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Some natural operators in linear vector fields

Abstract. The higher order tangent bundles of vector bundles are a modification of the usual dual to jets of functions, restricted to those linear along the fibres. The paper shows, roughly speaking, that these bundles are more rigid than their full version.

Introduction. The category of vector bundles with m-dimensional bases and vector bundle maps with local diffeomorphisms as base maps will be denoted by $\mathcal{VB}_m$. The category of vector bundles with m-dimensional bases and n-dimensional fibers and vector bundle isomorphisms onto open vector subbundles will be denoted by $\mathcal{VB}_{m,n}$.

Given a vector bundle E there are two (depending functorially on E) vector r-tangent bundles of E. Namely, the vector (r)-tangent bundle $T^{(r)fl}E = (J^r_{fl}(E, \mathbf{R})_0)^*$, where

$$J^r_{fl}(E, \mathbf{R})_0 = \{j^r_x\gamma \mid \gamma : E \to \mathbf{R} \text{ is fiber linear, } \gamma_x = 0,\ x \in M\},$$

and the vector $[r]$-tangent bundle $T^{[r]fl}E = E \otimes (J^r(M, \mathbf{R})_0)^*$.

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In this paper we deduce that for integers $m \geq 2$, $n \geq 1$ and $r \geq 1$ any $\mathcal{VB}_{m,n}$-natural operator A lifting linear vector fields X from vector bundles E into vector fields $A(X)$ on $T^{(r)fl}E$ is a linear combination with real coefficients of the flow operator $\mathcal{T}^{(r)fl}X$ and the Liouville vector field. As corollaries we deduce the same facts for $T^{(r)fl}E^*$, $(T^{(r)fl}E)^*$ and $(T^{(r)fl}E^*)^*$ instead of $T^{(r)fl}E$. Using similar methods we remark the same results for $T^{[r]fl}E$ instead of $T^{(r)fl}E$. The above result shows that $T^{(r)fl}$ is more rigid than their full version $T^{(r)}$ because we have $(r+2)$-linearly independent natural operators lifting vector fields to $T^{(r)}$, see [4].

Natural operators lifting vector fields are used practically in all papers in which problem of prolongations of geometric structures was studied. That is why such natural operators are classified in papers [1], [2], [4], [5] and others.

The trivial vector bundle $\mathbf{R}^m \times \mathbf{R}^n$ over $\mathbf{R}^m$ with standard fiber $\mathbf{R}^n$ will be denoted by $\mathbf{R}^{m,n}$. The coordinates on $\mathbf{R}^m$ will be denoted by $x^1, \ldots, x^m$. The fiber coordinates on $\mathbf{R}^{m,n}$ will be denoted by $y^1, \ldots, y^n$.

All manifolds are assumed to be finite dimensional and smooth. Maps are assumed to be smooth, i.e. of class $\mathcal{C}^\infty$.

1. The vector (r)-tangent bundle functor. Given a $\mathcal{VB}_m$-object $p : E \to M$ the vector (r)-tangent bundle $T^{(r)fl}E$ of E is the vector bundle

$$T^{(r)fl}E = (J^r_{fl}(E, \mathbf{R})_0)^*$$

over M, where

$$J^r_{fl}(E, \mathbf{R})_0 = \{j^r_x\gamma \mid \gamma : E \to \mathbf{R} \text{ is fiber linear, } \gamma_x = 0,\ x \in M\}.$$

Every $\mathcal{VB}_m$-map $f : E_1 \to E_2$ covering $\underline{f} : M_1 \to M_2$ induces a vector bundle map $T^{(r)fl}f : T^{(r)fl}E_1 \to T^{(r)fl}E_2$ covering $\underline{f}$ such that

$$\left\langle T^{(r)fl}f(\omega), j^r_{\underline{f}(x)}\xi \right\rangle = \langle \omega, j^r_x(\xi \circ f) \rangle ,$$

$\omega \in T^{(r)fl}_x E_1$, $j^r_{\underline{f}(x)}\xi \in J^r_{fl}(E_2, \mathbf{R})_0$, $x \in M_1$.

The correspondence $T^{(r)fl} : \mathcal{VB}_m \to \mathcal{VB}_m$ is a fiber product preserving gauge bundle functor.

2. The vector $[r]$-tangent bundle functor. Given a $\mathcal{VB}_m$-object $p : E \to M$ the vector $[r]$-tangent bundle $T^{[r]fl}E$ of E is the vector bundle $T^{[r]fl}E = E \otimes (J^r(M, \mathbf{R})_0)^*$ over M. Every $\mathcal{VB}_m$-map $f : E_1 \to E_2$ covering $\underline{f} : M_1 \to M_2$ induces (in obvious way) a vector bundle map $T^{[r]fl}f : T^{[r]fl}E_1 \to T^{[r]fl}E_2$ covering $\underline{f}$.

The correspondence $T^{[r]fl} : \mathcal{VB}_m \to \mathcal{VB}_m$ is a fiber product preserving gauge bundle functor.

Remark 1. The bundle $T^{(r)}M = (J^r(M, \mathbf{R})_0)^*$ is called the vector (r)-tangent bundle of a manifold M, see [2]. This justifies the name (r)-tangent bundle of a vector bundle. One can show that $T^{(r)fl}E$ and $T^{[r]fl}E$ have a very similar construction and that only $T^{(r)fl}E$ and $T^{[r]fl}E$ admit this construction, see [3]. This justifies the name $[r]$-tangent bundle of a vector bundle.

3. Examples of natural operators $T_{lin|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{(r)fl}$. Let $p : E \to M$ be a $\mathcal{VB}_{m,n}$-object. A projectable vector field X on E is called linear if $X : E \to TE$ is a vector bundle map from $p : E \to M$ into $Tp : TE \to TM$. Equivalently, the flow Fl_t^X of X is formed by $\mathcal{VB}_{m,n}$-maps. The space of linear vector fields on E will be denoted by $\mathcal{X}_{lin}(E)$.

A natural operator $A : T_{lin|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{(r)fl}$ is an $\mathcal{VB}_{m,n}$-invariant family of regular operators $A : \mathcal{X}_{lin}(E) \to \mathcal{X}(T^{(r)fl}E)$ for any $\mathcal{VB}_{m,n}$-object E. The $\mathcal{VB}_{m,n}$-invariance means that for any $\mathcal{VB}_{m,n}$-map $f : E_1 \to E_2$ and any f-conjugate linear vector fields X and Y on E_1 and E_2 the vector fields $A(X)$ and $A(Y)$ are $T^{(r)fl}f$-conjugate. The regularity means that A transforms smoothly parameterized families of linear vector fields into smoothly parameterized families of vector fields.

Example 1. (*The flow operator*) Let X be a linear vector field on a $\mathcal{VB}_{m,n}$-object $p : E \to M$. The flow Fl_t^X of X is formed by $\mathcal{VB}_{m,n}$-maps on E. Applying functor $T^{(r)fl}$ we obtain a flow $T^{(r)fl}(Fl_t^X)$ on $T^{(r)fl}E$. The vector field $\mathcal{T}^{(r)fl}X$ on $T^{(r)fl}E$ corresponding to the flow $T^{(r)fl}(Fl_t^X)$ is called the flow prolongation of X. The correspondence $\mathcal{T}^{(r)fl} : T_{lin|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{(r)fl}$, $X \to \mathcal{T}^{(r)fl}X$, is a natural operator.

Example 2. (*The Liouville vector field*) Let $p : E \to M$ be a $\mathcal{VB}_{m,n}$-object. Let L be the Liouville vector field on the vector bundle $T^{(r)fl}E$, $L_y = y \in T_x^{(r)fl}E \cong T_y(T_x^{(r)fl}E) \subset T_yT^{(r)fl}E$, $y \in T_x^{(r)fl}E$, $x \in M$. The correspondence $L : T_{lin|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{(r)fl}$, $X \to L$, is a natural operator.

4. A classification theorem. We have the following classification theorem.

Theorem 1. *Let $r \geq 1$, $m \geq 2$ and $n \geq 1$ be integers. Any natural operator $A : T_{lin|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{(r)fl}$ is a linear combination with real coefficients of the flow operator $\mathcal{T}^{(r)fl}$ and the Liouville vector field L.*

The proof of Theorem 1 will occupy the Sections 5–8. As a corollary we obtain

Corollary 1. *Let $r \geq 1$, $m \geq 2$ and $n \geq 1$ be integers. Any natural linear operator $A : T_{lin|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{(r)fl}$ is a constant multiple of the flow operator.*

5. A reducibility lemma.

Lemma 1. *Let* $A : T_{lin|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{(r)fl}$ *be a natural operator. The operator* A *is uniquely determined by the restriction* $\tilde{A} = A\left(\frac{\partial}{\partial x^1}\right) | T_0^{(r)fl}\mathbf{R}^{m,n}$ *of* $A\left(\frac{\partial}{\partial x^1}\right)$ *to the fibre* $T_0^{(r)fl}\mathbf{R}^{m,n}$ *of* $T^{(r)fl}\mathbf{R}^{m,n}$ *over* $0 \in \mathbf{R}^m$.

Proof. The lemma follows standardly from the regularity and invariance of A with respect to $\mathcal{VB}_{m,n}$-morphisms and the fact that any linear vector field X on $p : E \to M$ covering a non-vanishing vector field on M is locally $\mathcal{VB}_{m,n}$-conjugate with $\frac{\partial}{\partial x^1}$. □

6. A decomposition lemma.

Lemma 2. *Let* $A : T_{lin|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{(r)fl}$ *be a natural operator. Then there exists* $\alpha \in \mathbf{R}$ *such that* $A - \alpha\mathcal{T}^{(r)fl}$ *is a vertical type operator.*

Proof. Put $\tilde{\tilde{A}} = T\pi \circ \tilde{A} : T_0^{(r)fl}\mathbf{R}^{m,n} \to T_0\mathbf{R}^m$, where $\tilde{A}$ is as in Lemma 1 and $\pi : T^{(r)fl}\mathbf{R}^{m,n} \to \mathbf{R}^m$ is the bundle projection.

Using the invariance of A with respect to the fiber homotheties b_τ for $\tau \neq 0$ and then putting $\tau \to 0$ we see that $\tilde{\tilde{A}}(y) = \tilde{\tilde{A}}(0)$ for any $y \in T_0^{(r)fl}\mathbf{R}^{m,n}$.

Write $\tilde{\tilde{A}}(0) = \sum_i \alpha_i \left(\frac{\partial}{\partial x^i}\right)_0$ for some $\alpha_i \in \mathbf{R}$, $i = 1, \ldots, m$. Using the invariance of A with respect to $a_\tau = (x^1, \tau x^2, \ldots, \tau x^m, y^1, \ldots, y^n)$ for $\tau \neq 0$ we deduce that $\alpha_2 = \cdots = \alpha_m = 0$. Then $\tilde{\tilde{A}}(y) = \alpha\left(\frac{\partial}{\partial x^1}\right)_0$ for any $y \in T_0^{(r)fl}\mathbf{R}^{m,n}$, where $\alpha = \alpha_1$. Then $(A - \alpha\mathcal{T}^{(r)fl})\left(\frac{\partial}{\partial x^1}\right) | T_0^{(r)fl}\mathbf{R}^{m,n}$ is vertical. Then $A - \alpha\mathcal{T}^{(r)fl}$ is vertical because of Lemma 1. □

Replacing A by $A - \alpha\mathcal{T}^{(r)fl}$, where α is from the decomposition lemma, we can assume that A is a vertical type operator.

7. Some preparation.

Lemma 3. *Let* $m \geq 2$, $n \geq 1$ *and* $r \geq 1$ *be integers. Let* $A : T_{lin|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{(r)fl}$ *be a natural operator of vertical type. Define a map* $\overline{A} : T_0^{(r)fl}\mathbf{R}^{m,n} \to T_0^{(r)fl}\mathbf{R}^{m,n}$ *by*

$$\tilde{A}(y) = (y, \overline{A}(y)) \in T_0^{(r)fl}\mathbf{R}^{m,n} \times T_0^{(r)fl}\mathbf{R}^{m,n} \cong (VT^{(r)fl})_0\mathbf{R}^{m,n}\ , \tag{1}$$

$y \in T_0^{(r)fl}\mathbf{R}^{m,n}$, *where* $\tilde{A}$ *is as in Lemma 1. Then* A *is uniquely determined by* $\overline{A}$. *Moreover,* $\overline{A}$ *is linear and the dual map* $B = (\overline{A})^* : (T_0^{(r)fl}\mathbf{R}^{m,n})^* \to (T_0^{(r)fl}\mathbf{R}^{m,n})^*$ *satisfies the following conditions.*

(i) *For any local* $\mathcal{VB}_{m,n}$*-map* $f : \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ *preserving* $germ_0\left(\frac{\partial}{\partial x^1}\right)$

$$(T_0^{(r)fl}f^{-1})^* \circ B = B \circ (T_0^{(r)fl}f^{-1})^*\ . \tag{2}$$

(ii) *For any $\beta \in (\mathbf{N} \cup \{0\})^m$ with $1 \le |\beta| \le r$ and $l = 1, \dots, n$*

$$B(j_0^r(x^\beta y^l)) = \sum_{k=1}^{n} \sum_{1 \le |\sigma| \le |\beta|} c_\sigma^{\beta,l,k} j_0^r(x^\sigma y^k) \tag{3}$$

for some $c_\sigma^{\beta,l,k} \in \mathbf{R}$, where the second sum is over all $\sigma \in (\mathbf{N} \cup \{0\})^m$ with $1 \le |\sigma| \le |\beta|$.

Proof. Since $\tilde{A}$ is uniquely determined by $\overline{A}$, A is uniquely determined by $\overline{A}$ because of Lemma 1.

For any $t \in \mathbf{R}$ define $\overline{A}_t : T_0^{(r)fl}\mathbf{R}^{m,n} \to T_0^{(r)fl}\mathbf{R}^{m,n}$ by $A\left(t\frac{\partial}{\partial x^1}\right)(y) = (y, \overline{A}_t(y))$, $y \in T_0^{(r)fl}\mathbf{R}^{m,n}$. Clearly $\overline{A} = \overline{A}_1$. By the invariance of A with respect to the fiber homotheties we get the homogeneous condition $\overline{A}_t(\tau y) = \tau \overline{A}_t(y)$ for any $y \in T_0^{(r)fl}\mathbf{R}^{m,n}$ and $\tau \neq 0$. So, $\overline{A}_t$ is linear because of the homogeneous function theorem. In particular $\overline{A}$ is linear.

From the invariance of A with respect to a $\mathcal{VB}_{m,n}$-map $f : \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ preserving $germ_0\left(\frac{\partial}{\partial x^1}\right)$ we obtain (2).

Let $\beta \in (\mathbf{N} \cup \{0\})^m$ with $1 \le |\beta| \le r$ and $l = 1, \dots, n$. We can write

$$(\overline{A}_t)^*(j_0^r(x^\beta y^l)) = \sum_{k=1}^{n} \sum_{1 \le |\sigma| \le r} c_\sigma^{\beta,l,k}(t) j_0^r(x^\sigma y^k)$$

for some smooth maps $c_\sigma^{\beta,l,k} : \mathbf{R} \to \mathbf{R}$. By the invariance of A with respect to the base homotheties $(\tau x^1, \dots, \tau x^m, y^1, \dots, y^n)$ we obtain the homogeneous condition $c_\sigma^{\beta,l,k}(\tau t)\frac{1}{\tau^{|\beta|}} = c_\sigma^{\beta,l,k}(t)\frac{1}{\tau^{|\sigma|}}$ for $\tau \neq 0$. Then $c_\sigma^{\beta,l,k} = 0$ if $|\sigma| > |\beta|$. □

8. The main lemma. By Lemma 3, A is uniquely determined by $\overline{A}$. So, Theorem 1 will be proved after proving the following lemma.

Lemma 4. *Let $r \ge 1$, $m \ge 2$ and $n \ge 1$ be integers. Suppose that $B : (T_0^{(r)fl}\mathbf{R}^{m,n})^* \to (T_0^{(r)fl}\mathbf{R}^{m,n})^*$ is a linear map satisfying the conditions (i) and (ii) of Lemma 3. Then there is $\gamma \in \mathbf{R}$ such that $B = \gamma id_{(T_0^{(r)fl}\mathbf{R}^{m,n})^*}$.*

Proof. We start with a preparation.

Let $\alpha \in (\mathbf{N} \cup \{0\})^m$, $|\alpha| = r$, $l = 1, \dots, n$. We prove that

$$B(j_0^r(x^\alpha y^l)) = c j_0^r(x^\alpha y^l) \tag{4}$$

for some real number c (independent of α and l).

For, we write

$$B(j_0^r((x^1)^r y^1)) = \sum_{k=1}^{n} \sum_{1 \le |\sigma| \le r} c_\sigma^k j_0^r(x^\sigma y^k) \tag{5}$$

for some $c^k_\sigma \in \mathbf{R}$. By the invariance of A with respect to (locally defined) $(x^1, \dots, x^{i-1}, x^i + \tau(x^2)^2, x^{i+1}, \dots, x^m, y^1, \dots, y^n)^{-1}$ for $\tau \in \mathbf{R}$ and $i = 1, \dots, m$ (see condition (i)) we obtain

$$B(j^r_0((x^1)^r y^1)) = \sum_k \sum_\sigma (c^k_\sigma j^r_0(x^\sigma y^k) + \tau\sigma_i c^k_\sigma j^r_0(x^{\sigma - e_i + e_2 + e_2} y^k) + \dots) ,$$

where the dots is the finite sum of monomials in τ of degree ≥ 2. Then

$$c^k_\sigma = 0 \text{ for } 1 \leq |\sigma| < r . \tag{6}$$

More, by the invariance of B with respect to $(x^1, \dots, x^m, y^1, \tau y^2, \dots, \tau y^n)$ (see condition (i)) for $\tau \neq 0$ we deduce that

$$c^k_\sigma = 0 \text{ for } k \neq 1 . \tag{7}$$

Then by (5), (6) and (7) and the invariance of B with respect to $(x^1, \tau x^2, \dots, \tau x^m, y^1, \dots, y^n)$ (see condition (i)) we deduce that

$$B(j^r_0((x^1)^r y^1)) = c j^r_0((x^1)^r y^1) \tag{8}$$

for $c = c^1_{(r,0,\dots,0)} \in \mathbf{R}$. Then using the invariance of B with respect to $(x^1 + \tau^2 x^2 + \dots + \tau^m x^m, x^2, \dots, x^m, y^1, \dots, y^n)^{-1}$ for $\tau^2, \dots, \tau^m \in \mathbf{R}$ (see condition (i)) we get

$$\begin{aligned} &B(j^r_0((x^1 + \tau^2 x^2 + \dots + \tau^m x^m)^r y^1)) \\ &\quad = c j^r_0((x^1 + \tau^2 x^2 + \dots + \tau^m x^m)^r y^1) . \end{aligned} \tag{9}$$

Both sides of (9) are polynomials in $\tau^2, \dots, \tau^m$. Considering the coefficients of the polynomials in $(\tau^2)^{\alpha_2}, \dots, (\tau^m)^{\alpha_m}$ we get

$$B(j^r_0(x^\alpha y^1)) = c j^r_0(x^\alpha y^1) . \tag{10}$$

Then using the invariance of B with respect to the permutations of fibered coordinates (see condition (i)) we get (4).

Now, we will proceed by the induction with respect to r.

The case $r = 1$.

The Lemma 4 for $r = 1$ follows from (4) for $r = 1$.

The inductive step.

By (4) we have a linear map

$$[B] : (T^{(r-1)fl}_0 \mathbf{R}^{m,n})^* \to (T^{(r-1)fl}_0 \mathbf{R}^{m,n})^*$$

factorizing B.

By the assumptions (i) and (ii) of B we see that $[B]$ satisfies the conditions (i) and (ii) for $r - 1$.

Then, by the inductive assumption,

$$[B] = \gamma id_{(T_0^{(r-1)fl}\mathbf{R}^{m,n})^*} \tag{11}$$

for some $\gamma \in \mathbf{R}$. It remains to prove that $B = \gamma id_{(T_0^{(r)fl}\mathbf{R}^{m,n})^*}$, where γ is as above. By the assumption (ii) on B and by the equality (11) we have that

$$B(j_0^r(x^\beta y^l)) = \gamma j_0^r(x^\beta y^l) \tag{12}$$

for any $\beta \in (\mathbf{N} \cup \{0\})^m$ with $1 \leq |\beta| < r$ and $l = 1, \dots, n$

So, it remains to prove that for any $\alpha \in (\mathbf{N} \cup \{0\})^m$ with $|\alpha| = r$ and $l = 1, \dots, n$ we have

$$B(j_0^r(x^\alpha y^l)) = \gamma j_0^r(x^\alpha y^l)\ . \tag{13}$$

So, by (4) it remains to prove that $c = \gamma$, where c is as in (4). Using (12) for $\beta = (r-1, 0, \dots, 0) \in (\mathbf{N} \cup \{0\})^m$ and the invariance of B with respect to $(x^1 + (x^2)^2, x^2, \dots, x^m, y^1, \dots, y^n)^{-1}$ (see condition (i)) we deduce that

$$B(j_0^r((x^1)^{r-2}(x^2)^2 y^l)) = \gamma j_0^r((x^1)^{r-2}(x^2)^2 y^l)\ . \tag{14}$$

Then from (14) and (4) with $\alpha = (r-2, 2, 0, \dots, 0)$ we get that $c = \gamma$. The inductive step is complete.

The proof of Theorem 1 is complete. □

9. Some versions of Theorem 1. In this section we present some versions of Theorem 1. We start with the following proposition.

Proposition 1. *Let A be a $\mathcal{VB}_{m,n}$-natural operator lifting a linear vector field X on a vector bundle E into a vector field $A(X)$ on $T^{(r)fl}E$ (or $T^{(r)fl}E^*$ or $(T^{(r)fl}E)^*$ or $(T^{(r)fl}E^*)^*$). Then $A(X)$ is a linear vector field on $T^{(r)fl}E$ (or $T^{(r)fl}E^*$ or $(T^{(r)fl}E)^*$ or $(T^{(r)fl}E^*)^*$) for any linear vector field X on E.*

Proof. It is easy to see this for $X = \frac{\partial}{\partial x^1}$. (More precisely, the flow of $A(\frac{\partial}{\partial x^1})$ is invariant with respect to the fiber homotheties of $T^{(r)fl}\mathbf{R}^{m,n}$ because of $\frac{\partial}{\partial x^1}$ is invariant with respect to the fiber homotheties of $\mathbf{R}^{m,n}$.) Next we use the same arguments as in the proof of Lemma 1. For $T^{(r)fl}E^*$, $(T^{(r)fl}E)^*$ and $(T^{(r)fl}E^*)^*$ instead of $T^{(r)fl}E$ we use the same method. □

There is a natural involution (dualization) $()^* : \mathcal{VB}_{m,n} \to \mathcal{VB}_{m,n}$, $E \to E^*$, $f \to (f^{-1})^*$. So, using Proposition 1 and Theorem 1 we obtain easily the following versions of Theorem 1.

Theorem 2. *For $m \geq 2$, $n \geq 1$ and $r \geq 1$ any $\mathcal{VB}_{m,n}$-natural operator lifting a linear vector field X on E into a vector field $A(X)$ on $T^{(r)fl}E^*$ is a linear combination of $\mathcal{T}^{(r)fl}X^*$ and the Liouville vector field L on $T^{(r)fl}E^*$, where X^* is the dual to X linear vector field on E^* (if f_t is the flow of X, then $(f_t^{-1})^*$ is the flow of X^*).*

Theorem 3. *For $m \geq 2$, $n \geq 1$ and $r \geq 1$ any $\mathcal{VB}_{m,n}$-natural operator lifting a linear vector field X on E into a vector field $A(X)$ on $(T^{(r)fl}E)^*$ is a linear combination of $(\mathcal{T}^{(r)fl}X)^*$ and the Liouville vector field on $(T^{(r)fl}E)^*$.*

Theorem 4. *For $m \geq 2$, $n \geq 1$ and $r \geq 1$ any $\mathcal{VB}_{m,n}$-natural operator lifting a linear vector field X on E into a vector field $A(X)$ on $(T^{(r)fl}E^*)^*$ is a linear combination of $(\mathcal{T}^{(r)fl}X^*)^*$ and the Liouville vector field L on $(T^{(r)fl}E^*)^*$.*

10. The natural operators $T_{lin|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{[r]fl}$. Quite similarly as $\mathcal{VB}_{m,n}$-natural operators $A : T_{lin|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{(r)fl}$ one can define $\mathcal{VB}_{m,n}$-natural operators $A : T_{lin|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{[r]fl}$ lifting a linear vector field X from a vector bundle E into a vector field $A(X)$ on $T^{[r]fl}E$.

Using the same proofs as in Sections 5–8 with $T^{[r]fl}$ instead of $T^{(r)fl}$ (in particular with $y_0^l \otimes j_0^r x^\alpha$ instead of $j_0^r(x^\alpha y^l)$) one can obtain the following classification theorem.

Theorem 5. *Let $r \geq 1$, $m \geq 2$ and $n \geq 1$ be integers. Any natural operator $A : T_{lin|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{[r]fl}$ is a linear combination with real coefficients of the flow operator $\mathcal{T}^{[r]fl}$ and the Liouville vector field L.*

As a corollary one can obtain.

Corollary 2. *Let $r \geq 1$, $m \geq 2$ and $n \geq 1$ be integers. Any natural linear operator $A : T_{lin|\mathcal{VB}_{m,n}} \rightsquigarrow TT^{[r]fl}$ is a constant multiple of the flow operator.*

One can deduce the following versions of Theorem 5.

Theorem 6. *For $m \geq 2$, $n \geq 1$ and $r \geq 1$ any $\mathcal{VB}_{m,n}$-natural operator lifting a linear vector field X on E into a vector field $A(X)$ on $T^{[r]fl}E^*$ is a linear combination of $\mathcal{T}^{[r]fl}X^*$ and the Liouville vector field L on $T^{[r]fl}E^*$.*

Theorem 7. *For $m \geq 2$, $n \geq 1$ and $r \geq 1$ any $\mathcal{VB}_{m,n}$-natural operator lifting a linear vector field X on E into a vector field $A(X)$ on $(T^{[r]fl}E)^*$ is a linear combination of $(\mathcal{T}^{[r]fl}X)^*$ and the Liouville vector field on $(T^{[r]fl}E)^*$.*

Theorem 8. *For $m \geq 2$, $n \geq 1$ and $r \geq 1$ any $\mathcal{VB}_{m,n}$-natural operator lifting a linear vector field X on E into a vector field $A(X)$ on $(T^{[r]fl}E^*)^*$ is a linear combination of $(\mathcal{T}^{[r]fl}X^*)^*$ and the Liouville vector field L on $(T^{[r]fl}E^*)^*$.*

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