac{x+Tx}{2} \right\| \leq \frac{\eta_0(X)}{4} d.$$

On the other hand we have

$$\|Ty - x\| = \left\|Ty - T^2x\right\| \leq k\,\|y - Tx\| = \frac{k}{2}d,$$

$$\|Ty - Tx\| \leq k\,\|y - x\| = \frac{k}{2}d,$$

implying

$$\left\| Ty - \frac{x+Tx}{2} \right\| \leq \frac{k}{2}d.$$

Hence

$$\begin{aligned} \|Ty - y\| &\leq \left\| Ty - \frac{x+Tx}{2} \right\| + \left\| \frac{x+Tx}{2} - y \right\| \\ &\leq \frac{k}{2}d + \frac{\eta_0(X)}{4}d = \left(\frac{k}{2} + \frac{\eta_0(X)}{4}\right) d = ad \end{aligned} \tag{2}$$

where, in view of (1), $a < 1$. Using the above scheme we can start an iteration procedure. Select $x_0 \in \operatorname{Fix} T^2$ and find $x_1 \in E(x_0, Tx_0) \cap \operatorname{Fix} T^2$. Then repeat the procedure by finding $x_2 \in E(x_1, Tx_1) \cap \operatorname{Fix} T^2$ and continue building a sequence $\{x_n\}$ satisfying $x_{n+1} \in E(x_n, Tx_n) \cap \operatorname{Fix} T^2$, $n = 0, 1, \ldots$. In view of (2) we have

$$\|x_n - Tx_n\| \leq a^n \|x_0 - Tx_0\|$$

and

$$\|x_n - x_{n+1}\| = \frac{1}{2}\|x_n - Tx_n\| \leq \frac{a^n}{2}\|x_0 - Tx_0\|.$$

This implies that the sequence $\{x_n\}$ converges to a fixed point of T. $\square$

The above theorem means that in case of $0 \le \eta_0(X) < 2$ not only nonexpansive self-mappings of C but also mappings of the larger class $\mathcal{L}(k,1)$ have fixed points, provided k is sufficiently close to 1. In other words, the fixed point property for nonexpansive mappings is stable in the sense that it carries over to the class of lipschitzian mappings with the nonexpansive square provided the Lipschitz constant of the mapping itself is not too large.

The iteration scheme described in the proof of Theorem 1 has one disadvantage. The method of selecting consecutive points is not explicitly defined. The situation becomes a little better under additional assumption of C being weakly compact. As mentioned, the result of Bruck [2] yields in this case the existence of a nonexpansive retraction $R : C \to \operatorname{Fix} T^2$. It is easy to check that for any $x \in \operatorname{Fix} T^2$ the set $E(x, Tx)$ is R-invariant, $R : E(x, Tx) \to E(x, Tx) \cap \operatorname{Fix} T^2$. This leads to the following.

Corollary 1. *Under assumptions of Theorem 1, if C is weakly compact and $R : C \to \operatorname{Fix} T^2$ is a nonexpansive retraction, then for any $x \in \operatorname{Fix} T^2$ the sequence*

$$x_n = \left(R \circ \frac{I+T}{2}\right)^n x$$

converges to a fixed point of T.

4. Involutions. There is a special case of mappings with nonexpansive square. We say that the mapping $T : C \to C$ is an *involution* or T is *2-periodic* if $T^2 = I$ on C. Lipschitzian involutions with the Lipschitz constant k form a subclass of $\mathcal{L}(k,1)$. Without any geometrical restrictions on the space X and without assuming that C is bounded the following is known (see [3], [4]).

Theorem 2. *If C is a nonempty, closed, and convex subset of a Banach space X and $T : C \to C$ is an involution such that $k(T) < 2$, than T has a fixed point in C.*

A hint for the proof: Define $F = \frac{1}{2}(I+T)$ and check that for any $x \in C$ the sequence $x_n = F^n x$ converges to a fixed point of T.

It is not known if the restriction $k(T) < 2$ is sharp and exact in the class of all Banach spaces. For spaces having $\delta_X(1) > 0$ there is a better estimate. The conclusion of Theorem 2 holds if only

$$k(T)\left(1 - \delta_X\left(\frac{2}{k(T)}\right)\right) < 1.$$

And for a Hilbert space the condition $k(T) < \sqrt{\pi^2 - 3} = 2.62...$ is sufficient (see [4]).

According to our knowledge no examples of lipschitzian, or only uniformly continuous, fixed point free involutions of bounded, closed and convex sets are known. However there are examples of such continuous involutions [4], [5].

Besides asking about fixed points of involutions one can ask even stronger question. Suppose $C \subset X$ is bounded, closed, and convex. If $T : C \to C$ is a lipschitzian involution, is $\inf \{\|x - Tx\| : x \in C\} = 0$? The answer is unknown to the authors, even for T satisfying $k(T) = 2$ in general case.

We mention these questions, since they are connected to the well known problem of nonlinear functional analysis concerning the uniform classification of spheres (see [1]). It is known that for any infinitely dimensional Banach space, the unit ball B and the unit sphere S are homeomorphic. The question reads: Does there exist a homeomorphism H of B onto S such that H and H^{-1} are lipschitzian?

Assuming that such homeomorphism do exist we can easily produce an involution $T : B \to B$ with $\inf \{\|x - Tx\| : x \in B\} > 0$. It is enough to define T by $Tx = H^{-1}(-Hx)$. Consequently, if for a given space X all the lipschitzian involutions $T : B \to B$ satisfy $\inf \{\|x - Tx\| : x \in C\} = 0$ the unit ball and the unit sphere in this space are not Lipschitz homeomorphic.

Let us end with the observation that the above stability properties can be possibly extended and studied for classes of mappings with nonexpansive n-th power $T^n, n > 2$. Defining analogously as above a class $\mathcal{L}(k_1, k_2, ..., k_{n-1}, 1)$ we may ask, for conditions on the set of constants $(k_1, k_2, ..., k_{n-1})$ which guarantee some fixed point property.

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