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Some remarks about the Goebel–Kirk–Thele mapping

*Dedicated to W.A. Kirk on the occasion of
His Honorary Doctorate of
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Abstract. Directly inspired by the well-known construction of a nonlinear self-mapping of the unit ball of the Hilbert space ℓ_2 due to K. Goebel and W.A. Kirk, we introduce a new class of uniformly lipschitzian fixed point free mappings.

1. Introduction. A considerable part of metric fixed point theory is devoted to the study of *nonexpansive mappings*, (those which have Lipschitz constant $k = 1$) in closed convex bounded subsets of Banach spaces. If C is such a set and $k > 0$, a mapping $T : C \to C$ is *k-uniformly lipschitzian* on C if all the iterates T^n of T have the same Lipschitz constant k. This class of mappings was introduced by K. Goebel and W.A. Kirk [5], and it is strictly larger than the class of nonexpansive mappings. They obtained a fixed point theorem for k-uniformly lipschitzian mappings whenever k is sufficiently close to 1 (but greater than 1) in uniformly convex Banach spaces, and a bit later both authors together with R.E. Thele gave a similar result in Banach spaces with characteristic of convexity less than 1.

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Other fixed point theorems for uniformly lipschitzian mappings were given by Lifschitz [11], Casini–Maluta [1], and Domínguez Benavides [2] among many others (see the survey [7] for details). Roughly speaking, all of them claim that, under suitable geometrical conditions for the Banach space $(X, \|\cdot\|)$, there exists $k(X) > 1$ such that if $\lambda < k(X)$, each uniformly λ-lipschitzian self-mapping of any weakly compact convex subset of X has a fixed point. *A surprising characteristic of all these results is that none of them seems to be sharp*, that is, no maximal value for $k(X)$ is known. Even in the Hilbert spaces case, to fill this gap is a famous open problem: the greatest value known for $k(H)$ is $\sqrt{2}$ but no fixed point free k-uniformly lipschitzian mapping is known with $k < \frac{\pi}{2}$.

Thus, one can say that the fixed point theory for uniformly lipschitzian mappings needs an enlargement of the class of the examples of the fixed point free ones. The aim of these notes is to start this enlargement, by doing certain modifications on a celebrated example due to K. Goebel, W.A. Kirk and R.E. Thele given in [8].

2. Preliminaries. All the results of this paper are established in ℓ_2, the classical real space of all sequences $x = (x_n)$ for which $\sum_{i=1}^{\infty} x_i^2 < \infty$. The Euclidean norm $\|x\|_2 := \sqrt{\sum_{i=1}^{\infty} x_i^2}$ is associated to the ordinary inner product $\langle x, y\rangle = \sum_{n=1}^{\infty} x_n y_n$. The standard Schauder basis of $(\ell_2, \|\cdot\|_2)$ will be denoted by (e_n).

If $\|\cdot\|$ is a norm on ℓ_2 equivalent to $\|\cdot\|_2$, we will say that $\|\cdot\|$ is a renorming of ℓ_2.

We will denote the closed balls and the spheres as follows:

$$B_{\|\cdot\|} := \{x \in \ell_2 : \|x\| \leq 1\}, \quad S_{\|\cdot\|} := \{x \in \ell_2 : \|x\| = 1\}.$$

Also we will be concerned with the sets

$$B^+_{\|\cdot\|} := \{x \in B_{\|\cdot\|} : x_i \geq 0, \ i = 1, 2, \ldots\}$$

and $S^+_{\|\cdot\|} := B^+_{\|\cdot\|} \cap S_{\|\cdot\|}$.

In particular, $B_2 := B_{\|\cdot\|_2}$, $S_2 := S_{\|\cdot\|_2}$, $B_2^+ := B^+_{\|\cdot\|_2}$ and $S_2^+ := S^+_{\|\cdot\|_2}$.

If C is a closed convex subset of ℓ_2, $\|\cdot\|$ a renorming of ℓ_2 and $T : C \to C$ a lipschitzian mapping, by $\mathrm{Lip}\,(T, C, \|\cdot\|)$ we will denote the Lipschitz constant of T on C with respect to the metric associated to the norm $\|\cdot\|$. Moreover if T is uniformly lipschitzian on C, the symbol $\mathrm{Ulip}\,(T, C, \|\cdot\|)$ will be used instead of $\sup_n \mathrm{Lip}\,(T^n, C, \|\cdot\|)$.

The *right shift* operator $S : \ell_2 \to \ell_2$, is

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

Of course B_2, B_2^+, S_2 and S_2^+ are S-invariant sets.

Recall that two non zero vectors $v, w \in \ell_2$ are said to be *isosceles orthogonal* (with respect to the renorming $\|\cdot\|$), provided that $\|v+w\| = \|v-w\|$.

We complete this preliminary section giving a summary of well-known results (although we include some proof for the sake of readability).

2.1. The Kakutani mapping. Let $\varepsilon \in (0,1]$. By *Kakutani mapping* we will mean the transformation $\varphi_\varepsilon : B_2 \to B_2$ given by

$$\varphi_\varepsilon(x) = \varepsilon(1 - \|x\|_2)e_1 + S(x)$$

where $e_1 = (1, 0, \ldots) \in \ell_2$. In fact, the original example given by Kakutani in [10] was $\varphi_{\frac{1}{2}}$. Note that

$$\varphi_\varepsilon(x) = \begin{cases} (1 - \|x\|_2)(\varepsilon e_1) + \|x\|_2 S\left(\frac{1}{\|x\|_2}x\right), & x \neq 0_{\ell_2} \\ \varepsilon e_1, & x = 0_{\ell_2}. \end{cases}$$

It is straightforward to see that the map φ_ε has the $\|\cdot\|_2$-Lipschitz constant $\sqrt{1+\varepsilon^2}$. Moreover, the following facts are well-known.

(1) The mapping φ_ε is fixed-point free.
(2) The set B_2^+ is φ_ε-invariant.
(3) The Lipschitz constant of φ_ε in B_2 with respect to any other renorming is greater or equal to $\sqrt{1+\varepsilon^2}$ (see [14]).
(4) $\inf\{\|x - \varphi_\varepsilon(x)\| : x \in B_2^+\} = 0$. In fact, one can say a bit more:

$$\inf\{\|x - \varphi_\varepsilon(x)\| : x \in S_2^+\} = 0$$

(hence $\inf\{\|x - S(x)\| : x \in S_2^+\} = 0$).
(5) The mapping φ_ε is not uniformly lipschitzian in B_2^+ (see [15]).
(6) $\|\varphi_\varepsilon(x)\|_2 = \left[(1+\varepsilon^2)\|x\|_2^2 - 2\varepsilon^2\|x\|_2 + \varepsilon^2\right]^{1/2} \geq \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.

2.2. The K. Goebel, W.A. Kirk and R.L. Thele mapping. By Goebel–Kirk–Thele mapping we shall refer to the one defined in [8] as $R : B_2^+ \to S_2^+$ given by

$$R(x) = \frac{1}{\|\varphi_1(x)\|_2}\varphi_1(x).$$

It was claimed in [8] that R is *uniformly*-2-lipschitzian on B_2^+ (with respect to the Euclidean norm).

For convenience we will slightly modify this mapping here and we will consider the self-mappings of B_2^+ given by

$$R_\varepsilon(x) = \frac{1}{\|\varphi_\varepsilon(x)\|_2}\varphi_\varepsilon(x).$$

It is straightforward to see that every mapping R_ε is fixed point free.

Lemma 1. *If $x, y \in \ell_2$, $x \neq 0_{\ell_2} \neq y$, and $\|\cdot\|$ is a norm in this space associated to a scalar product $\langle \cdot, \cdot \rangle$, then*

$$\left\| \frac{1}{\|x\|} x - \frac{1}{\|y\|} y \right\| \leq \frac{1}{\|x\| \wedge \|y\|} \|x - y\|. \tag{1}$$

Proof. If $x, y \in \ell_2$, $x \neq 0_{\ell_2} \neq y$,

$$\begin{aligned}
\left\| \frac{1}{\|x\|} x - \frac{1}{\|y\|} y \right\|^2 &= 2 - \frac{2}{\|x\|\|y\|} \langle x, y \rangle \\
&= \frac{2\|x\|\|y\| - 2 \langle x, y \rangle}{\|x\|\|y\|} \\
&= \frac{2\|x\|\|y\| + (\|x - y\|^2 - \|x\|^2 - \|y\|^2)}{\|x\|\|y\|} \\
&= \frac{\|x - y\|^2 - (\|x\| - \|y\|)^2}{\|x\|\|y\|} \\
&\leq \frac{\|x - y\|^2}{\|x\|\|y\|} \\
&\leq \frac{\|x - y\|^2}{(\|x\| \wedge \|y\|)^2} \ .
\end{aligned}$$

□

Proposition 1. *For every $x, y \in B_2$ and $n \in \mathbb{N}$,*

$$\|R_\varepsilon^n(x) - R_\varepsilon^n(y)\|_2 \leq \frac{1 + \varepsilon^2}{\varepsilon} \|x - y\|_2 \tag{2}$$

Proof.

$$\begin{aligned}
\|R_\varepsilon(x) - R_\varepsilon(y)\|_2 &= \left\| \frac{1}{\|\varphi_\varepsilon(x)\|_2} \varphi_\varepsilon(x) - \frac{1}{\|\varphi_\varepsilon(y)\|_2} \varphi_\varepsilon(y) \right\| \\
&\leq \frac{1}{\|\varphi_\varepsilon(x)\|_2 \wedge \|\varphi_\varepsilon(y)\|_2} \|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)\|_2 \\
&\leq \frac{1}{\frac{\varepsilon}{\sqrt{1+\varepsilon^2}}} \|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)\|_2 \\
&\leq \frac{\sqrt{1 + \varepsilon^2}}{\varepsilon} \sqrt{1 + \varepsilon^2} \|x - y\|_2 \\
&= \frac{1 + \varepsilon^2}{\varepsilon} \|x - y\|_2 \ .
\end{aligned}$$

Since

$$R_\varepsilon^{n+1} = S^n \circ R_\varepsilon.$$

for $n \geq 1$ we see that all the iterates R_ε^n have the same Lipschitz constant as R_ε. □

One can check that the real function $\varepsilon \mapsto \frac{1+\varepsilon^2}{\varepsilon}$, $(\varepsilon > 0)$ attains its minimum value 2 at $\varepsilon = 1$. Therefore, in this sense, the original GKT mapping R_1 is the best choice among all that may be constructed in this way.

3. Modifying the norm. In this section we will introduce a class of nonlinear mappings in ℓ_2, which is, in fact, an elementary generalization of the Kakutani and GKT mappings.

3.1. The generalized Kakutani mappings. For $\varepsilon > 0$ and $\|\cdot\|$ an arbitrary renorming of ℓ_2, let $\varphi_{\varepsilon,\|\cdot\|} : \ell_2 \to \ell_2$ be the mapping given by

$$\varphi_{\varepsilon,\|\cdot\|}(x) := \varepsilon(1 - \|x\|)e_1 + S(x).$$

If $x \neq 0_{\ell_2}$, one can write

$$\varphi_{\varepsilon,\|\cdot\|}(x) := (1 - \|x\|)(\varepsilon e_1) + \|x\| S\left(\frac{1}{\|x\|}x\right).$$

Hence, $\|\varphi_{\varepsilon,\|\cdot\|}(x)\| \leq 1$ whenever $\|\varepsilon e_1\| \leq 1$, $\|S\| \leq 1$ and $\|x\| \leq 1$. Thus, $\varphi_{\varepsilon,\|\cdot\|}$ leaves invariant the unit ball of $(\ell_2, \|\cdot\|)$, (as well as its positive part $B^+_{\|\cdot\|}$), provided that the above first two conditions are fulfilled.

Again it is straightforward to see that $\varphi_{\varepsilon,\|\cdot\|}$ has no fixed points.

Proposition 2. *There exists $m > 0$ (depending on ε and $\|\cdot\|$), such that for every $x \in B_{\|\cdot\|}$,*

$$\|\varphi_{\varepsilon,\|\cdot\|}(x)\| \geq m.$$

Proof. Assuming it is not so, there exists a sequence (x^n) in $B_{\|\cdot\|}$ such that $\|\varphi_{\varepsilon,\|\cdot\|}(x^n)\| \to 0$, that is,

$$\varepsilon(1 - \|x^n\|)e_1 + S(x^n) \to 0_{\ell_2}.$$

If P_1 is the projection $P_1(x_1, x_2, \ldots) = x_1$, then

$$P_1[\varepsilon(1 - \|x^n\|)e_1 + S(x^n)] \to 0,$$

which implies that $\|x^n\| \to 1$. Therefore,

$$\begin{aligned}\|S(x^n)\| &\leq \|\varepsilon(1 - \|x^n\|)e_1 + S(x^n)\| + \|\varepsilon(1 - \|x^n\|)e_1\| \\ &= \|\varphi_{\varepsilon,\|\cdot\|}(x^n)\| + \|\varepsilon(1 - \|x^n\|)e_1\| \to 0,\end{aligned}$$

and this forces that $\|x^n\|_2 = \|S(x^n)\|_2 \to 0$, a contradiction. □

It is easy to see that $\mathrm{Lip}\left(\varphi_{\varepsilon,\|\cdot\|}, B_{\|\cdot\|}, \|\cdot\|\right) \leq \varepsilon\|e_1\| + \|S\|$. Nevertheless, it is shown in [15] that the mapping $\varphi_{1,\|\cdot\|_2}$ is not uniformly lipschitzian on B_2. We do not know if the same is true for an arbitrary $\varphi_{\varepsilon,\|\cdot\|}$ on $B_{\|\cdot\|}$.

3.2. The Goebel–Kirk–Thele generalized mappings. For $\varepsilon > 0$ and $\|\cdot\|$ an arbitrary renorming of ℓ_2 we define the mapping $R_{\varepsilon,\|\cdot\|} : B_{\|\cdot\|} \to S_{\|\cdot\|}$ as

$$\begin{aligned} R_{\varepsilon,\|\cdot\|}(x) &:= \frac{1}{\|\varepsilon(1-\|x\|)e_1 + S(x)\|}[\varepsilon(1-\|x\|)e_1 + S(x)] \\ &= \frac{1}{\|\varphi_{\varepsilon,\|\cdot\|}(x)\|}\varphi_{\varepsilon,\|\cdot\|}(x). \end{aligned}$$

Notice that this mapping is well-defined, because $m := \inf\{\|\varphi_{\varepsilon,\|\cdot\|}(x)\| : x \in B_{\|\cdot\|}\} > 0$, as it was noted in Proposition 2. Even for $\varepsilon > 1$ these mappings are well-defined too. In fact, $R_{\varepsilon,\|\cdot\|}(B^+_{\|\cdot\|}) \subset S^+_{\|\cdot\|}$.

These mappings have not fixed points in $B_{\|\cdot\|}$. Indeed, if $R_{\varepsilon,\|\cdot\|}(x) = x$ for some $x \in B_{\|\cdot\|}$ then one has that $\|x\| = 1$ and hence

$$x = \frac{1}{\|S(x)\|}S(x).$$

If $x = (x_1, x_2, x_3, \ldots)$ then

$$\begin{aligned} x_1 &= 0, \\ x_2 &= \frac{1}{\|S(x)\|}x_1 = 0, \\ x_3 &= \frac{1}{\|S(x)\|}x_2 = 0, \\ &\vdots \end{aligned}$$

It follows that $x = 0_{\ell_2}$, a contradiction.

On the other hand,

$$R^2_{\varepsilon,\|\cdot\|}(x) = \frac{1}{\|S(R_{\varepsilon,\|\cdot\|}(x))\|}S(R_{\varepsilon,\|\cdot\|}(x)).$$

If S is a $\|\cdot\|$-isometry then $\|S(R_{\varepsilon,\|\cdot\|}(x))\| = \|R_{\varepsilon,\|\cdot\|}(x)\| = 1$ and

$$\mathrm{Ulip}\left(R_{\varepsilon,\|\cdot\|}, B_{\|\cdot\|}, \|\cdot\|\right) = \mathrm{Lip}\left(R_{\varepsilon,\|\cdot\|}, B_{\|\cdot\|}, \|\cdot\|\right).$$

(Of course, it is impossible that $\|S\| < 1$ because, under this assumption, S would be a strict contraction leaving invariant the closed set S_2, and then S must have a fixed point in S_2, which is absurd). If $\|S\| > 1$ then the mapping $R_{\varepsilon,\|\cdot\|}$ is still well-defined, as well as if $\varepsilon > 1$, but it is unclear whether or not $R_{\varepsilon,\|\cdot\|}$ is uniformly lipschitzian in this case.

Remark 1. If the right shift S is an isometry then the restriction of the mapping $R_{\varepsilon,\|\cdot\|}$ to the sphere $S_{\|\cdot\|}$ is in fact the shift S. This right shift is fixed point free and has minimal displacement $d_S := \inf\{\|x - S(x)\| : x \in S_2\}$ equal to 0. We surely can expect a similar behavior of $R_{\varepsilon,\|\cdot\|}$ on $S_{\|\cdot\|}$. For some interesting comments about the minimal displacement of uniformly lipschitzian mappings one can see [9].

Let us to point out that from the Massera–Schaffer inequality (see [12]), for $x, y \in B_{\|\cdot\|}$ we have

$$\|R_{\varepsilon,\|\cdot\|}(x) - R_{\varepsilon,\|\cdot\|}(y)\| = \left\| \frac{\varphi_{\varepsilon,\|\cdot\|}(x)}{\|\varphi_{\varepsilon,\|\cdot\|}(x)\|} - \frac{\varphi_{\varepsilon,\|\cdot\|}(y)}{\|\varphi_{\varepsilon,\|\cdot\|}(y)\|} \right\|$$
$$\leq \frac{2}{\max\{\|\varphi_{\varepsilon,\|\cdot\|}(x)\|, \|\varphi_{\varepsilon,\|\cdot\|}(y)\|\}} \|\varphi_{\varepsilon,\|\cdot\|}(x) - \varphi_{\varepsilon,\|\cdot\|}(y)\|.$$

Since

$$\|\varphi_{\varepsilon,\|\cdot\|}(x) - \varphi_{\varepsilon,\|\cdot\|}(y)\| \leq (\varepsilon\|e_1\| + \|S\|)\|x - y\|,$$

we conclude that

$$\operatorname{Lip}\left(R_{\varepsilon,\|\cdot\|}, B_{\|\cdot\|}, \|\cdot\|\right) \leq \frac{2(\varepsilon\|e_1\| + \|S\|)}{m}.$$

where $m := \inf\{\|\varphi_{\varepsilon,\|\cdot\|}(y)\| : y \in B_{\|\cdot\|}\}$.

The following result will give us lower bounds for the Lipschitz constant of $R_{\varepsilon,\|\cdot\|}$. We will show that it is hard to improve the Lipschitz constant 2, just the one which has $R_{1,\|\cdot\|}$ in the Euclidean case.

With more precision, depending on the existence of a vector in the $\|\cdot\|$-sphere such that $S(v)$ is in some sense orthogonal to e_1, all these mappings have a *bad* Lipschitz constant k with respect to $\|\cdot\|$ in the unit ball, in the sense that this is greater or equal to 2.

Theorem 1. *Let $R_{\varepsilon,\|\cdot\|} : B^+_{\|\cdot\|} \longrightarrow S^+_{\|\cdot\|}$ be the Goebel–Kirk–Thele type mapping given by*

$$R_{\varepsilon,\|\cdot\|}(x) := \frac{1}{\|\varphi_{\varepsilon,\|\cdot\|}(x)\|} \varphi_{\varepsilon,\|\cdot\|}(x).$$

If there exists $v \in S^+_{\|\cdot\|}$ such that

1) $S(v)$ is isosceles orthogonal to εe_1,

2) for all positive real numbers a, b,

$$\|a\varepsilon e_1 + bS(v)\| = \|b\varepsilon e_1 + aS(v)\|,$$

then

$$\operatorname{Lip}\left(R_{\varepsilon,\|\cdot\|}, B_{\|\cdot\|}, \|\cdot\|\right) \geq 2.$$

Proof. Let $v \in S^+_{\|\cdot\|}$ be the vector whose existence is assumed.

Then, if k is the Lipschitz constant of $R_{\varepsilon,\|\cdot\|}$, for $\eta > 0$ small enough,

$$\begin{aligned}
k &\geq \frac{\|R_{\varepsilon,\|\cdot\|}\big((1/2+\eta)v\big) - R_{\varepsilon,\|\cdot\|}\big((1/2-\eta)v\big)\|}{\|(1/2+\eta)v-(1/2-\eta)v\|} \\
&= \frac{1}{2\eta}\|R_{\varepsilon,\|\cdot\|}\big((1/2+\eta)v\big) - R_{\varepsilon,\|\cdot\|}\big((1/2-\eta)v\big)\| \\
&= \frac{1}{2\eta}\left\|\frac{\varepsilon(1/2-\eta)e_1+(1/2+\eta)S(v)}{\|(1/2-\eta)\varepsilon e_1+(1/2+\eta)S(v)\|} - \frac{\varepsilon(1/2+\eta)e_1+(1/2-\eta)S(v)}{\|(1/2+\eta)\varepsilon e_1+(1/2-\eta)S(v)\|}\right\| \\
&= \frac{\|((1/2-\eta)\varepsilon e_1+(1/2+\eta)S(v)) - ((1/2+\eta)\varepsilon e_1+(1/2-\eta)S(v))\|}{2\eta\|(1/2-\eta)\varepsilon e_1+(1/2+\eta)S(v)\|} \\
&= \frac{\|2\eta(S(v)-\varepsilon e_1)\|}{2\eta\|(1/2-\eta)\varepsilon e_1+(1/2+\eta)S(v)\|} \\
&= \frac{\|S(v)-\varepsilon e_1\|}{\|(1/2-\eta)\varepsilon e_1+(1/2+\eta)S(v)\|}.
\end{aligned}$$

Letting $\eta \to 0$,

$$k \geq 2\frac{\|S(v)-\varepsilon e_1\|}{\|S(v)+\varepsilon e_1\|}.$$

If $S(v)$ is orthogonal to εe_1 in the isosceles sense, then $k \geq 2$ which completes the proof. □

The existence of such a vector v satisfying conditions (1) and (2) of Theorem 1 is unclear, in the sense that it depends on the norm considered in ℓ_2. In any case, condition (2) implies that $\varepsilon\|e_1\| = \|S(v)\|$.

Example 1. If we consider the James norms on ℓ_2

$$|x|_\beta := \max\{\|x\|_2, \beta\|x\|_\infty\},$$

(where $\beta > 1$), then taking $v = e_1$ we see that $S(v) = e_2$ is isosceles orthogonal to e_1, because

$$\|e_2+e_1\|_2 = \sqrt{2} = \|e_2-e_1\|_2,$$

$$\|e_2+e_1\|_\infty = 1 = \|e_2-e_1\|_\beta.$$

Hence,

$$|e_2+e_1|_\beta = \max\{\sqrt{2},\beta\} = |e_2-e_1|_\beta.$$

On the other hand, for positive a, b,

$$\|ae_2+be_1\|_2 = \sqrt{a^2+b^2} = \|be_2+ae_1\|_2,$$

$$\|ae_2+be_1\|_\infty = \max\{|a|,|b|\} = \|be_2+ae_1\|_\infty.$$

Hence $|ae_2 + be_1|_\beta = |be_2 + ae_1|_\beta$. Thus, $\mathrm{Lip}\left(R_{1,|\cdot|_\beta}, B_{|\cdot|_\beta}, |\cdot|_\beta\right) \geq 2$.

Remark 2. The key fact in the proof of Theorem 1 is that the GKT type mappings have their highest expansivity around the sphere of radius $\frac{1}{2}$. For example, if $e_1, e_2 \in S^+_{\|\cdot\|}$, and $\|\varepsilon e_1 + e_2\| = \|\varepsilon e_1 + e_3\|$, with the same notation as in the above proof we have,

$$\begin{aligned} k &\geq \frac{\|R_{\varepsilon,\|\cdot\|}\big((1/2)e_1\big) - R_{\varepsilon,\|\cdot\|}\big((1/2)e_2\big)\|}{\|(1/2)e_1 - (1/2)e_2\|} \\ &= \frac{\left\| \frac{\varepsilon(1/2)e_1 + (1/2)e_2}{\|(1/2)\varepsilon e_1 + (1/2)e_2)\|} - \frac{\varepsilon(1/2)e_1 + (1/2)e_3}{\|(1/2)\varepsilon e_1 + (1/2)e_3\|} \right\|}{\|(1/2)e_1 - (1/2)e_2\|} \\ &= \frac{\left\| \frac{\varepsilon e_1 + e_2}{\|\varepsilon e_1 + e_2\|} - \frac{\varepsilon e_1 + e_3}{\|\varepsilon e_1 + e_3\|} \right\|}{\|(1/2)e_1 - (1/2)e_2\|} \\ &= 2\frac{\|e_2 - e_3\|}{\|\varepsilon e_1 + e_2\|\|e_1 - e_2\|}. \end{aligned}$$

This bound is in general smaller than 2. In fact, one can say a bit more concerning the expansivity of the GKT type mappings: it is greater if we consider pairs of vectors with norm near to $\frac{1}{2}$, and both vectors belonging to a straight line passing through the origin.

Recall that a Banach space $(X, \|\cdot\|)$ has the WORTH property [16] if

$$\lim_{n\to\infty} |\|x_n - x\| - \|x_n + x\|| = 0$$

for all x in X and for all weakly null sequences (x_n).

If we have in $B^+_{\|\cdot\|}$ a weakly null sequence (v_n) satisfying condition (2) of the above theorem, then removing condition (1) we still can get a lower bound of the Lipschitz constant k.

Theorem 2. *Let $R_{\varepsilon,\|\cdot\|} : B^+_{\|\cdot\|} \longrightarrow S^+_{\|\cdot\|}$ be the Goebel–Kirk–Thele type mapping given by*

$$R_{\varepsilon,\|\cdot\|}(x) := \frac{1}{\|\varphi_{\varepsilon,\|\cdot\|}(x)\|}\varphi_{\varepsilon,\|\cdot\|}(x).$$

If there exists a weakly null sequence (v_n) in $S^+_{\|\cdot\|}$ such that
2') for all positive real numbers a, b, and positive integer n,

$$\|a\varepsilon e_1 + bS(v_n)\| = \|b\varepsilon e_1 + aS(v_n)\|$$

and if $(\ell_2, \|\cdot\|)$ has the WORTH property then

$$\mathrm{Lip}\left(R_{\varepsilon,\|\cdot\|}, B^+_{\|\cdot\|}, \|\cdot\|\right) \geq 2.$$

Proof. Repeating the same argument as in the above proof, we obtain

$$k \geq 2\frac{\|S(v_n) - \varepsilon e_1\|}{\|S(v_n) + \varepsilon e_1\|}.$$

By passing to subsequences if necessary we may suppose that there exist the real numbers $\lim_n \|S(v_n) - \varepsilon e_1\|$ and $\lim_n \|S(v_n) - \varepsilon e_1\|$.

Since S is a bounded operator, $(S(v_n))$ is a weakly null sequence. Moreover, WORTH property implies that

$$\lim_n \|S(v_n) - \varepsilon e_1\| = \lim_n \|S(v_n) + \varepsilon e_1\|.$$

Thus,

$$k \geq 2 \lim_n \frac{\|S(v_n) - \varepsilon e_1\|}{\|S(v_n) + \varepsilon e_1\|} = 2.$$

□

4. Behavior under renormings. If $T : C \longrightarrow C$ is fixed point free and uniformly lipschitzian on C with respect to $\|\cdot\|$, one could imitate the well-known Bielecki's approach, looking for a renorming $|\cdot|$ of X for which T becomes nonexpansive (and of course fixed point free) on C or, at least, with a smaller Lipschitz constant.

That was the underlying purpose of the authors of the recent papers [4], [14], [15], although they did not succeed.

The set C under consideration could be at least as relevant as the norm, in order to obtain reductions of the Lipschitz constant of a mapping. To illustrate this, we can regard a celebrated example due to T.C. Lim ([13]).

He defined a mapping $\mathcal{T}$ in the classical space ℓ_1, such that

$$\text{Lip}\,(\mathcal{T}, B_{\ell_1}[0_{\ell_1}, 1], \|\cdot\|_1) = 2$$

but

$$\text{Lip}\left(\mathcal{T}, B_1^+, \|\cdot\|\right) = 1$$

where

$$B_1^+ := \left\{ x \in \ell_1^+ : x_n \geq 0,\ \sum_{n=1}^{\infty} x_n \leq 1 \right\}$$

and $\|x\| := \max\{\|x^+\|_1, \|x^-\|_1\}$. (Here x^+, x^- is respectively the positive and the negative part of $x \in \ell_1$).

These facts show that dramatic reductions of the Lipschitz constant of a mapping T are possible by renormings of the underlying space, mainly when T is restricted to a suitable T-invariant subset of its domain.

Nevertheless, for the Kakutani mapping $\varphi_{\varepsilon,\|\cdot\|_2}$, it was shown in [14] that its Lipschitz constant on B_2, $\sqrt{1+\varepsilon^2}$, can not be reduced after renormings. But the following example shows that for the generalized Kakutani mappings some reductions are possible.

Example 2. Let $\|\cdot\|$ be the norm on ℓ_2 defined as

$$\|x\| := |x_1| + \|(x_2, x_3, \dots)\|_2.$$

It is straightforward to see that $\|x\|_2 \leq \|x\| \leq \sqrt{2}\|x\|_2$ for each $x \in \ell_2$.

We have that $\operatorname{Lip}\left(\varphi_{1,\|\cdot\|}, B_{\|\cdot\|}, \|\cdot\|\right) = 2$. Indeed, for $x, y \in B_{\|\cdot\|}$,

$$\begin{aligned}
\|\varphi_{1,\|\cdot\|}(x) - \varphi_{1,\|\cdot\|}(y)\| &= \|(\|y\| - \|x\|)e_1 + S(x-y)\| \\
&\leq |(\|y\| - \|x\|)|\|e_1\| + \|S(x-y)\| \\
&\leq \|y-x\| + \|x-y\|_2 \\
&\leq 2\|y-x\|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left\|\varphi_{1,\|\cdot\|}(e_1) - \varphi_{1,\|\cdot\|}\left(\frac{1}{2}\, e_1\right)\right\| &= \left\|e_2 - \left(\left(1 - \frac{1}{2}\right) e_1 + \frac{1}{2} e_2\right)\right\| \\
&= \frac{1}{2}\|e_2 - e_1\| \\
&= 2\|e_1 - \frac{1}{2} e_1\|.
\end{aligned}$$

But $\operatorname{Lip}\left(\varphi_{1,\|\cdot\|}, B_{\|\cdot\|}, \|\cdot\|_2\right) = \sqrt{3}$. Indeed, for $x, y \in B_{\|\cdot\|}$,

$$\begin{aligned}
\|\varphi_{1,\|\cdot\|}(x) - \varphi_{1,\|\cdot\|}(y)\|_2^2 &= \|(\|y\| - \|x\|)e_1 + S(x-y)\|_2 \\
&= |(\|y\| - \|x\|)|^2 + \|S(x-y)\|_2^2 \\
&\leq \|y-x\|^2 + \|x-y\|_2^2 \\
&\leq 3\|y-x\|_2^2,
\end{aligned}$$

and

$$\begin{aligned}
\left\|\varphi_{1,\|\cdot\|}\left(\frac{1}{2}, \frac{1}{2}, 0, \dots\right) - \varphi_{1,\|\cdot\|}(0_{\ell_2})\right\|_2 &= \left\|\left(-1, \frac{1}{2}, \frac{1}{2}, 0, \dots\right)\right\|_2 \\
&= \sqrt{\frac{3}{2}} \\
&= \sqrt{3}\left\|\left(\frac{1}{2}, \frac{1}{2}, 0, \dots\right)\right\|_2 .
\end{aligned}$$

Unfortunately, for the GKT type mappings these reductions have lower bounds, at least when the norm has some kind of regularity.

Theorem 3. *Let $R_{\varepsilon,\|\cdot\|} : B^+_{\|\cdot\|} \longrightarrow S^+_{\|\cdot\|}$ be the Goebel–Kirk–Thele type mapping given by*

$$R_{\varepsilon,\|\cdot\|}(x) := \frac{1}{\|\varphi_{\varepsilon,\|\cdot\|}(x)\|}\varphi_{\varepsilon,\|\cdot\|}(x).$$

Let us suppose that $|\cdot|$ is a renorming of ℓ_2 such that

$$\|w\| \le |w| \le \beta\|w\|$$

for each $w \in \ell_2$.

If there exists $v \in S^+_{\|\cdot\|}$ such that
1) $S(v)$ is $\|\cdot\|$-isosceles orthogonal to εe_1,
2) $\|a\varepsilon e_1 + bS(v)\| = \|b\varepsilon e_1 + aS(v)\|$ for all positive real numbers a, b, then

$$\text{Lip}\left(R_{\varepsilon,\|\cdot\|}, B_{\|\cdot\|}, |\cdot|\right) \ge \frac{2}{|v|}.$$

Proof. Let $v \in S_{\|\cdot\|}$ be the vector whose existence is assumed.

Then, if k is the Lipschitz constant of $R_{\varepsilon,\|\cdot\|}$, for $\eta > 0$ small enough,

$$\begin{aligned}
k &\ge \frac{|R_{\varepsilon,\|\cdot\|}\big((1/2+\eta)v\big) - R_{\varepsilon,\|\cdot\|}\big((1/2-\eta)v\big)|}{|(1/2+\eta)v - (1/2-\eta)v|}\\
&= \frac{|R_{\varepsilon,\|\cdot\|}\big((1/2+\eta)v\big) - R_{\varepsilon,\|\cdot\|}\big((1/2-\eta)v\big)|}{2\eta|v|}\\
&= \frac{1}{2\eta|v|}\left|\frac{\varepsilon(1/2-\eta)e_1+(1/2+\eta)S(v)}{\|(1/2-\eta)\varepsilon e_1+(1/2+\eta)S(v)\|} - \frac{\varepsilon(1/2+\eta)e_1+(1/2-\eta)S(v)}{\|(1/2+\eta)\varepsilon e_1+(1/2-\eta)S(v)\|}\right|\\
&= \frac{|((1/2-\eta)\varepsilon e_1 + (1/2+\eta)S(v)) - ((1/2+\eta)\varepsilon e_1 + (1/2-\eta)S(v))|}{2\eta\|(1/2-\eta)\varepsilon e_1 + (1/2+\eta)S(v)\||v|}\\
&= \frac{|2\eta(S(v) - \varepsilon e_1)|}{2\eta\|(1/2-\eta)\varepsilon e_1 + (1/2+\eta)S(v)\||v|}\\
&= \frac{|S(v) - \varepsilon e_1|}{\|(1/2-\eta)\varepsilon e_1 + (1/2+\eta)S(v)\||v|}\ .
\end{aligned}$$

Letting $\eta \to 0$, we obtain

$$k \ge 2\frac{|S(v) - \varepsilon e_1|}{\|S(v) + \varepsilon e_1\||v|}\ .$$

Since $S(v)$ is $\|\cdot\|$-orthogonal to εe_1 in the isosceles sense, then

$$k \ge 2\frac{|S(v) - \varepsilon e_1|}{\|S(v) + \varepsilon e_1\||v|} \ge 2\frac{\|S(v) - \varepsilon e_1\|}{\|S(v) + \varepsilon e_1\||v|} = \frac{2}{|v|}\ .$$

□

Note that Theorem 1 can be obtained as a particular case of Theorem 3 by taking $|\cdot| = \|\cdot\|$.

Let us recall the following results (see [14] for details).

Theorem 4. *Let* $(X, \|\cdot\|)$ *be a Banach space with an equivalent norm* $|\cdot|$, *and let* $T : B_{\|\cdot\|}[x_0, R] \to X$, *where* $x_0 \in X$ *and* $R > 0$. *Then for all* $\rho \in (0, R]$

$$\operatorname{Lip}\big(T, B_{\|\cdot\|}[x_0, R], |\cdot|\big) \geq \frac{d_T(\rho)}{\rho},$$

where

$$d_T(\rho) := \inf\{\|T(y) - T(x_0)\| : \|y - x_0\| = \rho\}.$$

In the case of the GKT mapping $R_{1,\|\cdot\|_2}$ we obtain

Corollary 1.

$$\operatorname{Lip}\Big(R_{\varepsilon,\|\cdot\|_2}, B_2, |\cdot|\Big) \geq \frac{1}{\rho}\sqrt{2 - \frac{2(1-\rho)\varepsilon}{\sqrt{\varepsilon^2(1-\rho)^2 + \rho^2}}}\ .$$

In particular for $\varepsilon = 1$, $\operatorname{Lip}\Big(R_{\varepsilon,\|\cdot\|_2}, B_2, |\cdot|\Big) \geq 1.57780$.

Remark 3. It should be noted that Theorem 4 does not hold in general if the domain of the mapping T is not a ball.

5. Further remarks on the Kakutani type mappings. It was shown in [15] that the mapping $\varphi_{1,\|\cdot\|_2}$ is not uniformly lipschitzian on B_2. In fact we do not know whether the same fact is true for the generalized mappings $\varphi_{1,\|\cdot\|}$, that is we do not know whether

$$\operatorname{Ulip}\big(\varphi_{1,\|\cdot\|}, B_{\|\cdot\|}, \|\cdot\|\big) < +\infty$$

for some renorming $\|\cdot\|$. Nevertheless, under quite natural assumptions if they have a uniform Lipschitz constant k with respect to the Euclidean norm (which in turn implies that the same is true with respect to any other equivalent norm), then this constant must be large enough. With more precision we have

Theorem 5. *Let* $\|\cdot\|$ *be a renorming of* ℓ_2 *such that* $\|S\| \leq 1$. *Let* $\delta(\cdot)$ *be the Clarkson modulus of convexity of* $(\ell_2, \|\cdot\|)$. *Suppose that* $\|e_n\| = 1$ *for each positive integer* n. *Then, if the mapping* $\varphi_{1,\|\cdot\|}$ *admits the uniform Lipschitz constant* k *on* $B_{\|\cdot\|}$ *with respect to the norm* $\|\cdot\|_2$, *one has*

$$k \geq \sqrt{2 + (2\delta(\|e_1 - e_2\|))^2}.$$

Proof. Since $\|e_1\| \leq 1$ and $\|S\| \leq 1$ we can assure that $\varphi_{1,\|\cdot\|}(B_{\|\cdot\|}) \subset B_{\|\cdot\|}$. Moreover, for every $x \in B_{\|\cdot\|}$,

$$\|\varphi_{1,\|\cdot\|}(x)\|_2 = \sqrt{(1-\|x\|)^2 + \|S(x)\|_2^2} \geq \|S(x)\|_2 = \|x\|_2.$$

Hence the sequence $\left(\|\varphi^n_{1,\|\cdot\|}(x)\|_2\right)$ is nondecreasing. In particular for $\lambda \in [0,1]$ it is well-defined $f(\lambda) := \lim_{n\to\infty} \|\varphi^n_{1,\|\cdot\|}(\lambda e_1)\|_2$. Moreover,

$$\begin{aligned} f(\lambda) &\geq \|\varphi^2_{1,\|\cdot\|}(\lambda e_1)\|_2 \\ &= \left\|(1-\|\varphi_{1,\|\cdot\|}(\lambda e_1)\|)e_1 + S(\varphi_{1,\|\cdot\|}(\lambda e_1))\right\|_2 \\ &= \left\|(1-\|\varphi_{1,\|\cdot\|}(\lambda e_1)\|)e_1 + (1-\lambda)e_2 + \lambda e_3\right\|_2 \\ &= \sqrt{(1-\lambda)^2 + \lambda^2 + (1-\|\varphi_{1,\|\cdot\|}(\lambda e_1)\|)^2}. \end{aligned} \tag{3}$$

On the other hand, our assumption $\|e_n\| = 1$ $(n \geq 1)$ implies that $\varphi^n_{1,\|\cdot\|}(e_1) = e_{n+1}$. Moreover, we claim that for every positive integer n

$$\left\langle \varphi^n_{1,\|\cdot\|}(e_1), \varphi^n_{1,\|\cdot\|}(\lambda e_1)\right\rangle = \lambda.$$

Indeed, it is obvious for $n = 1$, and

$$\begin{aligned} \left\langle \varphi^{n+1}_{1,\|\cdot\|}(e_1), \varphi^{n+1}_{1,\|\cdot\|}(\lambda e_1)\right\rangle &= \left\langle e_{n+2}, \varphi^{n+1}_{1,\|\cdot\|}(\lambda e_1)\right\rangle \\ &= \left\langle e_{n+2}, (1-\|\varphi^n_{1,\|\cdot\|}(\lambda e_1)\|)e_1 + S(\varphi^n_{1,\|\cdot\|}(\lambda e_1))\right\rangle \\ &= \left\langle e_{n+2}, S(\varphi^n_{1,\|\cdot\|}(\lambda e_1))\right\rangle \\ &= \left\langle e_{n+1}, \varphi^n_{1,\|\cdot\|}(\lambda e_1)\right\rangle \\ &= \left\langle \varphi^n_{1,\|\cdot\|}(e_1), \varphi^n_{1,\|\cdot\|}(\lambda e_1)\right\rangle. \end{aligned}$$

If there exists $k > 0$ such that

$$\|\varphi^n_{1,\|\cdot\|}(x) - \varphi^n_{1,\|\cdot\|}(y)\|_2 \leq k\|x-y\|_2$$

for all positive integer n and for all $x, y \in B_{\|\cdot\|}$, in particular one has that

$$\|\varphi^n_{1,\|\cdot\|_2}(e_1) - \varphi^n_{1,\|\cdot\|}(\lambda e_1)\|_2 \leq k(1-\lambda),$$

that is,

$$\sqrt{\|e_{n+1}\|_2^2 + \|\varphi^n_{1,\|\cdot\|}(\lambda e_1)\|_2^2 - 2\left\langle e_{n+1}, \varphi^n_{1,\|\cdot\|}(\lambda e_1)\right\rangle} \leq k(1-\lambda).$$

Hence,

$$\sqrt{1+\|\varphi^n_{1,\|\cdot\|}(\lambda e_1)\|_2^2-2\lambda}\le k(1-\lambda)$$

and then

$$\lim_{n\to\infty}\sqrt{1+\|\varphi^n_{1,\|\cdot\|}(\lambda e_1)\|_2^2-2\lambda}\le k(1-\lambda).$$

It follows that

$$1+f(\lambda)^2-2\lambda\le k^2(1-\lambda)^2. \tag{4}$$

Bearing in mind the inequality (3), we have that

$$1+[(1-\|\varphi_{1,\|\cdot\|}(\lambda e_1)\|)^2+(1-\lambda)^2+\lambda^2]-2\lambda\le 1+f(\lambda)^2-2\lambda\le k^2(1-\lambda)^2 \tag{5}$$

and hence

$$\frac{2(1-\lambda)^2+(1-\|\varphi_{1,\|\cdot\|}(\lambda e_1)\|)^2}{(1-\lambda)^2}\le k^2,$$

or, in other words,

$$\frac{2(1-\lambda)^2+(1-\|(1-\lambda)e_1+\lambda e_2\|)^2}{(1-\lambda)^2}\le k^2.$$

In particular for $\lambda=1/2$,

$$2+4(\delta(\|e_1-e_2\|))^2=\frac{\frac12+(\delta(\|e_1-e_2\|))^2}{\frac14}\le\frac{\frac12+(1-\frac12\|e_1+e_2\|)^2}{\frac14}\le k^2$$

which yields the conclusion. □

Remark 4. The above theorem is not sharp in the following sense. Let $\|\cdot\|$ be the renorming of ℓ_2 considered in the Example 2, and $\delta(\cdot)$ its modulus of convexity. Since $\|e_1-e_2\|=2=\|e_1+e_2\|$ one has that $\delta(\|e_1-e_2\|)=0$. Hence by Theorem 5

$$\mathrm{Ulip}\left(\varphi_{1,\|\cdot\|},B_{\|\cdot\|},\|\cdot\|_2\right)\ge\sqrt{2+(2\delta(\|e_1-e_2\|))^2}=\sqrt{2},$$

whereas we know that $\mathrm{Lip}\left(\varphi_{1,\|\cdot\|},B_{\|\cdot\|},\|\cdot\|_2\right)=\sqrt{3}$.

If we remember that the mapping $\varphi_{1,\|\cdot\|_2}$ has uniform Lipschitz constant $+\infty$ on B_2, the following result can be considered as a kind of stability of this constant. When a norm is close enough to $\|\cdot\|_2$ then $\mathrm{Ulip}\left(\varphi_{1,\|\cdot\|},B_{\|\cdot\|},\|\cdot\|\right)$, if it exists, must be very large.

Theorem 6. *Let $\|\cdot\|$ be a renorming of ℓ_2 such that $\|S\| \leq 1$. Let $\alpha > 0$, $\beta > 1$ such that for every $v \in \ell_2$*

$$\alpha\|v\|_2 \leq \|v\| \leq \beta\|v\|_2.$$

Suppose that $\|e_n\| = 1$ for each positive integer n. Then, if the mapping $\varphi_{1,\|\cdot\|}$ is uniformly k-lipschitzian on $B_{\|\cdot\|}$ with respect to the Euclidean norm $\|\cdot\|_2$, one has

$$k > \sqrt{\frac{\beta^2}{\beta^2 - 1}}.$$

Proof. We can repeat word by word the first part of the proof of Theorem 5. Let us observe that for every $x \in B_{\|\cdot\|}$

$$\begin{aligned}\|\varphi^{n+1}_{1,\|\cdot\|}(x)\|_2^2 &= (1 - \|\varphi^n_{1,\|\cdot\|}(x)\|)^2 + \|S(\varphi^n_{1,\|\cdot\|}(x))\|_2^2 \\ &= (1 - \|\varphi^n_{1,\|\cdot\|}(x)\|)^2 + \|\varphi^n_{1,\|\cdot\|}(x)\|_2^2\end{aligned}$$

and that the sequence $(\|\varphi^n_{1,\|\cdot\|}(x)\|_2)$ is nondecreasing. Taking limits when n tends to ∞ we obtain that

$$\lim_n \|\varphi^n_{1,\|\cdot\|}(x)\| = 1.$$

Since

$$\alpha\|\varphi^n_{1,\|\cdot\|}(\lambda e_1)\|_2 \leq \|\varphi^n_{1,\|\cdot\|}(\lambda e_1)\| \leq \beta\|\varphi^n_{1,\|\cdot\|}(\lambda e_1)\|_2,$$

we have that

$$\alpha f(\lambda) \leq 1 \leq \beta f(\lambda).$$

From inequalities (4) and (5) it follows that

$$1 + \frac{1}{\beta^2} - 2\lambda \leq 1 + f(\lambda)^2 - 2\lambda \leq k^2(1 - \lambda)^2.$$

In particular for $\lambda = 1/\beta^2$,

$$1 - \frac{1}{\beta^2} \leq k^2 \left(\frac{\beta^2 - 1}{\beta^2}\right)^2$$

which yields the conclusion. □

Questions

1. Does the conclusion of Theorem 1 hold without requirements (1) and/or (2)?

2. Is any mapping $\varphi_{\epsilon,\|\cdot\|}$ uniformly lipschitzian on $B_{\|\cdot\|}$? If the answer were yes, characterize the renormings with this property.

3. In $(L^1([0,1]), \|\cdot\|_1)$ there exist a weakly compact convex subset C and a fixed point free nonexpansive self-mapping T of C. By a result due to van Dulst, given $\varepsilon > 0$ there exists an equivalent norm $\|\cdot\|$ in $L^1([0,1])$ such that $\|f\|_1 \leq \|f\| \leq (1+\varepsilon)\|f\|_1$ for every $f \in L^1([0,1])$. Moreover the norm $\|\cdot\|$ has a very nice geometrical property, namely the so called Opial condition, which in turns implies the fixed point property for nonexpansive mappings. Thus, the Alspach mapping T is fixed point free and $(1+\varepsilon)$-uniformly lipschitzian with respect to a well-behaved norm. (Similar arguments can be repeated for each separable Banach space lacking the weak fixed point property). This seems to give a support to the following statement: a weakly compact convex subset C of a Banach space $(X, \|\cdot\|)$ lacks the fixed point property for nonexpansive mappings if and only if for every $\varepsilon > 0$ there exists a $(1+\varepsilon)$-uniformly lipschitzian fixed point free self-mapping of C. Our final question is whether (or not) this statement is true.

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