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On selections of the metric projection and best proximity pairs in hyperconvex spaces

Dedicated to W. A. Kirk on the occasion of his receiving an Honorary Doctorate from Maria Curie-Skłodowska University

Abstract. In this work we present new results on nonexpansive retractions and best proximity pairs in hyperconvex metric spaces. We sharpen the main results of R. Espínola et al. in [3] (*Nonexpansive retracts in hyperconvex spaces*, J. Math. Anal. Appl. **251** (2000), 557–570) on existence of nonexpansive selections of the metric projection. More precisely we characterize those subsets of a hyperconvex metric space with the property that the metric projection onto them admits a nonexpansive selection as a subclass of sets introduced in [3]. This is a rather exceptional property with a lot of applications in approximation theory, in particular we apply it to answer in the positive the main question posed by Kirk et al. in [5] (*Proximinal retracts and best proximity pair theorems*, Num. Funct. Anal. Opt. **24** (2003), 851–862).

1. Introduction. In [3] the author et al. introduced a subclass of subsets of a metric space, the so-called weakly externally hyperconvex subsets

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(see Section 2 for definitions), with the goal of characterizing those subsets of a metrically convex metric space for which there exists a nonexpansive selection of the metric projection. In this work we give the definitive solution to the main problems studied in that paper. More precisely it is proved that if M is a metrically convex metric space, A is a weakly externally hyperconvex of M and P_A is the metric projection on A (i.e. $P_A(x) = \{y \in A : \ d(x,y) = \inf\{d(x,u) : \ u \in A\}\}$) then there exists a nonexpansive selection R of P_A, this is

$$R(x) \in P_A(x) \text{ and } d(R(x), R(y)) \leq d(x,y) \text{ for } x, y \in M.$$

This is a rather exceptional property for a subset of a metric space which has a large number of nice consequences related to best approximation results (see for instance [3] and references therein). We apply this result to answer in the positive a question on best proximity pairs posed by Kirk et al. in [5]. Best proximity pairs raise in a very natural way in approximation theory when studying the proximity of two sets. For a proper motivation on best proximity pairs and their relation to fixed point theory the reader may check [5] and references therein. A subset E of a metric space M is said to be *proximinal* if given any $x \in M$ there exists $p_x \in E$ such that $d(x, p_x) = \text{dist}(x, E) = \inf\{d(x,y) : \ y \in E\}$. For A and B nonempty subsets of a metric space let $\text{dist}(A,B) = \inf\{d(x,y) : \ x \in A \text{ and } y \in B\}$.

Definition 1.1. Let X be a metric space and let A and B be nonempty subsets of X. Let

$$A_0 = \{x \in A : \ d(x,y) = \text{dist}(A,B) \text{ for some } y \in B\};$$
$$B_0 = \{x \in B : \ d(x,y) = \text{dist}(A,B) \text{ for some } y \in A\}.$$

A pair $(x,y) \in A_0 \times B_0$ for which $d(x,y) = \text{dist}(A,B)$ is called a best proximity pair for A and B.

In particular, it is proved in [5] that if M is a hyperconvex metric space and A and B are nonempty admissible subsets of M, then A_0 and B_0 are nonempty and hyperconvex (see also Proposition 2.14 in [5]). As a consequence of our main theorem we can extend this result to A and B weakly externally hyperconvex subsets of M. Next this is applied to answer in the positive a question posed in [5]. The last result of this work is another application of our main theorem, in this case we obtain the nonexpansive version of the Ky Fan's theorem given in [3] for hyperconvex spaces.

2. Definitions and preliminary results. This section contains the definitions and results that will be needed in the sequel. Hyperconvex metric spaces were introduced in 1956 by Aronszajn and Panitchpakdi in [1], for a detailed exposition on hyperconvex spaces the reader may consult the recent survey on them by the author and Khamsi [2].

Definition 2.1. A metric space M is said to be hyperconvex if given any family $\{x_\alpha\}$ of points of M and any family $\{r_\alpha\}$ of real numbers satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$$

it is the case that $\bigcap_\alpha B(x_\alpha; r_\alpha) \neq \emptyset$.

Next we give the definition of two subclasses of subsets of metric spaces.

Definition 2.2. A subset A of a metric space M is said to be admissible (in M) if it is an intersection of closed balls of M. Thus A is admissible if $A = \bigcap_{i \in I} B(x_i, r_i)$ where $x_i \in M$ and $r_i \geq 0$ for $i \in I$.

Definition 2.3. A subset E of a metric space M is said to be externally hyperconvex (relative to M) if given any family $\{x_\alpha\}$ of points in M and any family $\{r_\alpha\}$ of real numbers satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta \text{ and } \operatorname{dist}(x_\alpha, E) \leq r_\alpha$$

it follows that $\bigcap_\alpha B(x_\alpha; r_\alpha) \cap E \neq \emptyset$.

Externally hyperconvex subsets were shown in [4] to enjoy nice properties as, for instance, being always proximinal. The following theorem also gives a very important property of externally hyperconvex sets.

Theorem 2.4 ([4]). *Let M be hyperconvex, S a metric space and $T^\star$ a multivalued mapping from S into M such that $T^\star(x)$ is bounded nonempty externally hyperconvex for each $x \in S$, then there exists a selection $T : S \to M$ of T such that:*

$$d(T(x), T(y)) \leq d_H(T^\star(x), T^\star(y)) \text{ for all } x, y \in S,$$

where d_H denotes the usual Hausdorff metric on the family of nonempty bounded closed subsets of M.

The following notion plays a crucial role in this work.

Definition 2.5. A subset E of a metric space M is a proximinal nonexpansive retract of M if there exists a nonexpansive retraction R of M onto E for which

$$d(x, R(x)) = \operatorname{dist}(x, E)$$

for each $x \in M$. Thus $d(R(x), R(y)) \leq d(x, y)$ for each $x, y \in M$.

In an effort to characterize those subsets of a hyperconvex metric space which are proximinal nonexpansive retracts, the following definition was introduced in [3].

Definition 2.6. A subset E of a metric space M is said to be weakly externally hyperconvex (relative to M) if E is externally hyperconvex relative to $E \cup \{z\}$ for each $z \in M$. Precisely, given any family $\{x_\alpha\}$ of points in M

all but at most one of which lies in E, and any family $\{r_\alpha\}$ of real numbers satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta, \text{ with } \operatorname{dist}(x_\alpha, E) \leq r_\alpha \text{ if } x_\alpha \notin E,$$

it follows that $\bigcap_\alpha B(x_\alpha; r_\alpha) \cap E \neq \emptyset$.

It directly follows from the definition that weakly externally hyperconvex subsets are proximinal. At this point it is interesting to note that when the three classes of subsets so far presented are subsets of the same hyperconvex metric space M, then they are related in the following way: let A be a subset of M, then

A is admissible (in M) $\Rightarrow$ A is externally hyperconvex (relative to M)
$\Rightarrow$ A is weakly externally hyperconvex (relative to M)
$\Rightarrow$ A is hyperconvex.

The next definition was introduced in [6].

Definition 2.7. Let A be a subset of a metric space M. A mapping $R : A \to M$ is said to be ε-constant if $d(x, R(x)) \leq \varepsilon$ for each $x \in A$.

For A as above the ε-neighborhood of A is defined as follows:

$$N_\varepsilon(A) = \bigcup_{a \in A} B(a, \varepsilon).$$

The following fact will be needed.

Lemma 2.8 ([3])**.** *Let A be a weakly externally hyperconvex subset of a hyperconvex metric space M, then for any $\varepsilon > 0$ the set $N_\varepsilon(A)$ is weakly externally hyperconvex and there is an ε-constant nonexpansive retraction of $N_\varepsilon(A)$ on A.*

3. Proximinal nonexpansive retracts and best proximity pairs. We begin this section by recalling Theorem 3.1 in [3].

Theorem 3.1. *Suppose A is a weakly externally hyperconvex subset of a metrically convex metric space M. Then given any $\varepsilon > 0$ there exists a nonexpansive retraction $R : M \to A$ with the property that if $u \in M \setminus A$ there exists $v \in M \setminus A$ with $d(v, R(v)) = \operatorname{dist}(v, A)$ and $d(u, v) \leq \varepsilon$.*

Given A and M as above the ε-level set of A with respect to M is defined as follows:

$$S_\varepsilon = \{v \in M : \operatorname{dist}(v, A) = \varepsilon\}.$$

The following corollary, although not stated in [3], is however a consequence of the proof of the previous theorem.

Corollary 3.2. *Let $\varepsilon > 0$, A and M as above, and $S = \bigcup_{n \in \mathbb{N}} S_{n\varepsilon}$, then the retraction given by Theorem 3.1 can be chosen so that $d(v, R(v)) = \operatorname{dist}(v, A)$ for any $v \in S$.*

Our first result is the next technical lemma on Theorem 3.1.

Lemma 3.3. *Let A be a weakly externally hyperconvex subset of a metrically convex metric space M. For each $n \in \mathbb{N}$ let $\varepsilon_n = \frac{1}{2^n}$, then there exists a nonexpansive retraction r_n (associated to ε_n) as in Corollary 3.2 such that the sequence of retractions $\{r_n\}$ satisfies that*

$$d(r_n(x), r_m(x)) \leq \sum_{j=n+1}^{j=m} \frac{1}{2^j}$$

for $x \in M$ and $n < m$.

Proof. For $n = 1$ we take r_1 as the one given by Corollary 3.2. We prove next that given r_i for $1 \leq i \leq n$ as in the statement of the lemma we can construct r_{n+1} as required. We consider $S_{\varepsilon_{n+1}}$ and proceed as in Theorem 3.1. After applying Zorn's Lemma we may assume that $H_{\varepsilon_{n+1}}$ is the maximal subset of $S_{\varepsilon_{n+1}}$ where r_{n+1} can be extended as required. Then we need to prove that $S_{\varepsilon_{n+1}} = H_{\varepsilon_{n+1}}$. Suppose that there exits $v \in S_{\varepsilon_{n+1}} \setminus H_{\varepsilon_{n+1}}$ and let

$$P(v) = \left(\bigcap_{x \in A} B(x, d(x, v)) \right) \cap \left(\bigcap_{u \in H_{\varepsilon_{n+1}}} B(r_{n+1}(u), d(u, v)) \right)$$
$$\cap B\left(v, \frac{1}{2^{n+1}}\right) \cap B\left(r_n(v), \frac{1}{2^{n+1}}\right) \cap A.$$

All we need to prove is that $P(v) \neq \emptyset$. Since A is weakly hyperconvex and only one of the above balls is centered outside A, it is enough to check that each two of such balls have nonempty intersection. In a case-by-case check it only rests to study those cases involving the ball centered at $r_n(v)$, other cases were already studied in [3]. For these cases it is enough to recall that $d(x, r_n(v)) \leq d(x, v)$ for $x \in A$, now, since A is proximinal, let $p_v \in A$ such that $d(v, p_v) = \operatorname{dist}(v, A)$, so

$$\begin{aligned} d(v, r_n(v)) &\leq d(v, p_v) + d(p_v, r_n(v)) \\ &\leq 2 \operatorname{dist}(v, A) = \frac{1}{2^n}, \end{aligned}$$

and finally, for $u \in H_{\varepsilon_{n+1}}$,

$$d(r_{n+1}(u), r_n(v)) \leq d(r_{n+1}(u), r_n(u)) + d(r_n(u), r_n(v))$$

(by induction hypothesis)

$$\leq \frac{1}{2^{n+1}} + d(u, v).$$

So we can consider r_{n+1} defined on the whole $S_{\varepsilon_{n+1}}$ as required. Next we show how to extend r_{n+1} to $A \cup S_{\varepsilon_{n+1}} \cup S_{2\varepsilon_{n+1}}$. Let $v \in S_{2\varepsilon_{n+1}} = S_{\varepsilon_n}$, then the set

$$P(v) = \left(\bigcap_{x \in A} B(x, d(x, v)) \right) \cap \left(\bigcap_{u \in S_{\varepsilon_{n+1}}} B(r_{n+1}(u), d(u, v)) \right)$$
$$\cap B\left(v, \frac{1}{2^n}\right) \cap B\left(r_n(v), \frac{1}{2^{n+1}}\right) \cap A$$

is nonempty since $d(r_n(v), x) \leq d(v, x)$ for $x \in A$, $d(r_{n+1}(u), r_n(v)) \leq d(r_{n+1}(u), r_n(u)) + d(r_n(u), r_n(v)) \leq$ (by induction) $\frac{1}{2^{n+1}} + d(u, v)$, and, since the metric convexity of M implies that there exists $\hat{v} \in S_{\varepsilon_{n+1}}$ such that $d(v, \hat{v}) = \frac{1}{2^{n+1}}$, we have

$$d(v, r_n(v)) \leq d(v, \hat{v}) + d(\hat{v}, r_n(\hat{v})) + d(r_n(\hat{v}), r_n(v))$$
$$\leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n} + \frac{1}{2^{n+1}}.$$

Now, by selecting a point in $P(v)$ it is possible to extend r_{n+1} as required from $S_{\varepsilon_{n+1}}$ to $S_{\varepsilon_{n+1}} \cup \{v\}$. This same argument shows how to extend r_{n+1} to $A \cup S_{\varepsilon_{n+1}} \cup S_{2\varepsilon_{n+1}}$ onto A as required.

Let $S = \bigcup_{i=1}^{\infty} S_{i\varepsilon_{n+1}}$. By proceeding as above but selecting $\hat{v} \in S_{(i-1)\varepsilon_{n+1}}$ for $v \in S_{i\varepsilon_{n+1}}$, and using induction it follows that there exists a nonexpansive retraction r_{n+1} of $A \cup S$ onto A as required. Let $v \in M \setminus (A \cup S)$, then we consider the set

$$P(v) = \left(\bigcap_{x \in A \cup S} B(r_{n+1}(x), d(x, v)) \right) \cap B\left(r_n(v), \frac{1}{2^{n+1}}\right).$$

$P(v)$ is nonempty from the hyperconvexity of A and the fact that, by induction hypothesis,

$$d(r_{n+1}(x), r_n(v)) \leq d(r_{n+1}(x), r_n(x)) + d(r_n(x), r_n(v))$$
$$\leq d(x, v) + \frac{1}{2^{n+1}}.$$

Again, using induction it follows that r_{n+1} can defined on M as required. Hence

$$d(r_{n+1}(x), r_n(x)) \leq \frac{1}{2^{n+1}}$$

for $x \in M$. Now let $\{r_n\}$ be the sequence of retractions given by the above procedure, then for $m > n$ and $x \in M$

$$d(r_m(x), r_n(x)) \leq \sum_{j=n+1}^{j=m} d(r_j(x), r_{j-1}(x))$$
$$\leq \sum_{j=n+1}^{j=m} \frac{1}{2^j} .$$

Hence the proof of the lemma is completed. □

The following corollary is an immediate consequence of the previous lemma.

Corollary 3.4. *Let $\{r_n\}$ be the sequence of retractions given by Lemma 3.3, then $\{r_n(x)\}$ is convergent for each $x \in M$.*

Proof. To proof this corollary it is enough to recall that hyperconvex spaces are complete, hence A is complete. □

Next we present the main result of this work.

Theorem 3.5. *Let A be a complete weakly externally hyperconvex subset of a metrically convex metric space M, then A is a proximinal nonexpansive retract of M.*

Proof. Let $\{r_n\}$ be the sequence of retractions given by Lemma 3.3, then we define the mapping $r : M \to A$ as

$$r(x) = \lim_{n\to\infty} r_n(x).$$

Corollary 3.4 implies that r is a well-defined retraction on A. Moreover, since r_n is nonexpansive for each $n \in \mathbb{N}$, r is nonexpansive. Additionally we claim that $d(r(x), x) = \mathrm{dist}(x, A)$ for $x \in M$. For $x \in A$ there is nothing to prove, so let $x \in M \setminus A$. For each $n \in \mathbb{N}$ there exists $v_n \in M \setminus A$ such that

$$d(x, v_n) \leq \frac{1}{2^n} \quad \text{and}$$
$$d(v_n, r_n(v_n)) = \mathrm{dist}(v, A).$$

Hence we have

$$\begin{aligned} d(x, r_n(x)) &\leq d(x, v_n) + d(v_n, r_n(v_n)) + d(r_n(v_n), r_n(x)) \\ &\leq \frac{1}{2^n} + \mathrm{dist}(v_n, A) + \frac{1}{2^n} \\ &\leq \frac{1}{2^n} + \mathrm{dist}(x, A) + d(v_n, x) + \frac{1}{2^n} \\ &= \frac{3}{2^n} + \mathrm{dist}(x, A). \end{aligned}$$

Taking limit as $n \to \infty$ the conclusion follows. □

Since Theorem 2.1 of [3] implies that proximinal nonexpansive retracts of hyperconvex spaces are weakly externally hyperconvex, the previous theorem can be re-written in the following way.

Theorem 3.6. *Let M be a hyperconvex metric space and let $A \subseteq M$ be nonempty. Then A is a proximinal nonexpansive retract of M if, and only if, A is a weakly externally hyperconvex subset of M.*

Next we give applications of Theorems 3.5–3.6. We begin with an application to the existence of best proximity pairs, in particular we have the following extension of Proposition 2.8 in [5].

Corollary 3.7. *Let M be a hyperconvex metric space and let A and B be nonempty weakly externally hyperconvex subsets of M. Then A_0 and B_0 are nonempty and hyperconvex.*

Proof. The same proof of Proposition 2.8 in [5] carries over since A and B are proximinal nonexpansive retracts and, from Lemma 2.8, $N_\varepsilon(A)$ is weakly externally hyperconvex and there is and ε-constant nonexpansive retraction of $N_\varepsilon(A)$ on A. □

This corollary allows us to answer in the positive a question raised in [5], more precisely we obtain the following extension of Theorem 2.10 in [5].

Theorem 3.8. *Let A and B be two weakly externally hyperconvex subsets of a hyperconvex metric space M with A bounded, and suppose $T^* : A \to 2^B$ is such that:*

(i) *for each $x \in A$, $T^*(x)$ is a nonempty admissible (more generally, externally hyperconvex) subset of B;*
(ii) *$T^* : (A, d) \to (2^B, d_H)$ is nonexpansive (where d_H is the Hausdorff metric);*
(iii) *$T^*(A_0) \subseteq B_0$.*

Then there exists $x_0 \in A$ such that

$$\mathrm{dist}(x_0, T^*(x_0)) = \mathrm{dist}(A, B) = \inf\{\mathrm{dist}(x, T^*(x)) : x \in A\}.$$

Proof. The proof of this theorem follows the same steps as that of Theorem 2.10 in [5]. □

We finish this work with the nonexpansive version of the Fan's approximation principle given in [3] (Theorem 5.4). We omit its proof since it follows in a similar way as the proof of Theorem 5.4 in [3].

Corollary 3.9. *Let A be a bounded weakly externally hyperconvex subset of a hyperconvex metric space M and suppose that $T : A \to M$ is a nonexpansive mapping. Then there exists $x \in A$ such that*

$$d(x, T(x)) = \inf\{d(y, T(x)) : \ y \in A\}.$$

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