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On some classes of functions of Robertson type

ABSTRACT. Let Δ be the unit disc $|z| < 1$ and let $G(A,B)$, $-1 < A \leq 1$, $-A < B \leq 1$ be the class of functions of the form $g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$, holomorphic and nonvanishing in Δ and such that $\operatorname{Re}\left\{\frac{2zg'(z)}{g(z)} + \frac{1+Az}{1-Bz}\right\} > 0$ in Δ. It is known that the class $G = G(1,1)$ was introduced by M. S. Robertson. A. Lyzzaik has proved the Robertson conjecture on geometric properties of functions $g \in G$, $g \neq 1$.

In this paper we will investigate the properties of functions of the class $G(A,B)$. In particular when $A = B = 1$, we will obtain corresponding results of the class G.

1. Introduction. Let $\mathbb{C}$ denote the open complex plane, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc. In the sequel we will use the following well-known definitions. Let $S^*(\alpha)$, $0 \leq \alpha < 1$, denote the class of functions h holomorphic in Δ, normalized by $h(0) = h'(0) - 1 = 0$ and such that $\frac{h(z)}{z} \neq 0$ and

$$\operatorname{Re} \frac{zh'(z)}{h(z)} > \alpha, \quad z \in \Delta. \tag{1.1}$$

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Functions belonging to the class $S^*(\alpha)$ are called *starlike functions of order* α, while $S^* = S^*(0)$ is called the class of *starlike functions* (*with respect to the origin*).

Let $S^c(\alpha)$, $0 \le \alpha < 1$, denote the class of functions h of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \Delta$$

such that for every $z \in \Delta$ we have $h'(z) \neq 0$ and

$$\operatorname{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > \alpha. \tag{1.2}$$

Functions belonging to the class $S^c(\alpha)$ are called *convex functions of order* α.

It is noted that $h \in S^c(\alpha)$ if and only if $zh'(z) \in S^*(\alpha)$ for $0 \le \alpha < 1$ (see e.g. [3], vol. I, p. 140).

Let h be a holomorphic function in the disc Δ. We will say that h is *close-to-convex* in the unit disc Δ if and only if there is a function $\Phi \in S^c = S^c(0)$ such that

$$\operatorname{Re}\frac{h'(z)}{\Phi'(z)} > 0, \quad z \in \Delta. \tag{1.3}$$

It is known that the classes $S^*(\alpha)$ and $S^c(\alpha)$ were introduced by M. S. Robertson [10], while the class of normalized close-to-convex functions – by W. Kaplan [6]. We know also close-to-convex functions generally normalized (see e.g. [3], vol. II, p. 2).

Moreover, let $\wp$ denote the class of functions p holomorphic in Δ, $p(0)=1$ and such that $\operatorname{Re} p(z) > 0$ for $z \in \Delta$. This class is called *the class of Carathéodory functions with positive real part.*

In 1981 M. S. Robertson [11] introduced the class G of all functions g of the form

$$g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, \tag{1.4}$$

holomorphic and nonvanishing in Δ and such that

$$\operatorname{Re}\left\{\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z}\right\} > 0, \quad z \in \Delta. \tag{1.5}$$

Robertson also advanced a hypothesis (see [11]) on geometric interpretation of the functions of the family G. He assumed that if the function $g \in G$ and $g \neq 1$ then g is close-to-convex and univalent in Δ, $g(\Delta)$ is starlike with respect to the origin, $\lim_{r\to 1^-} g(r) = 0$ and for some $\alpha \in \mathbb{R}$ we have $\operatorname{Re}\{e^{i\alpha} g(z)\} > 0$, $z \in \Delta$. The above hypothesis was confirmed by A. Lyzzaik [8] in 1984.

A new analytic characterization of the class G has been presented in paper [7]. It is worth noticing that the analytic condition (1.5) was known to Styer [13].

In paper [4] there was introduced the class $G(M)$, $M > 1$, of functions g of the form (1.4) holomorphic and nonvanishing in Δ and such that

$$\operatorname{Re}\left\{\frac{2zg'(z)}{g(z)} + z\frac{P'(z;M)}{P(z;M)}\right\} > 0, \quad z \in \Delta, \tag{1.6}$$

where $P(\cdot;M)$ denotes the known Pick function. The class

$$G(1) = \left\{g \text{ of the form (1.4)} : g(z) \neq 0 \text{ and } \operatorname{Re}\left\{2z\frac{g'(z)}{g(z)} + 1\right\} > 0, z \in \Delta\right\} \tag{1.7}$$

was also considered.

Moreover, M. S. Obradović and S. Owa [9] investigated the class $G(\alpha)$, $0 \leq \alpha < 1$, of functions g of the form (1.4) holomorphic in the disc Δ, $g(z) \neq 0$ for $z \in \Delta$ and satisfying the condition

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)} + (1-\alpha)\frac{1+z}{1-z}\right\} > 0, \quad z \in \Delta. \tag{1.8}$$

The purpose of this paper is to introduce and investigate a new class of the aforesaid type.

2. Definition and some properties of the class $G(A, B)$.

Definition 2.1. Let $G(A,B)$, where $-1 < A \leq 1$, $-A < B \leq 1$, denote the class of functions g of the form (1.4) holomorphic and nonvanishing in disc Δ and such that

$$\operatorname{Re}\left\{\frac{2zg'(z)}{g(z)} + Q(z;A,B)\right\} > 0, \quad z \in \Delta, \tag{2.1}$$

where

$$Q(z;A,B) = \frac{1+Az}{1-Bz}\,, \quad z \in \Delta. \tag{2.2}$$

We note that the class $G(1,1)$ is identical to the known class G. Moreover, it is shown that $G(0,0) = G(1)$. If $B = -A$ then the function $Q(z;A,-A) \equiv 1$, so we have the class $G(1)$.

It is worth reminding in this place, that the function (2.2) was used in many papers, where different classes generated by the appropriate Carathéodory functions were considered.

It is known that the function Q of the form (2.2), when $B < 1$ maps conformally the disc Δ onto a disc situated on the right in the half-plane. If however $B = 1$, $Q(\Delta;A,B)$ is the half-plane $\{w : \operatorname{Re} w > \frac{1-A}{2}\}$, where $0 \leq \frac{1-A}{2} < 1$.

Let $Q(z) = Q(z; A, B)$ and let $F(z) = F(z; A, B)$ be a function satisfying the equation

$$\frac{zF'(z)}{F(z)} = Q(z), \tag{2.3}$$

where Q is of the form (2.2). Then $F \in S^*(A, B)$, $-1 < A \leq 1$, $-A < B \leq 1$ (see [5]), where

$$S^*(A, B) = \left\{ F : F(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \Delta \text{ and } \frac{zF'(z)}{F(z)} \prec Q(z) \right\}.$$

Furthermore, we have

$$F(z) = z \cdot \exp\left(\int_0^z \frac{Q(\zeta) - 1}{\zeta}\, d\zeta \right), \quad z \in \Delta. \tag{2.4}$$

If in the above-mentioned formula we put the function Q of the form (2.2), we will obtain the function of the form

$$F(z; A, B) = \begin{cases} z(1 - Bz)^{-\frac{A+B}{B}}, & z \in \Delta, \text{ for } B \neq 0, \\ z \exp(Az), & z \in \Delta, \text{ for } B = 0. \end{cases} \tag{2.5}$$

From (2.1) and (2.3) we conclude that for some function $g \in G(A, B)$ there exists a starlike function h of the class $S^* = S^*(1, 1)$ such that

$$g^2(z) \cdot F(z) = h(z), \quad z \in \Delta$$

and conversely. We have:

Property 2.1. *Let g be a holomorphic function in Δ such that $g(0) = 1$. Then $g \in G(A, B)$ if and only if there exists a function $h \in S^*$ such that*

$$g(z) = \sqrt{\frac{h(z)}{z}}(1 - Bz)^{\frac{A+B}{2B}}, \quad z \in \Delta,\ h \in S^*,\ \textit{for } B \neq 0, \tag{2.6}$$

$$g(z) = \sqrt{\frac{h(z)}{z}} \exp\left(-\frac{A}{2} z\right), \quad z \in \Delta,\ h \in S^*,\ \textit{for } B = 0. \tag{2.7}$$

Examples. It follows from Property 2.1 that the functions:

$$g_0(z; A, B) = \begin{cases} (1 - Bz)^{\frac{A+B}{2B}}, & z \in \Delta, \text{ for } B \neq 0, \\ \exp\left(-\frac{A}{2} z\right), & z \in \Delta, \text{ for } B = 0 \end{cases} \tag{2.8}$$

and

$$g_1(z; A, B) = \begin{cases} (1 - z)^{-1}(1 - Bz)^{\frac{A+B}{2B}}, & z \in \Delta, \text{ for } B \neq 0 \\ (1 - z)^{-1} \exp\left(-\frac{A}{2} z\right), & z \in \Delta, \text{ for } B = 0 \end{cases} \tag{2.9}$$

belong to the class $G(A,B)$. Furthermore, for $-1 < A \le 1$, $-A < B \le 1$, we have

$$g_1(z;A,B)$$
$$= 1+\left(1-\frac{1}{2}(A+B)\right)z+\left(1-\frac{1}{2}(A+B)+\frac{1}{8}(A^2-B^2)\right)z^2+\cdots, \ z\in\Delta.$$

The function

$$g_2(z) = \frac{1}{\sqrt{1-z^2}}, \quad z \in \Delta, \tag{2.10}$$

satisfies the condition

$$\mathrm{Re}\left(2z\frac{g_2'(z)}{g_2(z)}+1\right) > 0, \quad z\in\Delta,$$

so from (1.7) it follows that $g_2 \in G(1)$. Moreover, the function g_2 is not univalent, so $g_2 \notin G$.

Remark 2.1. Let us consider the function g_3, $g_3(0) = 1$, satisfying the equation

$$\frac{2zg_3'(z)}{g_3(z)} + \frac{1+Az}{1-Bz} = \frac{1+z^2}{1-z^2}, \quad z\in\Delta.$$

Because of (2.1) and (2.2) it is shown, that $g_3 \in G(A,B)$. We can check that if $B < 1$ then there exists a point $z_0 \in \Delta$ such that $g_3'(z) = 0$, i.e. g_3 is not a univalent function in Δ. Therefore $g_3 \notin G$.

We know the property (see e.g. [4], p. 56) that

$$f \in S^*\left(\frac{1}{2}\right) \Leftrightarrow h = \frac{f^2}{I}, \text{ where } I(z) \equiv z.$$

Hence from (2.6) and (2.7) we obtain:

Property 2.2. *Let g be a holomorphic function in Δ such that $g(0) = 1$. Then $g \in G(A,B)$ if and only if there exists a function $f \in S^*(\frac{1}{2})$ such that*

$$g(z) = \frac{f(z)}{z}(1-Bz)^{\frac{A+B}{2B}}, \quad z\in\Delta, \ \textit{for } B\neq 0, \tag{2.11}$$

$$g(z) = \frac{f(z)}{z}\exp\left(-\frac{A}{2}z\right), \quad z\in\Delta, \ \textit{for } B = 0. \tag{2.12}$$

From Property 2.1 and from the known estimates of the respective functionals in the class S^* we have:

Property 2.3. *If $g \in G(A,B)$, $-1 < A \le 1$, $-A < B \le 1$, $B \neq 0$, $0 \neq z = re^{i\varphi}$, $0 < r < 1$, $0 \le \varphi \le 2\pi$, then the following sharp estimates*

$$\frac{1}{1+|z|}\left|(1-Bz)^{\frac{A+B}{2B}}\right| \le |g(z)| \le \frac{1}{1-|z|}\left|(1-Bz)^{\frac{A+B}{2B}}\right|, \ |z| = r, \tag{2.13}$$

hold. The upper estimate is attained for the function g_ε of the form

$$g_\varepsilon(z) = (1 - Bz)^{\frac{A+B}{2B}} \cdot \sqrt{\frac{k_\varepsilon(z)}{z}}\ ,$$

where $k_\varepsilon(z) = \frac{z}{(1-\varepsilon z)^2}$, $\varepsilon = e^{-i\varphi}$, and the lower estimate for g_ε and $\varepsilon = -e^{-i\varphi}$.

If $g \in G(A, 0)$, then for $0 \neq z = re^{i\varphi}$, $0 < r < 1$, $0 \leq \varphi \leq 2\pi$ we have

$$\frac{1}{1+|z|} \exp\left(-\frac{A}{2}\mathrm{Re}\, z\right) \leq |g(z)| \leq \frac{1}{1-|z|} \exp\left(-\frac{A}{2}\mathrm{Re}\, z\right). \tag{2.14}$$

The extremal function for the upper estimate (2.14) is the function g^*_ε of the form

$$g^*_\varepsilon(z) = \exp\left(-\frac{A}{2} z\right) \sqrt{\frac{k_\varepsilon(z)}{z}}\ ,$$

where $\varepsilon = e^{-i\varphi}$, while for the lower estimate is the function g^*_ε for $\varepsilon = -e^{-i\varphi}$.

Let $0 < B < 1$. Then from (2.13) we have $|g(z)| \geq \frac{1}{2}(1 - B)^{\frac{A+B}{2B}}$ for $z \in \Delta$. If $-A < B < 0$ so $|g(z)| \geq \frac{1}{2}(1 - B)^{\frac{A+B}{2B}}$, but when $B = 0$ then from (2.14) $|g(z)| \geq \frac{1}{2}\exp\left(-\frac{1}{2}|A|\right)$ for $z \in \Delta$. In consequence we obtain:

Property 2.4. *If $g \in G(A, B)$, $g \neq 1$, $B < 1$, then there exists the constant $\delta > 0$ such that $|g(z)| > \delta$ for $z \in \Delta$.*

The point $w = 0$ is not the boundary point of the set $g(\Delta)$ for any function g from class $G(A, B)$, $B < 1$, and consequently $g \notin G$.

Property 2.5. *If $g \in G(A, B)$, $-1 < A \leq 1$, $-A < B \leq 1$, is of the form (1.4) then the sharp estimates*

$$|2d_1 + A + B| \leq 2, \tag{2.15}$$

$$\left|2d_2 + d_1^2 + 2d_1(A + B) + \frac{1}{2}(A + B)(A + 2B)\right| \leq 3 \tag{2.16}$$

hold. We obtain the equality in the above estimates for the function g_1 of the form (2.9).

Because for each function $h \in S^*$ the functions

$$z \to \frac{1}{\rho} h(\rho z), \quad z \to e^{i\varphi} h\left(e^{-i\varphi} z\right), \ 0 < \rho < 1, \ \varphi \in \mathbb{R}, \ z \in \Delta, \tag{2.17}$$

also belong to S^*, from Property 2.1 and estimation (2.15) we obtain:

Property 2.6. *The region of values of the coefficient d_1, i.e. $\{d_1 : g \in G(A, B),\ g(z) = 1 + d_1 z + \cdots\}$ has the form*

$$\left\{w \in \mathbb{C} : \left|w + \frac{A + B}{2}\right| \leq 1\right\}.$$

From the global formula (2.4) and Property 2.1 it follows:

Property 2.7. *If $g \in G(A,B)$, $-1 < A \le 1$, $-A < B \le 1$, then for $B \neq 0$*

$$g(z) = (1-Bz)^{\frac{A+B}{2B}} \cdot \exp\left(\frac{1}{2}\int_0^z \frac{p(\zeta)-1}{\zeta}\,d\zeta\right), \quad z \in \Delta,\ p \in \wp, \tag{2.18}$$

and for $B = 0$

$$g(z) = \exp\left(\frac{1}{2}\Big(-Az + \frac{1}{2}\int_0^z \frac{p(\zeta)-1}{\zeta}\,d\zeta\Big)\right), \quad z \in \Delta,\ p \in \wp, \tag{2.19}$$

and conversely, where $\wp$ denotes the aforesaid class of Carathéodory functions with positive real part.

We know that if $f \in S^*\left(\frac{1}{2}\right)$ then the function Φ defined by the formula

$$\Phi(z,\xi) = \frac{\xi}{f(\xi)} \cdot \frac{f(z)-f(\xi)}{z-\xi}, \quad z,\xi \in \Delta \tag{2.20}$$

satisfies the condition $\operatorname{Re}\Phi(z,\xi) > \frac{1}{2}$ (see [12], p. 121). Moreover, if $g \in G(A,B)$, $B \neq 0$ then from (2.11) the function

$$f(z) = zg(z)\,(1-Bz)^{-\frac{A+B}{2B}}, \quad z \in \Delta, \tag{2.21}$$

belongs to the class $S^*\left(\frac{1}{2}\right)$. We denote

$$d_0 = 1 = P_0(A,B), \quad (1-Bz)^{-\frac{A+B}{2B}} = 1 + \sum_{k=1}^{\infty} P_k(A,B) z^k, \quad z \in \Delta, \tag{2.22}$$

where

$$P_k(A,B) = \frac{(A+B)(A+3B)\cdot \ldots \cdot (A+(2k-1)B)}{k!2^k}, \quad k = 1,2,\ldots.$$

We prove:

Theorem 2.1. *Let $g \in G(A,B)$, $-1 < A \le 1$, $-A < B \le 1$, $B \neq 0$, $g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$, $z \in \Delta$ and let $R_n(z;A,B)$ denote the n-th partial sum of the power series expansion with the centre at the origin of the function $z \to g(z)(1-Bz)^{-\frac{A+B}{2B}}$, $R_0(z;A,B) \equiv 1$. Then the functions*

$$\Phi_n(z;A,B) = \frac{g(z) - (1-Bz)^{\frac{A+B}{2B}} \cdot R_{n-1}(z;A,B)}{z^n \cdot g(z)}, \tag{2.23}$$

$z \in \Delta$, $n = 1,2,\ldots$, are holomorphic in Δ and

$$|\Phi_n(z;A,B)| \le 1. \tag{2.24}$$

In particular

$$|\Phi_n(0;A,B)| = \left|\sum_{k=0}^{n} d_k P_{n-k}(A,B)\right| \le 1, \tag{2.25}$$

$$
\begin{aligned}
&\left|\Phi_n'(0;A,B)\right| \\
&\quad = \left|\sum_{k=0}^{n+1} d_k P_{n+1-k}\,(A,B) - (P_1(A,B)+d_1)\sum_{k=0}^{n} d_k P_{n-k}(A,B)\right| \\
&\quad \le 1 - \left|\sum_{k=0}^{n} d_k P_{n-k}(A,B)\right|^2 ,
\end{aligned} \tag{2.26}
$$

$$
|d_n - g_n(A,B)|^2 + \sum_{k=1}^{p} |d_{n+k} - g_{n+k}(A,B)|^2 \le 1 + \sum_{k=1}^{p} |d_k|^2, \quad p \ge 1, \tag{2.27}
$$

where

$$
g_k(A,B) = d_k, \quad k = 0,1,\dots,n-1,
$$

$$
\begin{aligned}
&d_0 P_{n+k}(A,B) + \cdots + d_{n-1} P_{k+1}(A,B) + g_n(A,B) P_k(A,B) \\
&\qquad + \cdots + g_{n+k}(A,B) P_0(A,B) = 0, \quad k = 0,1,\dots.
\end{aligned} \tag{2.28}
$$

Proof. By the assumption $g \in G(A,B)$, $B \neq 0$, so the function f of the form (2.21) belongs to the class $S^*\left(\frac{1}{2}\right)$. Let $z, \xi \in \Delta$. We consider the function Φ of the form (2.20). Hence we obtain

$$
\Phi(z,\xi) = \frac{1}{1-\frac{z}{\xi}} - \frac{1}{1-\frac{z}{\xi}} \cdot \frac{z}{\xi} \cdot \frac{g(z)(1-Bz)^{-\frac{A+B}{2B}}}{g(\xi)(1-B\xi)^{-\frac{A+B}{2B}}}\ , \quad z,\xi \in \Delta.
$$

The expansion of the function Φ in powers of z yields

$$
\Phi(z,\xi) = 1 + \sum_{n=1}^{\infty} \Phi_n(\xi;A,B) z^n, \quad z \in \Delta,
$$

where the functions $\Phi_n(\xi;A,B)$ are defined by formulas (2.23).

We notice that for all $n = 1,2,\dots$ the functions Φ_n are holomorphic in Δ. Moreover, because $\operatorname{Re}\Phi(z,\xi) > \frac{1}{2}$ then from the known estimate of the coefficients in the class $\wp$ we obtain the estimates (2.24).

On the other hand, because of (2.22) and the definition the function $R_{n-1}(\xi;A,B)$, from (2.23) we have

$$
\Phi_n(\xi;A,B) = \frac{S_n(A,B) + S_{n+1}(A,B)\xi + \cdots + S_{n+k}(A,B)\xi^k + \dots}{1 + S_1(A,B)\xi + \cdots}\ ,
$$

where

$$
S_n(A,B) = d_0 P_n(A,B) + \cdots + d_n P_0(A,B), \quad n = 1,2,\dots.
$$

Hence and from inequality (2.24) for $z = 0$ we obtain (2.25).

The inequality (2.26) is a consequence of the fact that if

$$
\Phi_n(\xi;A,B) = a_0 + a_1\xi + a_2\xi^2 + \cdots
$$

and

$$
|\Phi_n(\xi;A,B)| < 1 \text{ for } \ \xi \in \Delta,
$$

then $|a_1| \leq 1 - |a_0|^2$.

From (2.23) we have

$$\Phi_n(z; A, B) \cdot g(z) = \frac{g(z) - (1 - Bz)^{\frac{A+B}{2B}} \cdot R_{n-1}(z; A, B)}{z^n} \ ,$$

$z \in \Delta$, $n = 1, 2, \ldots$.

Put

$$G(z; A, B) = (1 - Bz)^{\frac{A+B}{2B}} \cdot R_{n-1}(z; A, B), \quad z \in \Delta$$

and let

$$G(z; A, B) = \sum_{n=0}^{\infty} g_n(A, B) z^n, \quad z \in \Delta.$$

Equating coefficients at the respective powers of z of identity

$$G(z; A, B) \cdot (1 - Bz)^{-\frac{A+B}{2B}} = R_{n-1}(z; A, B), \quad z \in \Delta,$$

we have (2.28). Then

$$G(z; A, B) = \sum_{k=0}^{n-1} d_k z^k + g_n(A, B) z^n + g_{n+1}(A, B) z^{n+1} + \cdots, \quad z \in \Delta.$$

From this and from (2.23) we have

$$\sum_{k=0}^{p} \left(d_{n+k} - g_{n+k}(A, B)\right) z^k + \sum_{k=p+1}^{\infty} a_k(A, B) z^k = \left(\sum_{k=0}^{p} d_k z^k\right) \cdot \Phi_n(z; A, B),$$

where $a_k(A, B)$ are the appropriate coefficients. From the inequality (2.24) we obtain

$$\left|\sum_{k=0}^{p} \left(d_{n+k} - g_{n+k}(A, B)\right) z^k + \sum_{k=p+1}^{\infty} a_k(A, B) z^k\right|^2 \leq \left|\sum_{k=0}^{\infty} d_k z^k\right|^2.$$

Let $z = re^{it}$, $0 < r < 1$, $0 \leq t \leq 2\pi$. Integrating the above inequality side-wise in the interval $[0, 2\pi]$ and making use of the equality $z\bar{z} = |z|^2$, $z \in \mathbb{C}$, we obtain

$$\sum_{k=0}^{p} |d_{n+k} - g_{n+k}(A, B)|^2 r^{2k} + \sum_{k=p+1}^{\infty} |a_k(A, B)|^2 r^{2k} \leq \sum_{k=0}^{p} |d_k|^2 r^{2k}.$$

Passing to the limit as $r \to 1^-$ and from the fact that $|a_k(A, B)|^2 \geq 0$, $k = p + 1, \ldots$, $p \geq 1$ we have (2.27). $\square$

We know that Theorem 2.1 has its equivalents in the classes G (see [1]) and $G(M)$, $M > 1$ (see [4]).

Similarly we do in case $B = 0$.

Let $g \in G(A,0)$. Then from (2.12) the function $f(z) = z \cdot g(z)\exp(\frac{A}{2}z)$, $z \in \Delta$ belongs to the class $S^*\left(\frac{1}{2}\right)$. Denote

$$d_0 = 1 = P_0(A), \quad \exp\left(\frac{A}{2}z\right) = 1 + \sum_{n=1}^{\infty} P_n(A)z^n, \quad z \in \Delta, \tag{2.29}$$

where $P_n(A) = \frac{A^n}{2^n \cdot n!}$. It is clear that $P_0(A) = P_0(A,0)$, $P_n(A) = P_n(A,0)$. We obtain:

Theorem 2.2. *Let $g \in G(A,0)$, $0 < A \leq 1$, $g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$, $z \in \Delta$ and let $R_n(z;A)$ denote the n-th partial sum of the power series expansion with the centre at the origin of the function $z \to g(z)\exp\left(\frac{A}{2}z\right)$, $R_0(z;A) \equiv 1$. Then the functions*

$$\Phi_n(z;A) = \frac{g(z) - \exp\left(-\frac{A}{2}z\right) \cdot R_{n-1}(z;A)}{z^n \cdot g(z)}, \quad z \in \Delta,\ n = 1, 2, \ldots, \tag{2.30}$$

are holomorphic in Δ and

$$|\Phi_n(z;A)| \leq 1. \tag{2.31}$$

In particular

$$|\Phi_n(0;A)| = \left|\sum_{k=0}^{n} d_k P_{n-k}(A)\right| \leq 1, \tag{2.32}$$

$$\begin{aligned} \left|\Phi_n'(0;A)\right| &= \left|\sum_{k=0}^{n+1} d_k P_{n+1-k}(A) - (P_1(A) + d_1)\sum_{k=0}^{n} d_k P_{n-k}(A)\right| \\ &\leq 1 - \left|\sum_{k=0}^{n} d_k P_{n-k}(A)\right|^2, \end{aligned} \tag{2.33}$$

$$|d_n - g_n(A)|^2 + \sum_{k=1}^{p} |d_{n+k} - g_{n+k}(A)|^2 \leq 1 + \sum_{k=1}^{p} |d_k|^2, \quad p \geq 1, \tag{2.34}$$

where

$$g_k(A) = d_k, \quad k = 0, 1, \ldots, n-1,$$

$$\begin{aligned} &d_0 P_{n+k}(A) + \cdots + d_{n-1}P_{k+1}(A) + g_n(A)P_k(A) \\ &\qquad + \cdots + g_{n+k}(A)P_0(A) = 0, \quad k = 0, 1, \ldots. \end{aligned} \tag{2.35}$$

From (2.23), (2.24) and (2.30), (2.31) for $n = 1$ we have:

Corollary 2.1. *If $g \in G(A,B)$, $-1 < A \leq 1$, $-A < B \leq 1$, $B \neq 0$, then*

$$\left|g(z) - \frac{(1 - Bz)^{\frac{A+B}{2B}}}{1 - |z|^2}\right| \leq \frac{\left|(1 - Bz)^{\frac{A+B}{2B}}\right| |z|}{1 - |z|^2}, \quad z \in \Delta. \tag{2.36}$$

If $g \in G(A, 0)$ *then*

$$\left| g(z) - \frac{\exp\left(-\frac{A}{2}\operatorname{Re} z\right)}{1-|z|^2} \right| \le \frac{\exp\left(-\frac{A}{2}\operatorname{Re} z\right)|z|}{1-|z|^2}\,, \quad z \in \Delta. \tag{2.37}$$

Remark 2.2. If in Theorem 2.1 we put $A = B = 1$ then we obtain the known theorem for the class G (see [1] p. 11). Furthermore, from (2.36) for $g \in G$ we have

$$\left| g(z) - \frac{1-z}{1-|z|^2} \right| \le \frac{|1-z||z|}{1-|z|^2}\,, \quad z \in \Delta.$$

Remark 2.3. If $g \in G(A, B)$, $B \neq 0$, $z \in \Delta$ any fixed then the values of the functional $H(g) = g(z)$, $g \in G(A, B)$ belong to $\overline{K(w_0, |zw_0|)}$ where $w_0 = \frac{(1-Bz)^{\frac{A+B}{2B}}}{1-|z|^2}$. Since $w_0 \neq 0$ and $|zw_0| < |w_0|$, we have $0 \notin \overline{K(w_0, |zw_0|)}$.

From (2.27), (2.28), (2.22) and (2.34), (2.35), (2.29) we have:

Corollary 2.2. *If* $g \in G(A, B)$, $-1 < A \le 1$, $-A < B \le 1$ *then*

$$\left| d_1 + \frac{A+B}{2} \right|^2 + \left| d_2 - \frac{1}{8}\left(A^2 - B^2\right) \right|^2 \le 1 + |d_1|^2. \tag{2.38}$$

The extremal function is the function g_1 *of the form* (2.9).

3. Application of classical Cluni method. In the following considerations we are using the so-called Cluni method (see [2]), i.e. without using the function (2.20).

Let the function g of the form (1.4) belong to the class $G(A, B)$. Thus the conditions (2.1), (2.2) are satisfied. It follows that there exists a function $p \in \wp$ such that

$$p(z) = \frac{2zg'(z)}{g(z)} + \frac{1+Az}{1-Bz}\,, \quad z \in \Delta. \tag{3.1}$$

It is known that if $p \in \wp$ then the function ω of the form

$$\omega(z) = \frac{p(z)-1}{p(z)+1}\,, \quad z \in \Delta,$$

belongs to the known class Ω (ω holomorphic in Δ, $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \Delta$). From this fact and from (3.1) we have

$$\begin{aligned}\Big(2zg'(z)(1-Bz) + 2g(z) + (A-B)zg(z)\Big)\omega(z) \\ = 2zg'(z)(1-Bz) + (A+B)zg(z), \quad z \in \Delta.\end{aligned}$$

Let $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n$. Considering the expansion of the function g in power series we get

$$\left(2+2\sum_{n=1}^{\infty}(n+1)d_n z^n+(A-B)\sum_{n=1}^{\infty} d_{n-1}z^n-2B\sum_{n=1}^{\infty}(n-1)d_{n-1}z^n\right)\left(\sum_{n=1}^{\infty}\omega_n z^n\right)$$
$$= 2\sum_{n=1}^{\infty} nd_n z^n - 2B\sum_{n=1}^{\infty}(n-1)d_{n-1}z^n + (A+B)\sum_{n=1}^{\infty} d_{n-1}z^n,\ z \in \Delta.$$

From this

$$\begin{aligned}&\left(2+\sum_{n=1}^{\infty}\left(2(n+1)d_n + (A+B-2Bn)d_{n-1}\right)z^n\right)\left(\sum_{n=1}^{\infty}\omega_n z^n\right)\\ &\qquad = \sum_{n=1}^{\infty}(2nd_n + (A+3B-2Bn)d_{n-1})z^n, \quad z \in \Delta.\end{aligned} \tag{3.2}$$

Let

$$p_n(A,B) = 2(n+1)d_n + (A+B-2Bn)d_{n-1}, \quad n = 1,2,\dots, \tag{3.3}$$

and

$$s_n(A,B) = 2nd_n + (A+3B-2Bn)d_{n-1}, \quad n = 1,2,\dots. \tag{3.4}$$

Then we obtain

$$2\sum_{n=1}^{\infty}\omega_n z^n + \sum_{n=2}^{\infty}\left(p_1(A,B)\omega_{n-1} + \dots + p_{n-1}(A,B)\omega_1\right)z^n$$
$$= \sum_{n=1}^{\infty} s_n(A,B)z^n, \quad z \in \Delta.$$

Equating coefficients on both sides of the above identity we have

$$2\omega_1 = 2d_1 + A + B, \tag{3.5}$$

$$2\omega_n + p_1(A,B)\omega_{n-1} + \dots + p_{n-1}(A,B)\omega_1 = s_n(A,B) \text{ for } n = 2,3,\dots. \tag{3.6}$$

Since $|\omega_1| \le 1$, from (3.5) we obtain

$$|2d_1 + A + B| \le 2,$$

which is identical to the estimate (2.15).

Next from (3.2)–(3.4) we have

$$\left(2+\sum_{k=1}^{n-1}p_k(A,B)z^k\right)\left(\sum_{k=1}^{\infty}\omega_k z^k\right) = \sum_{k=1}^{n} s_k(A,B)z^k + \sum_{k=n+1}^{\infty} a_k z^k,$$

where a_k are the appropriate coefficients. Since $|\omega(z)| < 1$ for $z \in \Delta$,

$$\left|\sum_{k=1}^{n} s_k(A,B)z^k + \sum_{k=n+1}^{\infty} a_k z^k\right|^2 < \left|2 + \sum_{k=1}^{n-1} p_k(A,B)z^k\right|^2, \quad z \in \Delta.$$

Similarly to the proof of the inequality (2.27) we get

$$\sum_{k=1}^{n} |s_k(A,B)|^2 \le 4 + \sum_{k=1}^{n-1} |p_k(A,B)|^2, \quad n = 2,3,\dots. \tag{3.7}$$

Since $|s_k(A,B)|^2 \ge 0$ for $k = 1,\dots,n-1$, then

$$|s_n(A,B)|^2 \le 4 + \sum_{k=1}^{n-1} |p_k(A,B)|^2, \quad n = 2,3,\dots.$$

If we adopt the notation (3.3), (3.4), we get:

Theorem 3.1. *If the function g of the form* (1.4) *belongs to the class $G(A,B)$, then the estimates*

$$\begin{aligned} &\left|2nd_n + (A+3B-2Bn)d_{n-1}\right|^2 \\ &\quad \le 4 + \sum_{k=1}^{n-1} |2(k+1)d_k + (A+B-2Bk)d_{k-1}|^2, \quad n = 2,3,\dots \end{aligned} \tag{3.8}$$

hold.

Remark 3.1. If we put $n = 2$ in (3.7), then we have

$$|2d_1 + A + B|^2 + |4d_2 + (A-B)d_1|^2 \le 4 + |4d_1 + (A-B)|^2. \tag{3.9}$$

This estimate is different from (2.38).

4. The class $G[\alpha] = G(1-2\alpha, 1)$. Relations between classes $G[0]$, $G[\alpha]$ and $G[1]$. We have recalled different applications of the function (2.2) in geometric theory of functions. In particular we often use it when $B = 1$ and $A = 1-2\alpha$, $0 \le \alpha < 1$. Hence we consider the class

$$G[\alpha] := G(1-2\alpha, 1), \quad 0 \le \alpha < 1.$$

Obviously, $G[0] = G(1,1) = G$ and $G[1] = G(0,0) = G(1)$. Furthermore, from the obtained properties of the class $G(A,B)$ we get the corresponding properties of the class $G[\alpha]$. In particular we have:

Property 4.1.

$$g \in G[\alpha] \Leftrightarrow g(z) = \sqrt{\frac{h(z)}{z}}(1-z)^{1-\alpha}, \quad h \in S^*; \tag{4.1}$$

$$g \in G[\alpha] \Rightarrow |d_1 + 1 - \alpha| \le 1; \tag{4.2}$$

$$g \in G[\alpha] \Rightarrow |g(z) - \frac{(1-z)^{1-\alpha}}{1-|z|^2}| \le \frac{|(1-z)^{1-\alpha}||z|}{1-|z|^2} \ . \tag{4.3}$$

Mutual relations between classes $G[0]$, $G[\alpha]$, $G[1]$, $0 < \alpha < 1$ are also worth considering.

Let

$$g_0(z) \equiv 1, \quad I(z) \equiv z, \quad z \in \Delta. \tag{4.4}$$

Obviously

$$g_0 \in G[0] \cap G[\alpha] \cap G[1], \quad 0 < \alpha < 1. \tag{4.5}$$

Since $I \in S^*$, from (4.1) the function g_1 of the form

$$g_1(z;\alpha) = (1-z)^{1-\alpha}, \ z \in \Delta, \ 0 \le \alpha < 1, \tag{4.6}$$

satisfies the conditions

$$g_1(\cdot;\alpha) \in G[\alpha], \ 0 \le \alpha < 1 \ \text{ and } \ g_1(\cdot;\alpha) \notin G[1]. \tag{4.7}$$

On the other hand for the function

$$g_2(z) = 1 - z, \ z \in \Delta, \tag{4.8}$$

we have

$$g_2 \in G[0] \ \text{ and } \ g_2 \notin G[\alpha] \ \text{ for } \ 0 < \alpha \le 1. \tag{4.9}$$

The function $h(z) = \frac{z}{(1-z)^2} \in S^*$, so from (4.1) for the function g_3 of the form

$$g_3(z;\alpha) = \frac{1}{(1-z)^\alpha} \ , \quad z \in \Delta, \tag{4.10}$$

the following conditions hold

$$g_3(\cdot;\alpha) \notin G[0] \ \text{ and } \ g_3(\cdot;\alpha) \in G[\alpha], \ 0 < \alpha \le 1. \tag{4.11}$$

We see that the point $z_0 = 0$ is not the boundary point of the set $g_3(\Delta)$ (it is an exterior point).

However for the function

$$g_4(z) = \frac{1}{1-z} \ , \quad z \in \Delta, \tag{4.12}$$

we have

$$g_4 \notin G[\alpha], \ 0 \le \alpha < 1 \ \text{ and } \ g_4 \in G[1]. \tag{4.13}$$

If we consider in the property (4.1) the function $h(z) = \frac{z}{(1+z)^2} \in S^*$, then we have the mapping

$$g_5(z;\alpha) = \frac{(1-z)^{1-\alpha}}{1+z} \ , \quad z \in \Delta, \tag{4.14}$$

satisfying the conditions

$$g_5(\cdot;\alpha) \notin G[0], \ g_5(\cdot;\alpha) \in G[\alpha], \ 0 < \alpha < 1 \ \text{ and } \ g_5(\cdot;\alpha) \notin G[1]. \tag{4.15}$$

From the above-mentioned examples (4.4), (4.6), (4.8), (4.10), (4.12), (4.14) and the obtained conditions (4.5), (4.7), (4.9). (4.11), (4.13), (4.15) we have

$$\begin{aligned} G[0] \cap G[\alpha] \cap G[1] &\neq \emptyset, \\ G[0] \setminus (G[\alpha] \cup G[1]) &\neq \emptyset, \\ G[1] \setminus (G[\alpha] \cup G[0]) &\neq \emptyset, \\ G[\alpha] \setminus (G[0] \cup G[1]) &\neq \emptyset, \\ (G[0] \cap G[\alpha]) \setminus G[1] &\neq \emptyset, \\ (G[1] \cap G[\alpha]) \setminus G[0] &\neq \emptyset. \end{aligned}$$

From the above relationships we get a question, whether a function $g \in G[0] \cap G[1]$, $g \neq 1$ exists. The answer is negative. We have:

Corollary 4.1. *The function $g \in G[0] \cap G[1]$, $g \neq 1$ does not exist. The intersection of classes $G[0]$, $G[\alpha]$ for $0 < \alpha < 1$, and $G[1]$ is a singleton, i.e. $G[0] \cap G[\alpha] \cap G[1] = \{g_0\}$.*

Indeed, suppose on the contrary that there exists a function $g \in G[0] \cap G[1]$, $g \neq 1$. Then from (4.1) we have

$$g \in G[0] \Leftrightarrow g^2(z) = (1-z)^2 \frac{h_1(z)}{z}\ ,\ z \in \Delta,\ h_1 \in S^*,$$

and

$$g \in G[1] \Leftrightarrow g^2(z) = \frac{h_2(z)}{z}\ ,\ z \in \Delta,\ h_2 \in S^*.$$

From this

$$h_2(z) = (1-z)^2 h_1(z),\ z \in \Delta,\ h_1, h_2 \in S^*.$$

We see that

$$\operatorname{Re}\left\{\frac{zh_2'(z)}{h_2(z)}\right\} = \operatorname{Re}\left\{\frac{-2z}{1-z} + \frac{zh_1'(z)}{h_1(z)}\right\}, \quad z \in \Delta,\ h_1, h_2 \in S^*.$$

But for $z \to 1^-$ we have $\operatorname{Re}\left\{\frac{zh_2'(z)}{h_2(z)}\right\} \to -\infty$, which contradicts the definition of the function $h_2 \in S^*$.

Unfortunately, we do not know so far any mutual relations between $G[\alpha_1]$ and $G[\alpha_2]$, where $\alpha_1 \neq \alpha_2$.

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